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ON EXCESS ENTROPIES FOR STATIONARY RANDOM FIELDS*

BY

WOJCIECH BUŁATEK AND BRUNON KAMIŃSKI (TORUŃ)

Abstract. We study the behaviour of the excess entropies of stationary random fields defined by Crutchfield and Feldman in two classes of random fields: Conze fields and product fields.

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INTRODUCTION

The excess entropy plays an important role in the theory of dynamical systems and information theory. It has been introduced by Crutchfield and Packard (cf. [2], [5], [6]) and then investigated also by other authors (see, e.g., [3], [9], [11]).

In [4] Crutchfield and Feldman defined three generalizations of the excess entropy to stationary random fields. Namely, these are defined as: the convergence excess entropy E_C, the mutual information excess entropy E_I, and the subextensive excess entropy E_S. In [4] there are studied entropies E_C and E_I for Ising models.

The purpose of this paper is to study these entropies on two classes of stationary random fields: the Conze fields induced by stationary processes and product fields induced by pairs of stationary processes.

It is well known (cf. [3], [7]) that the analogues of E_C, E_I and E_S for stationary processes coincide. We will show that this is not the case for random fields. Of course, all these entropies are equal for Bernoulli random fields. We prove that if one takes a slight and natural modification $\tilde{E}_I$ of E_I, then $E_C = \tilde{E}_I$ for Conze fields.

We also show that the behaviour of excess entropies for random fields is different than in the case of processes when the entropy of a random field is zero or if it has the Markov property.

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1. EXCESS ENTROPY FOR STOCHASTIC PROCESSES

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space and let $T : X \to X$ be an invertible, measurable, measure-preserving transformation (automorphism) of $(X, \mathcal{B}, \mu)$.

Any pair (P, T), where P is a finite measurable partition (shortly, *partition*) of X, is said to be a *stochastic process* (or, shortly, *process*) induced by P and T.

A consideration of the above pairs as processes instead of sequences of random variables forming strictly stationary processes with a finite state space is often applied. This approach is especially convenient when one makes use of entropy theory.

For a partition $P = \{P_1, P_2, \ldots, P_r\}$ of X we denote by $H_\mu(P)$ the entropy of P, i.e.

$$H_\mu(P) = \sum_{i=1}^{r} \eta\big(\mu(P_i)\big),$$

where $\eta(x) = -x \log x$, $x \in (0, 1]$ and $\eta(0) = 0$.

If $Q = \{Q_1, \ldots, Q_s\}$ is another partition of X, we denote by $H_\mu(P|Q)$ the conditional entropy of P given Q, i.e.

$$H_\mu(P|Q) = \sum_{j=1}^{s} \mu(Q_j) \sum_{i=1}^{r} \eta\big(\mu(P_i|Q_j)\big).$$

The number

$$I(P; Q) = H_\mu(P) - H_\mu(P|Q)$$

is called the *mutual information* of P and Q.

The partition being the join of partitions $P, T^{-1}P, \ldots, T^{-(n-1)}P$ is denoted by $\bigvee_{i=0}^{n-1} T^{-i}P$, $n \geqslant 1$.

We consider the sequence (h_n) defined by

$$h_0 = 0, \quad h_n = H\big(\bigvee_{i=0}^{n-1} T^{-i}P\big)$$

and let

$$\Delta h_n = h_n - h_{n-1}, \quad n \geqslant 1.$$

The limit

$$h_\mu = h_\mu(P, T) = \lim_{n\to\infty} \frac{h_n}{n}$$

is called the *entropy* of the process (P, T). It is well known that this limit exists (cf. [15]).

Moreover, the sequence (Δh_n) is non-increasing and

$$h_\mu \leqslant \Delta h_n \leqslant \frac{h_n}{n}, \quad n \geqslant 1.$$

On the other hand, the sequence $(h_n - nh_\mu)$ is non-negative and non-decreasing.

One associates with (P, T) the following three quantities:

$$E_C = E_C(P, T) = \sum_{n=1}^{\infty} (\Delta h_n - h_\mu),$$

$$E_I = E_I(P, T) = \lim_{n\to\infty} I\Big(\bigvee_{i=-n}^{-1} T^i P; \bigvee_{i=0}^{n-1} T^i P\Big) = \lim_{n\to\infty} (2h_n - h_{2n}),$$

$$E_S = E_S(P, T) = \lim_{n\to\infty} (h_n - nh_\mu).$$

Previous studies show (cf. [3], [7]) that these three numbers coincide. Their common value is denoted by $E = E(P, T)$ and called the *excess entropy* of the process (P, T).

It is easy to check that

(1.1) *If* (P, T) *is Markovian*, *then* $E(P, T) = h_1 - h_\mu$.
In particular, $E(P, T) = 0$ *iff* (P, T) *is Bernoullian.*

One can show more. Namely (cf. [7]):

(1.2) *If* (P, T) *is a function of a Markov process*, *then* $E(P, T) < \infty$.

It is shown in [7] that for any process (P, T) we have

$$E(P, T) \geqslant h_1 - h_\mu. \tag{1.3}$$

Now, let us suppose that the process (P, T) is aperiodic, i.e. for any set $A \in \bigvee_{i=-\infty}^{+\infty} T^i P$ of positive measure μ and any $n \in \mathbb{N} \setminus \{0\}$ there exists a set $B \in \bigvee_{i=-\infty}^{+\infty} T^i P$, $B \subset A$, with a positive measure μ with $T^n B \cap B = \emptyset$ (cf. [10]).

It is well known that for this class of processes the Rokhlin–Kakutani lemma is satisfied. It states that for any $n \in \mathbb{N} \setminus \{0\}$ and $\varepsilon > 0$ there exists a set $F \in \bigvee_{i=-\infty}^{+\infty} T^i P$ such that the sets $F, TF, \ldots, T^{n-1}F$ are pairwise disjoint and

$$\mu\Big(\bigcup_{i=0}^{n-1} T^i F\Big) > 1 - \varepsilon.$$

It follows that

(1.4) *If* (P, T) *is aperiodic and* $h_\mu = 0$, *then* $E(P, T) = \infty$.

P r o o f. Since $h_\mu = 0$, it is enough to notice the well-known fact:

$$E(P, T) = \lim_{n\to\infty} h_n = \infty.$$

Because we have not found its proof in literature, we show it here for completeness.

Let l be an arbitrary positive integer. It is enough to find $m = m_l \in \mathbb{N}$ such that

$$H_\mu\big(\bigvee_{i=-m}^{m} T^i P\big) > \log l - 1.$$

Choose $\delta > 0$ such that

$$\sum_{i=0}^{l-1} \eta(x_i) > \log l - \frac{1}{2}$$

for any $(x_0, \ldots, x_{l-1}) \in [0,1]^l$ with $|x_i - 1/l| < \delta$, $i = 0, \ldots, l-1$.

By the Rokhlin–Kakutani lemma there exists a set $F \in \bigvee_{i=-\infty}^{+\infty} T^i P$ such that the sets $F, TF, \ldots, T^{l-1}F$ are pairwise disjoint and

$$\mu\big(X \setminus \bigcup_{i=0}^{l-1} T^i F\big) < l \cdot \delta.$$

Let Q be the partition $\big\{F, TF, \ldots, T^{l-1}F, X \setminus \bigcup_{i=0}^{l-1} T^i F\big\}$. Since the numbers $x_i = \mu(T^i F)$ satisfy the inequality $|x_i - 1/l| < \delta$, $i = 0, \ldots, l-1$, we have

$$H_\mu(Q) \geqslant \sum_{i=0}^{l-1} \eta\big(\mu(T^i F)\big) = l \cdot \eta\big(\mu(F)\big) > \log l - \frac{1}{2}.$$

Applying the continuity of entropy $H_\mu(R)$ as a functional of a partition R with respect to the standard partition distance $|\cdot|$ (cf. [14]), set $\gamma > 0$ such that for any partition R with $\sharp R = l + 1$ and $|R - Q| < \gamma$ we have

$$|H_\mu(R) - H_\mu(Q)| < \frac{1}{2}.$$

Now, let $m = m_l \in \mathbb{N}$ and $\tilde{Q} = \{\tilde{F}_0, \tilde{F}_1, \ldots, \tilde{F}_l\}$, a partition of X, be such that

$$|Q - \tilde{Q}| < \gamma, \quad \tilde{F}_i \in \bigvee_{i=-m}^{m} T^i P, \quad i = 0, \ldots, l.$$

Therefore, $|H_\mu(Q) - H_\mu(\tilde{Q})| < \frac{1}{2}$, and so

$$H_\mu\big(\bigvee_{i=-m}^{m} T^i P\big) \geqslant H_\mu(\tilde{Q}) > H_\mu(Q) - \frac{1}{2} > \log l - 1, \quad l \geqslant 1,$$

which implies the desired property.

Let (P, T) and (Q, S) be two processes $P = \{P_1, \ldots, P_r\}$, $Q = \{Q_1, \ldots, Q_s\}$ on Lebesgue spaces $(X, \mathcal{B}, \mu)$, $(Y, \mathcal{C}, \nu)$, respectively. The process $(P \times Q, T \times S)$, where

$$P \times Q = \{P_i \times Q_j \colon 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s\},$$

is said to be the *product* of (P,T) and (Q,S). One can easily prove

$$E(P \times Q, T \times S) = E(P,T) + E(Q,S). \tag{1.5}$$

Applying (1.4) and (1.5) one can easily give examples of (P,T) with positive entropy and $E(P,T) = \infty$.

A process (P,T) is said to be the *Pinsker* one if for any non-trivial finite partition $Q \subset \bigvee_{n=-\infty}^{+\infty} T^n P$ the entropy $h_\mu(Q,T)$ is positive.

This study does not answer the following question: is it possible to find a Pinsker process with $E(P,T) = \infty$?

2. EXCESS ENTROPIES FOR RANDOM FIELDS

First, we introduce a notation for some subsets of $\mathbb{Z}^2$.

For positive integers m, n we put

$$R_{m,n} = \{(i,j) \in \mathbb{Z}^2 \colon |i| \leqslant m-1,\ |j| \leqslant n-1\}, \quad R_n = R_{n,n},$$

$$R_{m,n}^+ = \{(i,j) \in R_{m,n} \colon i \geqslant 0,\ j \geqslant 0\},$$

$$R_{m,n}^- = R_{m,n}^+ - (m,0), \quad R_{m,n}^0 = R_{m,n}^+ \cup R_{m,n}^-.$$

Let π^- (π^+) denote the set of all negative (non-negative) elements of $\mathbb{Z}^2$ with respect to the lexicographical order, i.e.

$$\pi^- = \{(i,j) \in \mathbb{Z}^2 \colon i \leqslant -1 \text{ or } i = 0 \text{ and } j \leqslant -1\}, \quad \pi^+ = \mathbb{Z}^2 \setminus \pi^-.$$

We put

$$\pi_n^\pm = R_n \cap \pi^\pm, \quad n \geqslant 1.$$

Let now $(X, \mathcal{B}, \mu)$ be, as in the previous section, a Lebesgue probability space and let Φ be a measure-preserving $\mathbb{Z}^2$-action on it, i.e. a homomorphism of the group $\mathbb{Z}^2$ into the group of all automorphisms of $(X, \mathcal{B}, \mu)$.

Any pair (P, Φ), where P is a partition of X, is said to be a *random field* induced by P and Φ.

As in the theory of stationary stochastic processes, one can think of random fields as stationary sequences of random variables indexed by elements of $\mathbb{Z}^2$ the state space of which is finite.

For a subset $A \subset \mathbb{Z}^2$ we put

$$P(A) = \bigvee_{g \in A} \Phi^g P.$$

First, we recall the concept of entropy of a random field.

Let (A_n) be a *Følner sequence* in $\mathbb{Z}^2$, i.e. every set A_n is non-empty, finite, $n \geqslant 1$, and

$$\lim_{n \to \infty} \frac{\sharp[(g + A_n) \cap A_n]}{\sharp A_n} = 1$$

for any $g \in \mathbb{Z}^2$.

For example, the sequence (R_n) defined above is the Følner one.

It is well known (cf. [13]) that for any Følner sequence (A_n) the limit

$$h_\mu = h_\mu(P,\Phi) = \lim_{n\to\infty} \frac{1}{\sharp A_n} H\big(P(A_n)\big)$$

exists, does not depend on the choice of (A_n) and is equal to the conditional entropy $H\big(P|P(\pi_\omega^-)\big)$, where π_ω^- denotes the set of all elements of $\mathbb{Z}^2$ negative with respect to any total order ω consistent with the addition in $\mathbb{Z}^2$.

The convergence of the above sequence in the case when $A_n,\ n \geqslant 1$, are parallelograms is proved in [1] .

In the sequel we shall use two general well-known classes of random fields.

A random field (P,Φ) is called *Markovian* if for any positive integer n the σ-algebras $P(R_{n-1})$ and $P(\mathbb{Z}^2 \setminus R_n)$ are relatively independent with respect to the σ-algebra $P(R_n \setminus R_{n-1}), n \geqslant 2$.

It is clear that (P,Φ) is Markovian iff for any $m,n \in \mathbb{N},\ m > n$, the σ-algebras $P(R_{n-1})$ and $P(R_m \setminus R_n)$ are relatively independent with respect to $P(R_n \setminus R_{n-1})$. This condition is of course equivalent to the equality

$$H_\mu\big(P(R_m)\big) = H_\mu\big(P(R_n)\big) + H_\mu\big(P(R_m \setminus R_{n-1})\big) - H_\mu\big(P(R_n \setminus R_{n-1})\big).$$

In particular, if the partitions $\Phi^g P,\ g \in \mathbb{Z}^2$, are independent, we say that (P,Φ) is a *Bernoulli random field.*

Now we recall the concepts of excess entropies given in [4] applying our notation.

The first excess entropy, called the *convergent excess entropy*, is defined as follows:

$$E_C = E_C(P,\Phi) = \sum_{n=1}^{\infty} \big(h_\mu(n) - h_\mu\big),$$

where $h_\mu(n) = H\big(P|P(\pi_n^-)\big),\ n \geqslant 1$. As $h_\mu(n) - h_\mu$ is positive for all $n \in \mathbb{N}$, the above infinite sum always makes sense (eventually being infinite).

The second excess entropy, called the *mutual information excess entropy*, is defined by the formula

$$E_I = E_I(P,\Phi) = \lim_{m,n\to\infty} I\big(P(R_{m,n}^+); P(R_{m,n}^-)\big).$$

This limit always exists (eventually being equal to $+\infty$).To see this observe first that

$$I\big(P(R_{m,n}^+); P(R_{m,n}^-)\big) = H_\mu\big(P(R_{m,n}^+)\big) + H_\mu\big(P(R_{m,n}^-)\big) - H_\mu\big(P(R_{m,n}^0)\big).$$

If now k,l are such that $m < k,\ n < l$, then

$$\begin{aligned} &I\big(P(R_{k,l}^+); P(R_{k,l}^-)\big) \\ &= H_\mu\big(P(R_{m,n}^+) \vee P(R_{k,l}^+ \setminus R_{m,n}^+)\big) + H_\mu\big(P(R_{m,n}^-) \vee P(R_{k,l}^- \setminus R_{m,n}^-)\big) \end{aligned}$$

$$
\begin{aligned}
&- H_\mu\big(P(R_{m,n}^+) \vee P(R_{k,l}^+ \setminus R_{m,n}^+) \vee P(R_{m,n}^-) \vee P(R_{k,l}^- \setminus R_{m,n}^-)\big)\\
&= H_\mu\big(P(R_{m,n}^+)\big) + H_\mu\big(P(R_{k,l}^+ \setminus R_{m,n}^+)|P(R_{m,n}^+)\big) + H_\mu\big(P(R_{m,n}^-)\big)\\
&+ H_\mu\big(P(R_{k,l}^- \setminus R_{m,n}^-)|P(R_{m,n}^-)\big) - H_\mu\big(P(R_{m,n}^+) \vee P(R_{m,n}^-)\big)\\
&- H_\mu\big(P(R_{k,l}^+ \setminus R_{m,n}^+) \vee P(R_{k,l}^- \setminus R_{m,n}^-)|P(R_{m,n}^+) \vee P(R_{m,n}^-)\big).
\end{aligned}
$$

Because the last term equals $H_\mu\big(P(R_{k,l}^+ \setminus R_{m,n}^+)|P(R_{m,n}^+) \vee P(R_{m,n}^-)\big) + H_\mu\big(P(R_{k,l}^- \setminus R_{m,n}^-)|P(R_{m,n}^+) \vee P(R_{m,n}^-) \vee P(R_{k,l}^+ \setminus R_{m,n}^+)\big)$, we can write

$$
\begin{aligned}
&I\big(P(R_{k,l}^+); P(R_{k,l}^-)\big)\\
&\quad = I\big(P(R_{m,n}^+); P(R_{m,n}^-)\big) + H_\mu\big(P(R_{k,l}^+ \setminus R_{m,n}^+)|P(R_{m,n}^+)\big)\\
&\quad - H_\mu\big(P(R_{k,l}^+ \setminus R_{m,n}^+)|P(R_{m,n}^+) \vee P(R_{m,n}^-)\big)\\
&\quad + H_\mu\big(P(R_{k,l}^- \setminus R_{m,n}^-)|P(R_{m,n}^-)\big)\\
&\quad - H_\mu\big(P(R_{k,l}^- \setminus R_{m,n}^-)|P(R_{m,n}^+) \vee P(R_{m,n}^-) \vee P(R_{k,l}^+ \setminus R_{m,n}^+)\big).
\end{aligned}
$$

Obviously, the last two differences are non-negative, hence the double sequence $I\big(P(R_{m,n}^+); P(R_{m,n}^-)\big)$ is non-decreasing (in the sense that $I\big(P(R_{m,n}^+); P(R_{m,n}^-)\big) \leqslant I\big(P(R_{m,n}^+); P(R_{m,n}^-)\big)$ for $m < k,\ n < l$), and therefore it is either convergent or diverges to $+\infty$.

The third excess entropy, called the *subextensive excess entropy*, is defined as follows:

$$
E_S = E_S(P, \Phi) = \lim_{m,n\to\infty} \big(h_\mu(m,n) - E_S^1 \cdot m - E_S^2 \cdot n - h_\mu \cdot mn\big),
$$

where $h_\mu(m,n) = H\big(P(R_{m,n}^+)\big)$ and E_S^1 (respectively, E_S^2) is the excess entropy of the process $(P, \Phi^{(1,0)})$ (respectively, $(P, \Phi^{(0,1)})$).

It is clear that for Bernoulli random field we have $E_C = E_I = E_S = 0$.

There are no natural conditions for $E_S(P, \Phi)$ to be well defined. Observe that if at least one of the processes $(P, \Phi^{(1,0)}), (P, \Phi^{(0,1)})$ has zero entropy, then $h_\mu = 0$, and so $E_S(P, \Phi) = -\infty$. (See also conclusion 3 in Section 5, where we give a class of random fields for which $E_S(P, \Phi)$ is not defined.)

Now we shall study the behaviour of E_C , E_I and E_S in two classes of random fields: Conze fields and product fields.

3. CONZE FIELDS

Let $(Y, \mathcal{C}, \nu)$ be a Lebesgue probability space and let (P, φ) be a process on it. We consider the product space

$$
(X, \mathcal{B}, \mu) = \prod_{n=-\infty}^{+\infty} (Y_n, \mathcal{C}_n, \nu_n), \qquad (Y_n, \mathcal{C}_n, \nu_n) = (Y, \mathcal{C}, \nu), \quad n \in \mathbb{Z}.
$$

Let $T : X \to X$ be the left shift transformation, i.e. $(Tx)_n = x_{n+1}$, and let $S_\varphi : X \to X$ be defined as follows:

$$(S_\varphi x)_n = \varphi x_n, \quad n \in \mathbb{Z}.$$

The transformations T and S_φ are commuting automorphisms of X.

We define the action Φ of $\mathbb{Z}^2$ on X by the formula (cf. [1])

$$\Phi_\varphi^{(m,n)} = T^m \circ S_\varphi^n$$

and let $P^* = \pi_0^{-1} P$, where $\pi_0 : X \to Y$ is the projection on the zero coordinate.

The random field (P^*, Φ_φ) is said to be the *Conze field* induced by the process (P, φ).

Observe that the entropy of (P^*, Φ_φ) coincides with the entropy of the process (P, φ). Indeed, we have

$$H_\mu\big(P^*(R_{n+1})\big) = H_\mu\Big(\bigvee_{i=-n}^{n} \bigvee_{j=-n}^{n} T^i S_\varphi^j P^*\Big).$$

Since μ is a product measure, the partitions

$$T^i\Big(\bigvee_{j=-n}^{n} S_\varphi^j P^*\Big), \quad -n \leqslant i \leqslant n,$$

are independent, and so applying the definition of S_φ we get

$$H_\mu\big(P^*(R_{n+1})\big) = (2n+1) \cdot H_\nu\Big(\bigvee_{j=-n}^{n} \varphi^j P\Big), \quad n \geqslant 1.$$

Therefore, we obtain

$$h_\mu(P^*, \Phi_\varphi) = h_\nu(P, \varphi).$$

Now we shall find the values of E_C, E_I and E_S.

THEOREM 3.1. *The Crutchfield–Feldman excess entropies for the Conze random field have the following values*:

$$E_C(P^*, \Phi_\varphi) = E_C(P, \varphi), \tag{3.1}$$

$$E_I(P^*, \Phi_\varphi) = 0, \tag{3.2}$$

and

$$E_S(P^*, \Phi_\varphi) = \lim_{m,n\to\infty} \Big[m\Big(H_\nu\Big(\bigvee_{j=0}^{n-1} \varphi^j P\Big) - n h_\nu(P, \varphi)\Big) - n E_S(P, \varphi)\Big]. \tag{3.3}$$

P r o o f. In order to show (3.1) observe that

$$h_\mu(n) = H_\mu\big(P^*|P^*(\pi_n^-)\big) = H_\mu\Big(P^*\big(\pi_n^- \cup \{(0,0)\}\big)\Big) - H_\mu\big(P^*(\pi_n^-)\big).$$

Since μ is the product measure, we have

$$H_\mu\Big(P^*\big(\pi_{n+1}^- \cup \{(0,0)\}\big)\Big) = H_\mu\big(\bigvee_{j=-n}^{0} S_\varphi^j P^* \vee \bigvee_{i=-n}^{-1} \bigvee_{j=-n}^{n} T^i S_\varphi^j P^*\big)$$

$$= H_\mu\big(\bigvee_{j=-n}^{0} S_\varphi^j P^*\big) + nH_\mu\big(\bigvee_{j=-n}^{n} S_\varphi^j P^*\big) = H_\nu\big(\bigvee_{j=-n}^{0} \varphi^j P\big) + nH_\nu\big(\bigvee_{j=-n}^{n} \varphi^j P\big).$$

Similarly we get

$$H_\mu\big(P^*(\pi_{n+1}^-)\big) = H_\nu\big(\bigvee_{j=-n}^{-1} \varphi^j P\big) + nH_\nu\big(\bigvee_{j=-n}^{n} \varphi^j P\big),$$

and so

$$h_\mu(n+1) = H_\nu\big(\bigvee_{j=-n}^{0} \varphi^j P\big) - H_\nu\big(\bigvee_{j=-n}^{-1} \varphi^j P\big)$$
$$= H_\nu\big(P|\bigvee_{j=-n}^{-1} \varphi^j P\big) = h_\nu(n+1).$$

Since

$$h_\mu = h_\mu(P^*, \Phi) = h_\nu(P, \varphi) = h_\nu,$$

we get

$$E_C(P^*, \Phi_\varphi) = \sum_{n=1}^{\infty} \big(h_\mu(n) - h_\mu\big) = \sum_{n=1}^{\infty} \big(h_\nu(n) - h_\nu\big) = E_C(P, \varphi),$$

i.e. (3.1) is satisfied.

In order to calculate $E_I(P^*, \Phi_\varphi)$ observe that proceeding as in the proof of (3.1) we have

$$H_\mu\big(P^*(R_{m,n}^+)\big) = mH_\nu\big(\bigvee_{j=0}^{n-1} \varphi^j P\big) = H_\mu\big(P^*(R_{m,n}^-)\big)$$

and

$$H_\mu\big(P^*(R_{m,n}^0)\big) = 2mH_\nu\big(\bigvee_{j=0}^{n-1} \varphi^j P\big).$$

Hence

$$I\big(P^*(R_{m,n}^+); P^*(R_{m,n}^-)\big) = H\big(P^*(R_{m,n}^+)\big) + H\big(P^*(R_{m,n}^-)\big) - H\big(P^*(R_{m,n}^0)\big) = 0,$$

and so

$$E_I(P^*, \Phi_\varphi) = 0,$$

which gives (3.2).

For any $m, n \geqslant 1$ we have

$$h_\mu(m,n) = H_\mu\big(P^*(R^+_{m,n})\big) = mH_\nu\big(\bigvee_{j=0}^{n-1} \varphi^j P\big).$$

Since the process (P^*, T) is Bernoullian, we have

$$H_\mu\big(\bigvee_{i=0}^{l-1} T^i P^*\big) = l \cdot H_\mu(P^*) = l \cdot H_\nu(P).$$

On the other hand,

$$H_\mu\big(\bigvee_{i=0}^{l-1} S^i_\varphi P^*\big) = H_\nu\big(\bigvee_{i=0}^{l-1} \varphi^i P\big), \quad l \geqslant 1.$$

Therefore,

$$h_\mu(P^*, T) = H_\nu(P), \qquad h_\mu(P^*, S_\varphi) = h_\nu(P, \varphi),$$

and so

$$E^1_S = 0, \quad E^2_S = E_S(P, \varphi).$$

Thus

$$\begin{aligned} E_S(P, \Phi_\varphi) &= \lim_{m,n\to\infty} \big(h_\mu(m,n) - E^1_S \cdot m - E^2_S \cdot n - h_\mu \cdot mn\big) \\ &= \lim_{m,n\to\infty} \Big(mH_\nu\big(\bigvee_{j=0}^{n-1} \varphi^j P\big) - E_S(P,\varphi)\cdot n - h_\nu(P,\varphi)\cdot mn\Big) \\ &= \lim_{m,n\to\infty} \Big[m\Big(H_\nu\big(\bigvee_{j=0}^{n-1} \varphi^j P\big) - nh_\nu(P,\varphi)\Big) - E_S(P,\varphi)\cdot n\Big], \end{aligned}$$

i.e. (3.3) is satisfied. ∎

It follows from Theorem 3.1 that in general $E_C(P^*, \Phi_\varphi) \neq E_I(P^*, \Phi_\varphi)$.

Now we propose a slight modification $\tilde{E}_I$ of E_I to get the equality $E_C = \tilde{E}_I$ in the class of Conze fields. Namely, for any random field (Q, Ψ) we put

$$\tilde{E}_I(Q, \Psi) = \lim_{n\to\infty} I\big(Q(\pi^-_n); Q(\pi^+_n)\big).$$

One can easily show that this limit exists.

Now we shall show the following

PROPOSITION 3.1. *If (P^*, Φ_φ) is a Conze random field generated by a process (P, φ), then*

$$\tilde{E}_I(P^*, \Phi_\varphi) = E_I(P, \varphi) = E_C(P^*, \Phi_\varphi). \tag{3.4}$$

P r o o f. Arguing as in the proof of (3.1) we have

$$H_\mu\big(P^*(R_{n+1,n+1})\big) = (2n+1)\cdot H_\nu\big(\bigvee_{j=-n}^{n}\varphi^j P\big),$$

$$H_\mu\big(P^*(\pi_{n+1}^-)\big) = H_\nu\big(\bigvee_{j=-n}^{-1}\varphi^j P\big) + n\cdot H_\nu\big(\bigvee_{j=-n}^{n}\varphi^j P\big),$$

$$H_\mu\big(P^*(\pi_{n+1}^+)\big) = H_\nu\big(\bigvee_{j=0}^{n}\varphi^j P\big) + n\cdot H_\nu\big(\bigvee_{j=-n}^{n}\varphi^j P\big), \quad n \geqslant 1.$$

Hence

$$\begin{aligned} I\big(P^*(\pi_n^+); P^*(\pi_n^-)\big) &= H_\mu\big(P^*(\pi_n^+)\big) + H_\mu\big(P^*(\pi_n^-)\big) - H_\mu\big(P^*(R_{n,n})\big) \\ &= I\big(\bigvee_{j=0}^{n-1}\varphi^j P; \bigvee_{j=-(n-1)}^{-1}\varphi^j P\big), \quad n \geqslant 1, \end{aligned}$$

and so

$$\tilde{E}_I(P^*, \Phi_\varphi) = E_I(P, \varphi).$$

The second equality in (3.4) follows at once from (3.1) and the known equality $E_I = E_C$ for processes. ∎

4. PRODUCT RANDOM FIELDS

Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be Lebesgue probability spaces equipped with automorphisms T and S, respectively.

Let $(Z, \mathcal{A}, \lambda)$ denote the product space $(X, \mathcal{B}, \mu) \times (Y, \mathcal{C}, \nu)$. We consider an action Φ of the group $\mathbb{Z}^2$ on $(Z, \mathcal{A}, \lambda)$, called the *product* one, defined as follows (cf. [8]):

$$\Phi^{(m,n)}(x, y) = (T^m x, S^n y), \quad (x, y) \in Z, \quad (m, n) \in \mathbb{Z}^2.$$

Let P and Q be partitions of X and Y, respectively, and let $P \times Q$ denote the product partition of Z defined in Section 1.

The pair $(P \times Q, \Phi)$ is said to be the *product random field* generated by the processes (P, T) and (Q, S).

PROPOSITION 4.1. *Any product random field is a Markov field with zero entropy.*

P r o o f. Let $(P \times Q, \Phi)$ be the product field induced by processes (P, T) and (Q, S). One easily checks that for any finite subset $A = \{(i_1, j_1), \ldots, (i_r, j_r)\}$ of $\mathbb{Z}^2$ we have

$$H_{\mu\times\nu}\big((P \times Q)(A)\big) = H_\mu\big(\bigvee_{k=1}^{r} T^{i_k} P\big) + H_\nu\big(\bigvee_{k=1}^{r} S^{j_k} Q\big). \tag{4.1}$$

Therefore, for any $n \geqslant 1$ we have

$$H_{\mu\times\nu}\big((P\times Q)(R_{n+1})\big) = H_\mu\big(\bigvee_{i=-n}^{n} T^i P\big) + H_\nu\big(\bigvee_{j=-n}^{n} S^j Q\big). \tag{4.2}$$

Hence

$$h_{\mu\times\nu}(P\times Q,\Phi) = \lim_{n\to\infty}\frac{1}{(2n+1)^2}H_{\mu\times\nu}\big((P\times Q)(R_{n+1})\big) = 0.$$

Now we shall verify the Markov property of the considered random field. From (4.1) for any $m \geqslant n \geqslant 1$ we obtain

$$\begin{aligned} H_{\mu\times\nu}\big((P\times Q)(R_{m+1})\big) &= H_{\mu\times\nu}\big((P\times Q)(R_{m+1}\setminus R_n)\big)\\ &= H_\mu\big(\bigvee_{i=-m}^{m} T^i P\big) + H_\nu\big(\bigvee_{j=-m}^{m} S^j Q\big). \end{aligned}$$

This implies the equality

$$\begin{aligned} &H_{\mu\times\nu}\big((P\times Q)(R_{n+1})\big) + H_{\mu\times\nu}\big((P\times Q)(R_{m+1}\setminus R_n)\big)\\ &\quad = H_{\mu\times\nu}\big((P\times Q)(R_{m+1})\big) + H_{\mu\times\nu}\big((P\times Q)(R_{n+1}\setminus R_n)\big), \end{aligned}$$

which means that the algebras $(P\times Q)(R_{n-1})$ and $(P\times Q)(R_m\setminus R_n)$ are relatively independent with respect to $(P\times Q)(R_n\setminus R_{n-1})$, i.e. the field (P,Φ) is Markovian. ■

THEOREM 4.1. *The Crutchfield–Feldman excess entropies for product random fields have the following values*:

$$E_C(P\times Q,\Phi) = 0, \tag{4.3}$$

$$E_I(P\times Q,\Phi) = E_I(P,T) + \lim_{n\to\infty} H_\nu\big(\bigvee_{j=0}^{n-1} S^j Q\big), \tag{4.4}$$

$$E_S(P\times Q,\Phi) = E_S(P,T) + E_S(Q,S). \tag{4.5}$$

P r o o f. For any $n \geqslant 0$ we have

$$\begin{aligned} h_{\mu\times\nu}(n+1) &= H_{\mu\times\nu}\big(P\times Q\,|\,(P\times Q)(\pi^-_{n+1})\big)\\ &= H_{\mu\times\nu}\Big((P\times Q)\big(\pi^-_{n+1}\cup\{(0,0)\}\big)\Big) - H_{\mu\times\nu}\big((P\times Q)(\pi^-_{n+1})\big)\\ &\overset{\text{df}}{=} \tilde h_{n+1} - \tilde h'_{n+1} \end{aligned}$$

and

$$\tilde h_{n+1} = H_{\mu\times\nu}\big(\bigvee_{i=-n}^{-1}\bigvee_{j=-n}^{n}(T^i\times S^j)(P\times Q)\ \vee\ P\times\bigvee_{j=-n}^{0}S^jQ\big)$$
$$= H_{\mu\times\nu}\big(\bigvee_{i=-n}^{0}T^iP\times\bigvee_{j=-n}^{n}S^jQ\big) = H_\mu\big(\bigvee_{i=-n}^{0}T^iP\big)+H_\nu\big(\bigvee_{j=-n}^{n}S^jQ\big).$$

Similar calculations give

$$\tilde h'_{n+1} = \tilde h_{n+1},$$

and so $h_\mu(n) = 0$, $n \geqslant 1$. Since $h_{\mu\times\nu}(P\times Q,\Phi) = 0$, we obtain (4.3).

In a similar manner we obtain

$$H_{\mu\times\nu}\big((P\times Q)(R^+_{m,n})\big) = H_\mu\big(\bigvee_{i=0}^{m-1}T^iP\big)+H_\nu\big(\bigvee_{j=0}^{n-1}S^jQ\big), \tag{4.6}$$

$$H_{\mu\times\nu}\big((P\times Q)(R^-_{m,n})\big) = H_\mu\big(\bigvee_{i=-m}^{-1}T^iP\big)+H_\nu\big(\bigvee_{j=0}^{n-1}S^jQ\big),$$

and

$$H_{\mu\times\nu}\big((P\times Q)(R^0_{m,n})\big) = H_\mu\big(\bigvee_{i=-m}^{m-1}T^iP\big)+H_\nu\big(\bigvee_{j=0}^{n-1}S^jQ\big).$$

Thus

$$I\big((P\times Q)(R^+_{m,n});(P\times Q)(R^-_{m,n})\big)$$
$$= H_\mu\big(\bigvee_{i=0}^{m-1}T^iP\big)+H_\mu\big(\bigvee_{i=-m}^{-1}T^iP\big)-H_\mu\big(\bigvee_{i=-m}^{m-1}T^iP\big)+H_\nu\big(\bigvee_{j=0}^{n-1}S^jQ\big)$$
$$= I\big(\bigvee_{i=0}^{m-1}T^iP\,;\bigvee_{i=-m}^{-1}T^iP\big)+H_\nu\big(\bigvee_{j=0}^{n-1}S^jQ\big),$$

and so

$$E_I(P\times Q,\Phi) = E_I(P,T)+\lim_{n\to\infty}H_\nu\big(\bigvee_{j=0}^{n-1}S^jQ\big),$$

which proves (4.4).

It is easy to check that

$$h_{\mu\times\nu}(P\times Q, T\times I) = h_\mu(P,T),$$

$$h_{\mu\times\nu}(P\times Q, I\times S) = h_\nu(Q,S),$$

and due to the equality $h_{\mu\times\nu}(P \times Q, \Phi) = 0$ and (4.6) we get

$$
\begin{aligned}
&E_S(P \times Q, \Phi) \\
&= \lim_{m,n\to\infty} \big[H_\mu\big(\bigvee_{i=0}^{m-1} T^i P\big) + H_\nu\big(\bigvee_{j=0}^{n-1} S^j Q\big) - m \cdot h_\mu(P, T) - n \cdot h_\nu(Q, S)\big] \\
&= E_S(P, T) + E_S(Q, S),
\end{aligned}
$$

i.e. (4.5) is true. ■

Applying the remark given in the proof of (1.4) we have

REMARK 4.1. *If the process* (Q, S) *is aperiodic*, *then* $E_I(P \times Q, \Phi) = +\infty$.

REMARK 4.2. *Similarly to* (4.4) *one proves*

$$
\tilde{E}_I(P \times Q, \Phi) = E_I(P \times Q, \Phi). \tag{4.7}
$$

5. CONCLUSIONS

1. All excess entropies for random fields defined in [4] by Crutchfield and Feldman coincide (are equal to 0) on the class of Bernoulli fields.

2. The behaviour of E_C and E_I on the two considered classes of random fields shows that they are in general incomparable. On the other hand, since $E_C = \tilde{E}_I$ for Conze fields and $E_C < \tilde{E}_I$ for product fields, one could propose the hypothesis that $E_C(P, \Phi) \leqslant \tilde{E}_I(P, \Phi)$ for any field (P, Φ). Unfortunately, actually we are not able to show (or to disprove) this inequality.

3. The formula (3.3) implies that in general the (double) limit defining E_S does not exist. Indeed, suppose that the process (P, φ) is Markovian but not Bernoulli. Then

$$
H\big(\bigvee_{i=0}^{n-1} \varphi^i P\big) = H(P) + (n-1) \cdot h(P, \varphi), \quad n \geqslant 1,
$$

and so

$$
E(P, \varphi) = H(P) - h(P, \varphi) > 0.
$$

Therefore,

$$
m\Big(H\big(\bigvee_{j=0}^{n-1} \varphi^j P\big) - nh(P, \varphi)\Big) - E_S(P, \varphi) \cdot n = (m-n)E_S(P, \Phi), \quad m, n \geqslant 1,
$$

which implies the non-existence of the above limit.

4. The formulas of the Crutchfield–Feldman entropies for product fields show that there are differences between the behaviour of excess entropies for stochastic processes and random fields.

Namely, as we know, any stationary Markov process with a finite state space is finitary, i.e. its excess entropy is finite. The formula (4.4) shows that under a natural and weak assumption of the aperiodicity of the process (Q, S) for any process (P, T) the excess entropy E_I of the product field $\big((P \times Q), \Phi\big)$ is infinite despite this product field is Markovian.

On the other hand, we know that for any aperiodic process with zero entropy its excess entropy is infinite. Taking (P, T) and (Q, S) aperiodic and finitary, we get an aperiodic product random field $(P \times Q, \Phi)$ with zero entropy for which, according to (4.3) and (4.5), the excess entropies E_C and E_S are finite.

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Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
ul. Chopina 12/18, 87-100 Toruń, Poland
E-mail: bulatek@mat.uni.torun.pl
E-mail: bkam@mat.uni.torun.pl

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