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INDEPENDENCE ARISING FROM INTERACTING FOCK SPACES AND RELATED CENTRAL LIMIT THEOREM

BY

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Abstract. We present the notion of projective independence, which abstracts, in an algebraic setting, the factorization rule for the vacuum expectation of creation-annihilations-preservation operators in interacting Fock spaces described in [3]. Furthermore, we give a central limit theorem based on such a notion and a Fock representation of the limit process.

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1. INTRODUCTION

In quantum probability one finds several central limit theorems (see, e.g., [1], [7], [9], [11], [12], [14], [15], [19]). There are analogies and differences between the classical and quantum cases. As in the classical setting, the features exhibited by each quantum central limit theorem are: a suitable notion of *independence* for the converging family of random variables and the property, for the limit law, to be infinitely divisible with respect to a proper convolution. On the contrary, in the non-commutative context, one requires the convergence in the sense of moments, which is generally weaker than the convergence in law. A striking inhomogeneity is also registered for the concept of independence. In fact, in quantum probability many inequivalent notions of independence emerge: as the random variables in general do not mutually commute, one can factorize the mixed moments in many ways, obtaining in each case that the joint states (distributions) can be computed once one knows the marginal ones.

In [8], [16]–[18] the authors developed an axiomatic study of quantum independence based on some universal prescriptions, thus limiting the number of non-commutative independencies to five.

In addition to this approach, it has been developed another one, more pragmatic, emphasizing concrete applications. Its underlying idea is that a factorization rule for the mixed moments is seen as a form of *independence* if it leads to a non-trivial central limit theorem. Within this context, in [3] the authors proved a central limit theorem with the help of the so-called one mode type interacting Fock spaces (1-MT IFS), i.e. standard interacting Fock spaces with constant interacting functions (see [6] for more details). Namely, it was shown that any probability measure on $\mathbb{R}$ with finite moments of any order can be obtained as central limit (in the sense of convergence of moments) of certain self-adjoint random variables, which are in fact a sum of creation, annihilation and preservation operators in 1-MT IFS.

This theorem, proved also in [13] by means of a different approach, concretely confirmed and furthermore generalized the result obtained by Accardi and Bożejko in [2] according to which any *symmetric* measure on $\mathbb{R}$, being infinitely divisible with respect to the so-called universal convolution, can be a central limit law.

Therefore, it is natural to ask whether it is possible to find a notion of independence, i.e. a factorization rule for the mixed moments, into the abstract setting of algebraic probability spaces, which underlies the central limit theorem presented in [3]. These notes offer an affirmative answer to such a question and we call the desired independence *projective*.

The paper is organized as follows. In Section 2 we recall some tools which will be useful later, e.g., the definition of algebraic probability space, singleton and uniform boundedness conditions.

In Section 3 we introduce the central notion of projective independence and analyze its relation with the factorization rule for the mixed moments in 1-MT IFS described in [3]. Moreover, the definition of *symmetric* projective independence as a particular case of the general one is given: we need it in order to split the proof of the algebraic central limit theorem into two parts. In fact, in Section 4, we firstly obtain our central limit theorem only in the symmetric case and then we use such a result to prove the general theorem as an extension of it.

In order to gain the "symmetric" central limit theorem, we need some technical conditions, namely the boundedness of the mixed moments and the existence of the limit for the mean covariance (see condition (4.5)). The former allows us to use a result by Accardi, Hashimoto and Obata in [5] and simplify our proof. The latter replaces and weakens a usual condition, in central limit theorems, that the family of converging random variables is identically distributed. In fact, it is easy to check that, in this case, condition (4.5) is automatically verified.

It is worth noticing that the singleton condition, an ingredient of many central limit theorems, is automatically satisfied by any *symmetric* projectively independent family of algebraic random variables but it is not required in our main theorem (Theorem 4.1). This is due to the fact that the singleton condition cannot give non-symmetric central limit laws. (As another example in which the emergence of non-symmetric distribution in the central limit theorem is a consequence of the lacking of the singleton condition, one can recall the Ulmann law arising from the φ_γ-independence in the central limit theorem by Accardi, Hashimoto and Obata in [5]).

By a theorem due to Accardi, Frigerio and Lewis (see [4]), the limit sequence of any algebraic central limit theorem is the sequence of the mixed moments of certain random variables in a suitable algebraic probability space. Hence in Section 5 we show that, under some conditions, a GNS representation for the limit process obtained in Theorem 4.1, is realized in the 1-mode type interacting Fock space.

We point out that a similar result is achieved in [10], where a central limit theorem and a representation for the limit process are given for a "perturbed" symmetric φ-projectively independent family in an algebraic probability space. Namely, in [10], for every random variable of the limiting process, the perturbation is given by the multiplication of a Riemann integrable function on the unit interval of $\mathbb{R}$.

2. THE SINGLETON AND THE BOUNDEDNESS OF THE MIXED MOMENTS CONDITIONS

Here we recall some definitions and properties which will be used in the successive sections of the paper.

Let S be a non-empty ordered set. A *partition* of S is a family $\sigma = \{V_1, \dots, V_l\}$ of mutually disjoint non-empty subsets of S, whose union is S.

If there occur two partitions $\sigma_1 = \{V_1, \dots, V_l\}$ and $\sigma_2 = \{U_1, \dots, U_l\}$ of S such that for any $i = 1, \dots, l$ there exists (a unique) $j = 1, \dots, l$ such that $V_i = U_j$, we identify the two partitions in the set $\mathcal{P}(S)$ of all partitions of S. Moreover, for any $q \in \mathbb{N}^*$, we define $\mathcal{P}(q) := \mathcal{P}(\{1, \dots, q\})$.

Any $V_i \in \sigma$ is called a *block of the partition* σ. A partition σ of S uniquely defines an equivalence relation $\sim_\sigma$ on S, where $i \sim_\sigma j$ if and only if i, j belong to the same block of σ.

$\sigma = \{V_1, \dots, V_l\} \in \mathcal{P}(S)$ is called a *pair partition* if $|V_i| = 2$ for any $i = 1, \dots, l$, where $|\cdot|$ denotes the cardinality.

An algebraic probability space is a pair $\{\mathcal{A}, \varphi\}$, where $\mathcal{A}$ is a unital $*$-algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear, normalized ($\varphi(1) = 1$) and positive ($\varphi(a^* a) \geqslant 0$ for all $a \in \mathcal{A}$) functional.

Throughout the paper we will deal with sets of the type $\{a_i^\varepsilon;\, i \in \mathcal{I},\, \varepsilon \in F\}$, where F is a finite set such that $F = F_s \cup F_a$ with $F_s \cap F_a = \emptyset$. The family $\{a_i^\varepsilon;\, i \in \mathcal{I},\, \varepsilon \in F\}$ will be also thought closed with respect to the involution $*$ and called *self-adjoint*. The reason for introducing the upper suffixes in (a_i^ε) is derived from concrete examples of central limit theorems (see [5]) and it is furthermore natural whenever $\{\mathcal{A}, \varphi\}$ is constructed starting from the 1-MT IFS, as shown in the example below.

EXAMPLE 2.1. Given a separable Hilbert space $\mathcal{H}$, consider the 1-MT IFS $\Gamma(\mathcal{H}, \{\lambda_n\}_{n\in\mathbb{N}})$ over $\mathcal{H}$ with interacting sequence $\{\lambda_n\} \subset \mathbb{R}_+$ and vacuum vector Φ, i.e.

$$\Gamma(\mathcal{H}, \{\lambda_n\}_{n\in\mathbb{N}}) := \mathbb{C}\Phi \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n},$$

where for any $n, m \geqslant 1$, for any $f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathcal{H}$, the pre-scalar product on the tensor product space $\mathcal{H}^{\otimes n}$ is obtained by

$$\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_m \rangle := \delta_m^n \lambda_n \prod_{j=1}^{n} \langle f_j, g_j \rangle$$

with the requirements $\lambda_0 = \lambda_1 := 1$ and $\lambda_n = 0 \Longrightarrow \lambda_m = 0$ for any $m \geqslant n$ (see [3] and [6] for details on 1-MT IFS). Let us define the creation operator $A^+(f)$ with the test function $f \in \mathcal{H}$:

$$A^+(f)\,\Phi := f,$$

$$\left(A^+(f)\,(f_1 \otimes \ldots \otimes f_n)\right) := f \otimes f_1 \otimes \ldots \otimes f_n, \ \ \forall n \in \mathbb{N}, \ \ \forall\, f_1, \ldots, f_n \in \mathcal{H}.$$

The annihilation operator $A(f)$ is defined as the adjoint of the creation operator:

$$A(f)\,\Phi := 0,$$

$$\left(A(f)\,(f_1 \otimes \ldots \otimes f_n)\right) := \frac{\lambda_n}{\lambda_{n-1}} \langle f, f_1 \rangle\, f_2 \otimes \ldots \otimes f_n, \ \ \forall n \in \mathbb{N}, \ \ \forall\, f_1, \ldots, f_n \in \mathcal{H},$$

with the convention $0/0 = 0$. Given $\alpha := (\alpha_m) \subset l^\infty(\mathbb{R})$ with $\alpha_0 = 0$ and the identity operator $I \in \mathbf{B}(\mathcal{H})$, define the preservation operator with intensity (α, I):

$$\Lambda_\alpha(I)\,(f_1 \otimes \ldots \otimes f_n) := \alpha_n\,(f_1 \otimes \ldots \otimes f_n), \quad \forall n \in \mathbb{N}, \ \ \forall\, f_1, \ldots, f_n \in \mathcal{H}.$$

Take $\mathcal{A}$ as

$$\mathcal{A} := *\text{-alg}\,\{I, A(f), \Lambda_\alpha(I) : f \in \mathcal{H}, \alpha = (\alpha_m) \in l^\infty(\mathbb{R})\},$$

$$\varphi := \langle \Phi, \cdot \Phi \rangle.$$

Hence it follows that $\{\mathcal{A}, \varphi\}$ is an algebraic probability space. If $\varepsilon \in \{-1, 0, 1\}$ and

$$A^\varepsilon(f, \alpha, I) := \begin{cases} A(f) & \text{if } \varepsilon = -1, \\ \Lambda_\alpha(I) & \text{if } \varepsilon = 0, \\ A^+(f) & \text{if } \varepsilon = 1, \end{cases}$$

then $F = \{-1, 0, 1\}, F_s = \{-1, 1\}, F_a = \{0\}$.

Let $q \in \mathbb{N}^*$ and consider the map $k : \{1, \ldots, q\} \to \mathcal{I}$; we write indifferently k_l or $k(l)$, and put

$\mathrm{Range}(k) := \left\{\overline{k}_1, \ldots, \overline{k}_m\right\} \subset \mathcal{I}, \ \ m \in \mathbb{N}^*, \ m \leqslant q, \ \overline{k}_i \neq \overline{k}_j$ for $i \neq j$;

$V_{k,j} := k^{-1}\left(\overline{k}_j\right) = \left\{l \in \{1, \ldots, q\} : k(l) = \overline{k}_j\right\}$ for $j = 1, \ldots, m$.

Clearly, k induces a partition $\sigma \in \mathcal{P}(q)$ whose blocks are the $V_{k,j}$'s. For a fixed $V_{k,j} = k^{-1}\left(\overline{k}_j\right)$, define $\varepsilon_{V_{k,j}} := \{\varepsilon_l :\ l \in V_{k,j}\}$, the restriction of ε to $V_{k,j}$, where $j = 1, \ldots, m$.

If $\mathcal{I}^{\{1,\ldots,q\}}$ is the set of all mappings from $\{1, \ldots, q\}$ into $\mathcal{I}$ and $k, l \in \mathcal{I}^{\{1,\ldots,q\}}$, we say that k is *equivalent* to l and write $k \approx l$ if they induce the same partition of $\{1, \ldots, q\}$, namely:

(i) $|\mathrm{Range}\,(k)| = |\mathrm{Range}\,(l)| =: m$;

(ii) $k^{-1}(\overline{k}_j) = l^{-1}\left(\overline{l}_{\sigma(j)}\right)$ for all $j = 1, \ldots, m$, where σ is a permutation on $\{1, \ldots, m\}$.

We define $[k] := \{l \in \mathcal{I}^{\{1,\ldots,q\}}$ such that $k \approx l\}$, the $\approx$-equivalence class of k.

Conversely, any partition $\sigma \in \mathcal{P}(q)$ defines a unique equivalence class $[k]$, where k is any function which is constant on the blocks of σ. Therefore, we have a natural identification $\mathcal{I}^{\{1,\ldots,q\}}{}_{/\approx} \equiv \mathcal{P}(q)$ which will be exploited in the following by taking $[k] \equiv \sigma$.

A (unital) algebraic stochastic process is the quadruple $\left\{\{\mathcal{A}, \varphi\}, \mathcal{B}, (j_i)_{i\in\mathcal{I}}\right\}$, where $\{A, \varphi\}$ is an algebraic probability space, $\mathcal{I}$ a set, $\mathcal{B}$ a unital $*$-algebra and, for any $i \in \mathcal{I}$, $j_i : \mathcal{B} \to \mathcal{A}$ an identity preserving $*$-homomorphism.

The following definitions will be useful in the paper.

DEFINITION 2.1. Let $\{a_i^\varepsilon;\ i \in \mathcal{I}, \varepsilon \in F\}$ be a self-adjoint family in $\{\mathcal{A}, \varphi\}$. It is said to satisfy the *singleton condition* (with respect to φ) if for any $n \geqslant 1$, for any choice of $i_1, \ldots, i_n \in \mathcal{I}$, $\varepsilon_1, \ldots, \varepsilon_n \in F$

$$\varphi(a_{i_n}^{\varepsilon_n} \ldots a_{i_1}^{\varepsilon_1}) = 0 \tag{2.1}$$

whenever $\{i_1, \ldots, i_n\}$ has a singleton i_s (i.e. $i_s \neq i_j, j \neq s$) and $\varphi(a_{i_s}^{\varepsilon_s}) = 0$.

DEFINITION 2.2. The self-adjoint family $\{a_i^\varepsilon;\ \varepsilon \in F,\ i \in \mathcal{I}\}$ in $\{\mathcal{A}, \varphi\}$ is said to satisfy the *condition of uniform boundedness of mixed moments* if for each $m \in \mathbb{N}^*$ there exists a positive constant D_m such that

$$|\varphi(a_{i_m}^{\varepsilon_m} \ldots a_{i_1}^{\varepsilon_1})| \leqslant D_m$$

for any choice of $i_1, \ldots, i_m \in \mathcal{I}$ and $\varepsilon_1, \ldots, \varepsilon_m \in F$.

3. PROJECTIVE INDEPENDENCE

Let $\{\mathcal{A}, \varphi\}$ be an algebraic probability space and assume that a self-adjoint family $\{a_i^\varepsilon;\ \varepsilon \in F,\ i \in \mathcal{I}\}$ of elements of $\mathcal{A}$ is given. Then for any $q \in \mathbb{N}^*$, $\varepsilon = \{\varepsilon_q, \ldots, \varepsilon_1\} \subset F$, we take $\varepsilon' := \varepsilon \cap F_s$. Moreover, for any $k : \{1, \ldots, q\} \to \mathcal{I}$, any $a_{k_q}^{\varepsilon_q}, \ldots, a_{k_1}^{\varepsilon_1} \in \mathcal{A}$ we write:

$\prod^{\leftarrow} a^{\varepsilon'}$ to denote the product of all $a_{k_l}^{\varepsilon_l}$, with $\varepsilon_l \in F_s$, in the same order as they appear in $a_{k_q}^{\varepsilon_q} \ldots a_{k_1}^{\varepsilon_1}$ (we use the convention $\varphi\left(\prod^{\leftarrow} a^{\varnothing}\right) := 1$);

$a^{\varepsilon V_{k',j}} := \prod^{\leftarrow}_{s\in V_{k',j}} a^{\varepsilon_s}_{\overline{k}'_j}$, where $\prod^{\leftarrow}$ denotes the product of $a^{\varepsilon_l}_{k_l}$'s in the same order appearing in the product $a^{\varepsilon_q}_{k_q} \dots a^{\varepsilon_1}_{k_1}$ and $V_{k',j} = k^{-1}(\overline{k}'_j)$ for any j.

DEFINITION 3.1 (Projective independence). Let $\{\mathcal{A}, \varphi\}$ be an algebraic probability space. The self-adjoint family $\{a^{\varepsilon}_i;\ \varepsilon \in F,\ i \in \mathcal{I}\}$ in $\mathcal{A}$ is called *φ-projectively independent* if for any $q \in \mathbb{N}^*$, $\varepsilon = (\varepsilon_q, \dots, \varepsilon_1) \in F^q$, any $k : \{1, \dots, q\} \to \mathcal{I}$, and $a^{\varepsilon_q}_{k_q}, \dots, a^{\varepsilon_1}_{k_1} \in \mathcal{A}$ there exist $\alpha\,(k, \varepsilon) \in \mathbb{C}$ and $\omega\,(k', \varepsilon') \geqslant 0$ such that

$$\begin{aligned} \varphi(a^{\varepsilon_q}_{k_q} \dots a^{\varepsilon_1}_{k_1}) &= \alpha(k,\varepsilon)\varphi\big(\overleftarrow{\prod} a^{\varepsilon'}\big) \\ &= \alpha\,(k,\varepsilon)\,\omega(k',\varepsilon') \prod_{j=1}^{|\mathrm{Range}(k')|} \varphi(a^{\varepsilon V_{k',j}}). \end{aligned} \tag{3.1}$$

From now on we will use indifferently the following notation:

$$\alpha\,(k,\varepsilon) = \alpha\,(\sigma,\varepsilon) \quad \text{and} \quad \omega\,\big(k',\varepsilon'\big) = \omega\,\big(\tau,\varepsilon'\big),$$

where σ and τ are the partitions induced by the maps k and k', respectively, on $\{1, \dots, q\}$; and $k' := k|_{\{\{1,\dots,q\}\setminus\{l|\ \varepsilon_l \in F_a\}\}}$. Obviously, $\mathrm{Range}\,(k') \subseteq \mathrm{Range}\,(k)$.

REMARK 3.1. The definition above abstracts, in an algebraic setting, the situation described in the paper [3], relative to 1-mode type interacting Fock spaces. In that case, as already remarked, the explicit form of F is $\{-1, 0, 1\}$, $F_s = \{-1, +1\}$ and $F_a = \{0\}$. Moreover, the coefficients $\alpha(k, \varepsilon)$ and $\omega\,(k', \varepsilon')$ are products of the non-symmetric and symmetric Jacobi coefficients $\{\alpha_n\}$ and $\{\omega_n\}$ of the distribution uniquely associated with the interacting Fock space (see [2]). In order to be more explicit we present the following example.

EXAMPLE 3.1. Use the notation of Example 2.1, and let μ be the one-dimensional distribution associated with the IFS with Jacobi parameters $\alpha := (\alpha_n)$ and $(\omega_n := \lambda_n/\lambda_{n-1})_n$ (see [2], Theorem 5.2). Then, if $f_1, f_2 \in \mathcal{H}$, I the identity operator, we have

$$\begin{aligned} &\langle \Phi, A\,(f_1)\,\Lambda_\alpha\,(I)\,A\,(f_2)\,A^+\,(f_2)\,A^+\,(f_1)\,\Phi\rangle \\ &\qquad = \alpha_1\langle \Phi, A\,(f_1)\,A\,(f_2)\,A^+\,(f_2)\,A^+\,(f_1)\,\Phi\rangle \\ &\qquad = \alpha_1 \cdot \omega_1 \cdot \omega_2 \langle \Phi, A\,(f_1)\,A^+\,(f_1)\,\Phi\rangle\langle \Phi, A\,(f_2)\,A^+\,(f_2)\,\Phi\rangle. \end{aligned}$$

Hence $\alpha(k, \varepsilon) = \alpha_1$ and $\omega\,(k', \varepsilon') = \omega_1\omega_2$.

The following definition is a particular case of Definition 3.1 and will be useful in the proof of our central limit theorem.

DEFINITION 3.2. Let $\{\mathcal{A}, \varphi\}$ be an algebraic probability space and suppose $F = F_s$. Then the self-adjoint family $\{a^{\varepsilon}_i;\ \varepsilon \in F_s,\ i \in \mathcal{I}\}$ in $\mathcal{A}$ is called *φ-symmetric projectively independent* if for any $q \in \mathbb{N}^*$, any $\varepsilon' = (\varepsilon_q, \dots, \varepsilon_1) \in (F_s)^q$, any

$k' : \{1, \dots, q\} \to \mathcal{I}$ there exists $\omega(k', \varepsilon') \geqslant 0$ such that, in the notation of Definition 3.1,

$$\varphi(a_{k'_q}^{\varepsilon_q} \dots a_{k'_1}^{\varepsilon_1}) = \omega(k', \varepsilon') \prod_{j=1}^{|\mathrm{Range}(k')|} \varphi(a^{\varepsilon_{V_{k',j}}}).$$

LEMMA 3.1. *Let $\{a_i^\varepsilon; \varepsilon \in F_s,\ i \in \mathcal{I}\}$ be a self-adjoint family of φ-symmetric projectively independent elements of an algebraic probability space $\{\mathcal{A}, \varphi\}$ with mean zero, i.e. $\varphi(a_i^\varepsilon) = 0$ for any $\varepsilon \in F_s$, $i \in \mathcal{I}$. Then it satisfies the singleton condition.*

Proof. Let us fix $q \in \mathbb{N}^*$ and consider the product $a_{k'_q}^{\varepsilon'_q} \dots a_{k'_1}^{\varepsilon'_1}$. If there exists $l \in \{1, \dots, q\}$ such that $|V_{k',l}| = 1$, then, by φ-symmetric projective independence, we have

$$\varphi(a_{k'_q}^{\varepsilon'_q} \dots a_{k'_1}^{\varepsilon'_1}) = \omega(k', \varepsilon') \varphi(a_{k'_l}^{\varepsilon'_l}) \prod_{j=1, j \neq l}^{|\mathrm{Range}(k')|-1} \varphi(a^{\varepsilon'_{V_{k',j}}}) = 0.$$

Hence the singleton condition is fulfilled. ∎

REMARK 3.2. If the family is φ-projectively independent, the singleton condition is not generally satisfied. In fact, under the assumptions of Examples 2.1 and 3.1, by such a condition of independence, if $\alpha_1 \neq 0$, for any $f_1 \neq 0$, we have

$$\langle \Phi, A(f_1) \Lambda_\alpha(I) A^+(f_1) \Phi \rangle = \alpha_1 \langle \Phi, A(f_1) A^+(f_1) \Phi \rangle \neq 0$$

even if $\langle \Phi, \Lambda_\alpha(I) \Phi \rangle = 0$.

4. QUANTUM CENTRAL LIMIT THEOREM

Let $\{a_n^\varepsilon; \varepsilon \in F,\ n \in \mathbb{N}\}$ be a self-adjoint family in an algebraic probability space $\{\mathcal{A}, \varphi\}$ and consider the centered sum

$$S_N(a^\varepsilon) := \sum_{n=1}^{N} a_n^\varepsilon$$

for $\varepsilon \in F$ and $N \in \mathbb{N}$.

In order to prove our central limit theorem, we need to consider the following "normalized family" as in [3] and [13]:

$$\widetilde{a}_n^\varepsilon = \begin{cases} a_n^\varepsilon & \text{if } \varepsilon \in F_s, \\ c_n a_n^\varepsilon & \text{if } \varepsilon \in F_a, \end{cases} \tag{4.1}$$

where $\{c_n\}$ is a bounded sequence in $\mathbb{R}$ satisfying the condition

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} c_n \to 1 \quad \text{as } N \to +\infty, \tag{4.2}$$

and we have to study the asymptotic behavior of the expression

$$\frac{1}{N^{q/2}}\varphi\big(S_N\,(\widetilde{a}^{\varepsilon_q})\ldots S_N\,(\widetilde{a}^{\varepsilon_1})\big)$$

for any $q\in\mathbb{N}^*$. Suppose that $\{a_n^\varepsilon;\varepsilon\in F,\,n\in\mathbb{N}\}$ is φ-projectively independent. If $\varepsilon=(\varepsilon_q,\ldots,\varepsilon_1)\in F^q$, we make the partition

$$\{1,\ldots,q\}=\{z_1,\ldots,z_m\}\cup\{z_1^{'},\ldots,z_h^{'}\}, \tag{4.3}$$

where $q=m+h$, and

$$\{z_1,\ldots,z_m\}:=\big\{d\in\{1,\ldots,q\}\,;\;\varepsilon_d\in F_s\big\},$$
$$\big\{z'_1,\ldots,z'_h\big\}:=\big\{d\in\{1,\ldots,q\};\;\varepsilon_d\in F_a\big\}.$$

Then (3.1) and (4.1) give

$$\varphi(\widetilde{a}_{k_q}^{\varepsilon_q}\ldots\widetilde{a}_{k_1}^{\varepsilon_1})=\alpha\,(\sigma,\varepsilon)\,\omega(\tau,\varepsilon')\prod_{i=1}^{h}c_{k_{z'_i}}\prod_{j=1}^{|\mathrm{Range}(k')|}\varphi(a^{{\varepsilon'}_{V_{k',j}}}). \tag{4.4}$$

THEOREM 4.1 (Central limit theorem). *Let $\{\mathcal{A},\varphi\}$ be an algebraic probability space and $\{a_n^\varepsilon;\varepsilon\in F,\,n\in\mathbb{N}\}$ be a φ-projectively independent self-adjoint family in $\mathcal{A}$ such that $\varphi\,(a_n^\varepsilon)=0$ for all $\varepsilon\in F$ and $n\in\mathbb{N}$. Suppose that $\{a_i^\varepsilon;$ $\varepsilon\in F_s,\,n\in\mathbb{N}\}$ satisfies the uniform boundedness condition and the limit*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\varphi(a_k^{\varepsilon_1}a_k^{\varepsilon_2})=:C\,(\varepsilon_1,\varepsilon_2) \tag{4.5}$$

exists for any $\varepsilon_1,\varepsilon_2\in F_s$. Then:

(i) *As $N\to\infty$, the limit of the expression*

$$\frac{1}{N^{q/2}}\varphi\big(S_N\,(\widetilde{a}^{\varepsilon_q})\ldots S_N\,(\widetilde{a}^{\varepsilon_1})\big)=\frac{1}{N^{q/2}}\sum_{1\leqslant k_1,\ldots,k_q\leqslant N}\varphi(\widetilde{a}_{k_q}^{\varepsilon_q}\ldots\widetilde{a}_{k_1}^{\varepsilon_1}) \tag{4.6}$$

is equal to zero if m is odd and, if $m=2p$, is equal to

$$\sum_{\sigma\in\mathcal{P}_\tau(q;2p)}\alpha\,(\sigma,\varepsilon)\,\omega(\tau,\varepsilon')\prod_{j=1}^{p}C(\varepsilon_{l_j},\varepsilon_{r_j}), \tag{4.7}$$

where

$$\mathcal{P}_\tau(q;2p):=\big\{\sigma\in\mathcal{P}\,(q):\sigma=\tau\cup\gamma,\tau\cap\gamma=\emptyset,\tau\in\mathrm{P.P.}\,(\{z_1,\ldots,z_{2p}\})\big\},$$

$\mathrm{P.P.}\,(\{z_1,\ldots,z_{2p}\})$ *denotes the set of all pair partitions of $\{z_1,\ldots,z_{2p}\}$ and $\{l_j,r_j\}_{j=1}^p$ the left-right index set relative to the pair partition τ such that $r_1<r_2<\ldots<r_p$.*

(ii) *More generally, if* $s, t > 0$, $s < t$, *putting, for any* $N \in \mathbb{N}$, $S_N^{(s,t)}(\widetilde{a}^{\varepsilon}) := \sum_{n=[sN]+1}^{[tN]} \widetilde{a}_n^{\varepsilon}$, *the limit*

$$\lim_{N\to\infty} \frac{1}{N^{q/2}} \varphi\big(S_N^{(s_q,t_q)}(\widetilde{a}^{\varepsilon_q}) \dots S_N^{(s_1,t_1)}(\widetilde{a}^{\varepsilon_1})\big),$$

where $s_j, t_j > 0$, $s_j < t_j$, *vanishes when* m *is odd and, if* $m = 2p$, *is equal to*

$$\sum_{\sigma \in \mathcal{P}_\tau(q;2p)} \alpha(\sigma, \varepsilon)\, \omega(\tau, \varepsilon') \Big[\prod_{d=1}^{h} (\sqrt{t_{z'_d}} - \sqrt{s_{z'_d}}) \Big] \times \Big[\prod_{j=1}^{p} C(\varepsilon_{l_j}, \varepsilon_{r_j}) \langle \chi_{(s_{l_j}, t_{l_j})}, \chi_{(s_{r_j}, t_{r_j})} \rangle_{\mathbf{L}^2(\mathbb{R}_+)} \Big],$$

where $\chi_{(s,t)}$ *is the indicator function of the interval* (s, t) *in* $\mathbb{R}_+$.

In order to prove the theorem we firstly restrict to the case when $F = F_s$.

LEMMA 4.1 (Symmetric central limit theorem). *Let* $\{a_n^{\varepsilon}; \varepsilon \in F_s, n \in \mathbb{N}\}$ *be a self-adjoint family of elements of an algebraic probability space* $\{\mathcal{A}, \varphi\}$, *symmetric* φ*-projectively independent and with mean zero. We suppose that such a family satisfies the uniform boundedness condition and the condition* (4.5). *Then*:

(i) *The limit*

$$\lim_{N\to\infty} \frac{1}{N^{m/2}} \varphi\big(S_N(a^{\varepsilon_m}) \dots S_N(a^{\varepsilon_1})\big)$$

is zero if m *is odd and, if* $m = 2p$, *is equal to*

$$\lim_{N\to\infty} \frac{1}{N^p} \sum_{\substack{k':\{1,\dots,2p\}\to\{1,\dots,p\} \\ \text{2-1 map}}} \sum_{\substack{1 \leqslant k'_{l_j} = k'_{r_j} \leqslant N \\ j=1,\dots,p}} \varphi(\dots a_{k'_{l_j}}^{\varepsilon_{l_j}} \dots a_{k'_{r_j}}^{\varepsilon_{r_j}} \dots) = \sum_{\tau \in \mathrm{P.P.}(2p)} \omega(\tau, \varepsilon) \prod_{j=1}^{p} C(\varepsilon_{l_j}, \varepsilon_{r_j}), \tag{4.8}$$

where we use the same notation as in Theorem 4.1 *and* $\tau \in \mathrm{P.P.}(2p)$ *denotes the set of all pair partitions of* $\{1, \dots, 2p\}$.

(ii) *More generally, if* $s, t > 0$, $s < t$ *and putting* $S_N^{(s,t)}(a^{\varepsilon}) := \sum_{n=[sN]+1}^{[tN]} a_n^{\varepsilon}$, *the limit*

$$\lim_{N\to\infty} \frac{1}{N^{m/2}} \varphi\big(S_N^{(s_m,t_m)}(a^{\varepsilon_m}) \dots S_N^{(s_1,t_1)}(a^{\varepsilon_1})\big),$$

where $s_j, t_j > 0$, $s_j < t_j$, *vanishes when* m *is odd and, if* $m = 2p$, *is equal to*

$$\sum_{\tau \in \mathrm{P.P.}(2p)} \omega(\tau, \varepsilon) \prod_{j=1}^{p} [C(\varepsilon_{l_j}, \varepsilon_{r_j}) \langle \chi_{(s_{l_j}, t_{l_j})}, \chi_{(s_{r_j}, t_{r_j})} \rangle_{\mathbf{L}^2(\mathbb{R}_+)}],$$

where $\chi_{(s,t)}$ *is the indicator function of the interval* (s, t) *in* $\mathbb{R}_+$.

P r o o f. (i) Firstly we have

$$\frac{1}{N^{m/2}}\varphi\big(S_N\,(a^{\varepsilon_m})\dots S_N\,(a^{\varepsilon_1})\big)=\frac{1}{N^{m/2}}\sum_{1\leqslant k'_1,\dots,k'_m\leqslant N}\varphi(a^{\varepsilon_m}_{k'_m}\dots a^{\varepsilon_1}_{k'_1}). \tag{4.9}$$

From Lemma 3.1 we know that the family $\{a^{\varepsilon}_n;\varepsilon\in F,\,n\in\mathbb{N}\}$ satisfies the singleton condition. Moreover, by assumption, the uniform boundedness condition is fulfilled. Then, from [5], Lemma 2.4, it follows that the limit of (4.9) can be different from zero only if $m=2p$ and $k'\colon\{1,\dots,2p\}\to\{1,\dots,p\}$ is a 2-1 map, whose range is denoted by $\{\overline{k}'_1,\dots,\overline{k}'_p\}$. It is well known that such a map induces a pair partition on $\{1,\dots,2p\}$ for which we use the notation $\{l_j,r_j\}:=k'^{-1}(\overline{k}'_j)$ with $l_j>r_j$ for all $j=1,\dots,p$, $r_1<\dots<r_p$. The limit for $N\to\infty$ of (4.9) can be written as follows:

$$\lim_{N\to\infty}\frac{1}{N^p}\sum_{\substack{k'\colon\{1,\dots,2p\}\to\{1,\dots,p\}\\ \text{2-1 map}}}\ \sum_{\substack{1\leqslant k'_{l_j}=k'_{r_j}\leqslant N\\ j=1,\dots,p}}\varphi(\dots a^{\varepsilon_{l_j}}_{k'_{l_j}}\dots a^{\varepsilon_{r_j}}_{k'_{r_j}}\dots).$$

Since our family is φ-symmetric projectively independent, this quantity is

$$\lim_{N\to\infty}\sum_{\tau:=\{l_j,r_j\}_{j=1}^{p}\in\text{P.P.}(2p)}\omega\,(\tau,\varepsilon)\,\frac{1}{N^p}\sum_{k'_{l_1}=k'_{r_1}=1}^{N}\cdots\sum_{k'_{l_p}=k'_{r_p}=1}^{N}\prod_{j=1}^{p}\varphi(a^{\varepsilon_{l_j}}_{k'_{l_j}}a^{\varepsilon_{r_j}}_{k'_{r_j}}) \tag{4.10}$$

where we used the natural identification $[k']\equiv\tau$. By (4.5) we see that the limit (4.10) exists and is equal to

$$\sum_{\tau\in\text{P.P.}(2p)}\omega\,(\tau,\varepsilon)\prod_{j=1}^{p}C(\varepsilon_{l_j},\varepsilon_{r_j}).$$

Let us prove (ii):

$$\begin{aligned}\frac{1}{N^{m/2}}\varphi\big(S_N^{(s_m,t_m)}\,(a^{\varepsilon_m})\dots S_N^{(s_1,t_1)}\,(a^{\varepsilon_1})\big)\qquad&\\ =\frac{1}{N^{m/2}}\sum_{k'_1=[s_1N]+1}^{[t_1N]}\cdots\sum_{k'_m=[s_mN]+1}^{[t_mN]}\varphi(a^{\varepsilon_m}_{k'_m}\dots a^{\varepsilon_1}_{k'_1}).&\end{aligned} \tag{4.11}$$

Using the same arguments as in (i), one can check that the limit for $N\to\infty$ in (4.11) can be different from zero only if $m=2p$. Moreover, after applying the

φ-symmetric projective independence we obtain

$$\lim_{N\to\infty}\frac{1}{N^p}\varphi\big(S_N^{(s_m,t_m)}(a^{\varepsilon_m})\dots S_N^{(s_1,t_1)}(a^{\varepsilon_1})\big)$$
$$=\lim_{N\to\infty}\sum_{\tau:=\{l_j,r_j\}_{j=1}^p\in\text{P.P.}(2p)}\omega(\tau,\varepsilon)$$
$$\times\frac{1}{N^p}\prod_{j=1}^p\Big(\sum_{k'_{l_j}=[s_{l_j}N]+1}^{[t_{l_j}N]}\ \sum_{k'_{r_j}=[s_{r_j}N]+1}^{[t_{r_j}N]}\varphi(a_{k'_{l_j}}^{\varepsilon_{l_j}}a_{k'_{r_j}}^{\varepsilon_{r_j}})\Big).$$

Since for any $j=1,\dots,p$, k'_{l_j} and k'_{r_j} are equal, putting $\overline{s}_j:=\max\{s_{l_j},s_{r_j}\}$ and $\overline{t}_j:=\min\{t_{l_j},t_{r_j}\}$ we have

$$\lim_{N\to\infty}\frac{1}{N^p}\varphi\big(S_N^{(s_m,t_m)}(a^{\varepsilon_m})\dots S_N^{(s_1,t_1)}(a^{\varepsilon_1})\big)$$
$$=\lim_{N\to\infty}\frac{1}{N^p}\sum_{\tau\in\text{P.P.}(2p)}\omega(\tau,\varepsilon)\prod_{j=1}^p\Big(\sum_{k'_{l_j}=k'_{r_j}=[\overline{s}_jN]+1}^{[\overline{t}_jN]}\varphi(a_{k'_{l_j}}^{\varepsilon_{l_j}}a_{k'_{r_j}}^{\varepsilon_{r_j}})\Big) \tag{4.12}$$

if $\overline{s}_j<\overline{t}_j$ and the limit vanishes if $\overline{s}_j\geqslant\overline{t}_j$. Now let us fix $j=1,\dots,p$ and consider

$$\frac{1}{N}\sum_{k'_{l_j}=k'_{r_j}=[\overline{s}_jN]+1}^{[\overline{t}_jN]}\varphi(a_{k'_{l_j}}^{\varepsilon_{l_j}}a_{k'_{r_j}}^{\varepsilon_{r_j}})$$
$$=\frac{1}{N}\sum_{k'_{l_j}=k'_{r_j}=1}^{[\overline{t}_jN]}\varphi(a_{k'_{l_j}}^{\varepsilon_{l_j}}a_{k'_{r_j}}^{\varepsilon_{r_j}})-\frac{1}{N}\sum_{k'_{l_j}=k'_{r_j}=1}^{[\overline{s}_jN]}\varphi(a_{k'_{l_j}}^{\varepsilon_{l_j}}a_{k'_{r_j}}^{\varepsilon_{r_j}}). \tag{4.13}$$

Notice that for any $t>0$, if $M:=[tN]$, there exists $0\leqslant\delta<1$ such that for any $\varepsilon_1,\varepsilon_2\in F_s$

$$\frac{1}{N}\sum_{k=1}^{[tN]}\varphi(a_k^{\varepsilon_1}a_k^{\varepsilon_2})=\frac{t}{M+\delta}\sum_{k=1}^{M}\varphi(a_k^{\varepsilon_1}a_k^{\varepsilon_2})$$

and, by using (4.5), we obtain

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{[tN]}\varphi(a_k^{\varepsilon_1}a_k^{\varepsilon_2})=tC(\varepsilon_1,\varepsilon_2).$$

Hence, from (4.13) we find

$$\lim_{N\to\infty}\sum_{k'_{l_j}=k'_{r_j}=[\overline{s}_jN]+1}^{[\overline{t}_jN]}\varphi(a_{k'_{l_j}}^{\varepsilon_{l_j}}a_{k'_{r_j}}^{\varepsilon_{r_j}})=(\overline{t}_j-\overline{s}_j)C(\varepsilon_{l_j},\varepsilon_{r_j}).$$

Iterating the same arguments for any $j = 1, \dots, p$, from (4.12) it follows that

$$\lim_{N\to\infty} \frac{1}{N^p} \varphi\big(S_N^{(s_m,t_m)}(a^{\varepsilon_m}) \dots S_N^{(s_1,t_1)}(a^{\varepsilon_1})\big)$$
$$= \sum_{\tau\in \mathrm{P.P.}(2p)} \omega(\tau,\varepsilon) \prod_{j=1}^{p} [C(\varepsilon_{l_j}, \varepsilon_{r_j}) \langle \chi_{(s_{l_j},t_{l_j})}, \chi_{(s_{r_j},t_{r_j})} \rangle_{\mathbf{L}^2(\mathbb{R}_+)}]. \quad \blacksquare$$

Proof of Theorem 4.1. Let $\{a_n^\varepsilon;\ \varepsilon \in F,\ n \in \mathbb{N}\}$ be a φ-projectively independent family of elements in an algebraic probability space $(\mathcal{A}, \varphi)$.

Let us prove (i). We have

$$\frac{1}{N^{q/2}} \varphi\big(S_N(\widetilde{a}^{\varepsilon_q}) \dots S_N(\widetilde{a}^{\varepsilon_1})\big) = \frac{1}{N^{q/2}} \sum_{1\leqslant k_1,\dots,k_q \leqslant N} \varphi(\widetilde{a}_{k_q}^{\varepsilon_q} \dots \widetilde{a}_{k_1}^{\varepsilon_1}).$$

From (3.1), (4.4) and (4.3) the quantity above becomes

$$\text{(4.14)} \qquad \frac{1}{N^{q/2}} \sum_{1\leqslant k_{z_1'},\dots,k_{z_h'},k'_{z_1},\dots,k'_{z_m}\leqslant N} \alpha(\sigma,\varepsilon) \prod_{d=1}^{h} c_{k_{z'_d}} \varphi\big(\overleftarrow{\prod} a^{\varepsilon'}\big)$$
$$= \frac{1}{N^{h/2}} \sum_{1\leqslant k_{z_1'},\dots,k_{z_h'}\leqslant N} \prod_{d=1}^{h} c_{k_{z'_d}} \alpha(\sigma,\varepsilon)$$
$$\times \left(\frac{1}{N^{m/2}} \sum_{1\leqslant k'_{z_1},\dots,k'_{z_m}\leqslant N} \omega(\tau,\varepsilon') \varphi\big(\overleftarrow{\prod} a^{\varepsilon'}\big)\right),$$

where σ and τ are the partitions induced by k and k', respectively. Now, recalling that the "symmetric" part of a φ-projectively independent family is a φ-symmetric projectively independent family, by Lemma 4.1 and condition (4.2), the limit for $N \to \infty$ of the left-hand side of (4.14) (or, equivalently, of (4.6)), is equal to zero if m is odd and, if $m = 2p$, is equal to

$$\sum_{\sigma\in\mathcal{P}_\tau(q;2p)} \alpha(\sigma,\varepsilon)\, \omega(\tau,\varepsilon') \prod_{j=1}^{p} C(\varepsilon_{l_j}, \varepsilon_{r_j}).$$

For (ii) we firstly observe that for any $t > 0$, $N \in \mathbb{N}$, there exist $M \in \mathbb{N}$, $\delta > 0$ such that

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{[tN]} c_k = \frac{\sqrt{t}}{\sqrt{M+\delta}} \sum_{k=1}^{M} c_k.$$

Hence, as a consequence of (4.2), we find

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{k=1}^{[tN]} c_k = \sqrt{t},$$

and the assertion can be achieved by using the same arguments as in (i) and (ii) in the proof of Lemma 4.1. $\blacksquare$

5. INTERACTING FOCK SPACE REPRESENTATIONS OF THE LIMIT PROCESSES

Throughout this section we will take $F := \{-1, 0, 1\}$, $F_s := \{-1, +1\}$ and $F_a := \{0\}$ and, for any $a \in \mathcal{A}$, $a^{-1} = (a^1)^*$, $a^0 = (a^0)^*$. Our goal consists in finding Fock representations for the limit processes coming out from the two results achieved in Theorem 4.1.

We firstly look at part (i) of this theorem. As a consequence of the reconstruction theorem by Accardi, Frigerio and Lewis (see [4]), there exist an algebraic probability space $(\mathcal{B}, \psi)$ and random variables $a_\psi^{-1}, a_\psi^1, a_\psi^0$ in it such that

$$\sum_{\sigma \in \mathcal{P}_\tau(q;2p)} \alpha(\sigma, \varepsilon)\, \omega(\tau, \varepsilon') \prod_{j=1}^{p} C(\varepsilon_{l_j}, \varepsilon_{r_j}) = \lim_{N\to\infty} \frac{1}{N^{q/2}} \varphi\big(S_N(\widetilde{a}^{\varepsilon_q}) \dots S_N(\widetilde{a}^{\varepsilon_1})\big) = \psi(a_\psi^{\varepsilon_q} \dots a_\psi^{\varepsilon_1}). \tag{5.1}$$

If $(\mathcal{H}_\psi, \Phi_\psi)$ is the GNS representation of $(\mathcal{B}, \psi)$, then

$$\psi(a_\psi^{\varepsilon_q} \dots a_\psi^{\varepsilon_1}) = \langle \Phi_\psi, A^{\varepsilon_q} \dots A^{\varepsilon_1} \Phi_\psi \rangle,$$

where the A^{ε_j}'s are operators in $\mathcal{H}_\psi$. We would like to write them concretely as operators of creation, annihilation and preservation in a suitable Fock space. To this purpose it is necessary to make some constraints on the family $\{a_i^\varepsilon;\ \varepsilon \in F, i \in \mathcal{I}\}$ in $\{\mathcal{A}, \varphi\}$, i.e. we need something more than the projective independence. Therefore we suppose that for any $\varepsilon_1, \varepsilon_2 \in F_s$

$$C(\varepsilon_1, \varepsilon_2) = \begin{cases} c > 0 & \text{if } \varepsilon_1 = -1,\ \varepsilon_2 = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{5.2}$$

and without loss of generality we take $c = 1$. Under this assumption, there are some terms which do not give any contribution to the sum $\sum_{\sigma \in \mathcal{P}_\tau(q;2p)}$ above. Namely, for $\varepsilon' = \{-1, 1\}^{2p}$, denote by $\{-1, 1\}_+^{2p}$ the subset of all ε' such that

- $\sum_{j=1}^{2p} \varepsilon'(j) = 0$;
- for any $k = 1, \dots, 2p$, $\sum_{j=1}^{k} \varepsilon'(j) \geqslant 0$.

Then it is easy to check that (5.2) implies that only a partition ε such that $\varepsilon' \in \{-1, 1\}_+^{2p}$ can give a nonzero term into summation on the left-hand side of (5.1). Let $\{l_j, r_j\}_{j=1}^p$ be the left-right index set relative to $\varepsilon' \in \{-1, 1\}_+^{2p}$. By (5.2) we also see that the nonzero contributions in the considered sum are determined exactly by $\varepsilon' \in \{-1, 1\}_+^{2p}$ such that, for any $j = 1, \dots, p$, $\varepsilon(l_j) = -1$ and $\varepsilon(r_j) = 1$. To avoid the introduction of new symbols, whenever we write $\varepsilon' \in \{-1, 1\}_+^{2p}$, we require $\varepsilon(l_j) = -1$, and $\varepsilon(r_j) = 1$ are satisfied. Moreover, we assume:

1. For any $q \in \mathbb{N}^*$, any $\varepsilon \in \{-1, 0, 1\}^q$, $k : \{1, \dots, q\} \to \mathcal{I}$

$$\varphi(a_{k_q}^{\varepsilon_q} \dots a_{k_1}^{\varepsilon_1}) = 0 \tag{5.3}$$

if there is a crossing in k, i.e. there exist $h < i < j < l$ such that $k_h = k_j$, $k_i = k_l$. Hence only $\tau \in$ P.P. $(2p)$ non-crossing can appear on the left-hand side of (5.1). We use the notation $\varepsilon \in \{-1, 0, 1\}^q_{2p,+}$ to indicate any $\varepsilon \in \{-1, 0, 1\}^q$, where $q = 2p + h$ and such that $\varepsilon' := \varepsilon|_{Fs} \in \{-1, 1\}^{2p}_+$. Since it is well known (see [6], Lemma 22.6, for details) that any non-crossing pair partition τ on $\{1, \ldots, 2p\}$ is uniquely determined by $\varepsilon' \in \{-1, 1\}^{2p}_+$, it follows that every $\sigma \in \mathcal{P}_\tau(q; 2p)$ such that τ is non-crossing is uniquely determined by $\varepsilon \in \{-1, 0, 1\}^q_{2p,+}$. As a consequence, from now on, we write $\alpha\,(\sigma, \varepsilon)$ as $\alpha\,(\varepsilon)$ and $\omega\,(\tau, \varepsilon')$ as $\omega\,(\varepsilon')$.

2 (Factorization principle). For any $\varepsilon \in \{-1, 1\}^{2q}_+$

$$\varphi(a^{\varepsilon_q}_{k_q} \ldots a^{\varepsilon_1}_{k_1}) = \varphi\Big(\prod_{h=l_{d_1}}^{r_{d_1}} a^{\varepsilon_h}_{k_h}\Big) \ldots \varphi\Big(\prod_{h=l_{d_{m+1}}}^{r_{d_{m+1}}} a^{\varepsilon_h}_{k_h}\Big), \tag{5.4}$$

where m and $\{d_j\}_{j=1}^{m+1}$ are determined by $\varepsilon = \{l_h, r_h\}_{h=1}^q$ such that $1 \leqslant m < q$ with $1 = d_1 < \ldots < d_{m+1} \leqslant q$, $r_{d_h} = l_{d_{h-1}+1}$ for any $h = 2, \ldots, m+1$ and $r_{d_1} = 1$, $l_{d_{m+1}} = 2q$. Each block $\{\varepsilon_{l_{d_j}}, \ldots, \varepsilon_{r_{d_j}}\}$ with $j = 1, \ldots, m+1$ is called a *connected component of the partition* ε.

3 (Rule to compute the mixed moments). Let us introduce the following notation:

$$\omega_1 := \omega(\varepsilon' = \{-1, 1\}),$$
$$\omega_2 := \omega(\varepsilon' = \{-1, -1, 1, 1\})$$

and, generally, for any $n \geqslant 3$

$$\omega_n := \omega(\varepsilon' = \{\underbrace{-1, \ldots, -1}_{n \text{ times}}, \underbrace{1, \ldots, 1}_{n \text{ times}}\}).$$

Take

$$\varphi(a_{k_1} a^+_{k_1}) = \omega_1 \tag{5.5}$$

and, if $\varepsilon \in \{-1, 1\}^{2p}_+$ and

$$\varphi(a^{\varepsilon_{2p}}_{k_{2p}} \ldots a^{\varepsilon_1}_{k_1}) = \prod_{j=1}^{r} \omega^{l_j}_j, \qquad r \leqslant p,\ l_j \in \mathbb{N},\ j = 1, \ldots r, \tag{5.6}$$

then

$$\varphi(a_{k_{2p+1}} a^{\varepsilon_{2p}}_{k_{2p}} \ldots a^{\varepsilon_1}_{k_1} a^+_{k_{2p+1}}) = \omega_1 \prod_{j=1}^{r} \omega^{l_j}_{j+1}. \tag{5.7}$$

For example, if

$$\varphi(a_3^{-1} a_3^1 a_2^{-1} a_1^{-1} a_1^1 a_2^1) = \omega_1^2 \omega_2,$$

then

$$\varphi(a_4^{-1} a_3^{-1} a_3^1 a_2^{-1} a_1^{-1} a_1^1 a_2^1 a_4^1) = \omega_1 \omega_2^2 \omega_3.$$

By means of (5.4)–(5.7), one can inductively compute all the mixed moments.

Fixing $\varepsilon \in \{-1, 0, 1\}^q_{2p,+}$, we introduce the depth function of the string ε as the map $d_\varepsilon : \{1, \ldots , q\} \to \{0, \pm 1, \ldots , \pm 2p\}$ such that for any $j \in \{1, \ldots , q\}$

$$d_\varepsilon (j) := \sum_{k=1}^{j} \varepsilon (k)$$

and consider the sequence of real numbers $\{\alpha_n\}_n$ satisfying the property

$$\prod_{j=1}^{h} \alpha_{d_\varepsilon(j)} = \alpha (\varepsilon), \tag{5.8}$$

where on the right-hand side the same α appears as in the first line of (5.1). It is worth noticing that (5.8) does not uniquely determine the sequence; in fact, for $\varepsilon_1 = \{-1, 0, 1\}$ and $\varepsilon_2 = \{-1, 1, -1, 0, 1\}$ we find $\alpha (\varepsilon_1) = \alpha (\varepsilon_2) = \alpha_1$.

Let $\mathbb{C}$ be the complex field. If $\lambda_0 := 1$, $\lambda_1 := \omega_1$ and, for any $n \geqslant 2$, $\lambda_n := \lambda_{n-1}\omega_n$, we introduce the 1-mode interacting Fock space $\Gamma (\mathbb{C}, \{\lambda_n\})$ as in [2]. Namely, let us take $K_n := \mathbb{C}^{\otimes n}$, where its elements are multiples of a vector $a^{+n}\Phi$ and $(v, w)_n := \lambda_n \overline{v} w$, $v, w \in \mathbb{C}$. Then $\{(\cdot, \cdot)_n\}$ is a sequence of pre-scalar products. After completing, K_n is equipped with a Hilbert space structure. The 1-mode interacting Fock space $\Gamma (\mathbb{C}, \{\lambda_n\})$ is given by the orthogonal sum

$$\bigoplus_{n\geqslant 0} \{K_n, (\cdot, \cdot)_n\}.$$

Since $\lambda_n = 0 \Longrightarrow \lambda_m = 0$ for all $m \geqslant n$, the following linear operator, called a *creation operator*, is well defined:

$$a^+ : a^{+n}\Phi \longmapsto a^{+(n+1)}\Phi,$$

while, using the convention $0/0 = 0$, we define the *annihilation operator* as a linear operator such that

$$a : a^{+n}\Phi \longmapsto \frac{\lambda_n}{\lambda_{n-1}} a^{+(n-1)}\Phi.$$

Then the commutation relation

$$aa^+ = \frac{\lambda_N}{\lambda_{N-1}} \tag{5.9}$$

is satisfied, where N is the number operator. Finally, the *preservation operator* with intensity $\{\alpha_n\}_n$ determined by (5.8) is defined as

$$\alpha_N(a^{+n}\Phi) := \alpha_n(a^{+n}\Phi),$$

where N is again the number operator.

LEMMA 5.1. *The family of creation and annihilation operators is symmetric projectively independent with respect to the vacuum state* $\langle \Phi, \cdot \Phi \rangle$ *in* 1*-mode IFS.*

P r o o f. In fact, for any $p \in \mathbb{N}$, any $\varepsilon = \big(\varepsilon(1), \ldots, \varepsilon(2p)\big) \in \{-1,1\}_+^{2p}$, by the definitions of 1-mode IFS and depth function, from (5.9) we have

$$\langle \Phi, a_{2p}^{\varepsilon(2p)} \ldots a_1^{\varepsilon(1)} \Phi \rangle = \prod_{j=1}^{p} \omega_{d_\varepsilon(r_j)},$$

where $\{l_j, r_j\}_{j=1}^p$ is the unique non-crossing pair partition induced by ε. The assertion follows after noticing that the terms of the product on the right-hand side above depend only on ε. ∎

Denote by $Q := a + a^+ + \alpha_N$ the non-symmetric field operator in 1-mode IFS. The following result gives us the Fock representation for the limit process.

THEOREM 5.1. *The limit process* $\{a_\psi^{-1}, a_\psi^1, a_\psi^0\}$ *is represented in* $\Gamma(\mathbb{C}, \{\lambda_n\})$, *that is*

$$\psi(a_\psi^{\varepsilon_q} \ldots a_\psi^{\varepsilon_1}) = \langle \Phi, Q_q \ldots Q_1 \Phi \rangle.$$

P r o o f. In fact, from [2], Theorem 5.1 and (5.8), we get

$$\langle \Phi, Q_q \ldots Q_1 \Phi \rangle = \sum_{NC\mathcal{P}(q;2p)} \alpha(\varepsilon) \prod_{j=1}^{p} \omega_{d_{\varepsilon'}(r_j)},$$

where

$$NC\mathcal{P}(q;2p) := \big\{\sigma \in \mathcal{P}(q) \colon \sigma = \tau \cup \gamma, \tau \cap \gamma = \emptyset, \tau \in NC\text{P.P.}(\{z_1, \ldots, z_{2p}\})\big\}$$

and NCP.P. $(\{z_1, \ldots, z_{2p}\})$ is the set of all the non-crossing pair partitions of $\{z_1, \ldots, z_{2p}\}$. On the other hand, we know from (5.1)–(5.3) and (5.8) that

$$\psi(a_\psi^{\varepsilon_q} \ldots a_\psi^{\varepsilon_1}) = \sum_{NC\mathcal{P}(q;2p)} \alpha(\varepsilon)\, \omega(\varepsilon').$$

Moreover, from (5.4)–(5.7) it follows that

$$\omega(\varepsilon') = \prod_{j=1}^{p} \omega_{d_{\varepsilon'}(r_j)},$$

where $\{l_j, r_j\}_{j=1}^p$ is the unique non-crossing pair partition induced by ε'. ∎

Finally, we turn to give a Fock representation for the limit process expressed in part (ii) of Theorem 4.1. As already noticed, one knows that there exist an algebraic probability space $(\mathcal{B},\psi)$ and random variables $a_{\psi}^{-1(s_j,t_j)}, a_{\psi}^{1(s_j,t_j)}, a_{\psi}^{0(s_j,t_j)}$, $0 \leqslant s_j < t_j$, $j = 1,\ldots,q$, in this space such that

$$(5.10) \qquad \sum_{\sigma\in\mathcal{P}_\tau(q;2p)} \alpha\,(\sigma,\varepsilon)\,\omega(\tau,\varepsilon')\big[\prod_{d=1}^{h}(\sqrt{t_{z'_d}}-\sqrt{s_{z'_d}})\big]$$
$$\times\big[\prod_{j=1}^{p}\langle\chi_{(s_{l_j},t_{l_j})},\chi_{(s_{r_j},t_{r_j})}\rangle_{\mathbf{L}^2(\mathbb{R}_+)}C(\varepsilon_{l_j},\varepsilon_{r_j})\big]$$
$$=\lim_{N\to\infty}\frac{1}{N^{q/2}}\varphi\big(S_N^{(s_q,t_q)}(\widetilde{a}^{\varepsilon_q})\ldots S_N^{(s_1,t_1)}(\widetilde{a}^{\varepsilon_1})\big)=\psi(a_{\psi}^{\varepsilon_q(s_q,t_q)}\ldots a_{\psi}^{\varepsilon_1(s_1,t_1)}).$$

As above, given $(\mathcal{H}_\psi,\Phi_\psi)$, the GNS space of $(\mathcal{B},\psi)$, we want to realize $\mathcal{H}_\psi$ as a Fock space, Φ_ψ as the vacuum on such a space and reach the equality

$$\psi(a_{\psi}^{\varepsilon_q(s_q,t_q)}\ldots a_{\psi}^{\varepsilon_1(s_1,t_1)})=\langle\Phi_\psi,A^{\varepsilon_q}\ldots A^{\varepsilon_1}\Phi_\psi\rangle,$$

where the A^{ε_j}'s are operators of creation, annihilation and preservation in $\mathcal{H}_\psi$.

Firstly we suppose the condition (5.2), assumptions 1–3 and (5.5)–(5.8) hold also in our case.

Let $\mathcal{H}:=\mathbf{L}^2(\mathbb{R}_+)$ and get the 1-MT IFS on it, together with creation and annihilation operators, as in Example 2.1. The preservation operator with intensity $\{\alpha_n\}_n$ determined by (5.8) and $X\in\mathbf{B}(\mathcal{H})$:

$$\Lambda_\alpha(X):\mathcal{H}_n\to\mathcal{H}_n,$$

is such that, for any $f_1,\ldots,f_n\in\mathcal{H}$

$$\Lambda_\alpha(X)(f_1\otimes\ldots\otimes f_n):=\alpha_n(Xf_1)f_2\otimes\ldots\otimes f_n.$$

If X is the identity operator, we will write $\Lambda_\alpha:=\Lambda_\alpha(I)$, while, for any $f\in\mathcal{H}$, $\Lambda_\alpha(f):=\Lambda_\alpha(M_f)$, where M_f is the multiplication operator by f.

LEMMA 5.2. *The family of creation and annihilation operators is symmetric projectively independent with respect to the vacuum state* $\langle\Phi,\cdot\Phi\rangle$ *in* 1*-mode type IFS.*

Proof. The assertion can be achieved by using the same arguments as in the proof of Lemma 5.1. ■

Let $s,t\geqslant 0$, $s<t$. Denote by $\sqrt{\mathbf{t}}-\sqrt{\mathbf{s}}$ the function on $\mathbb{R}_+$ with constant value $\sqrt{t}-\sqrt{s}$, $\chi_{[s,t]}$ the indicator function on the interval $[s,t]$, and $Q^{(s,t)}:=A(\chi_{[s,t]})+A^+(\chi_{[s,t]})+\Lambda_\alpha(\sqrt{\mathbf{t}}-\sqrt{\mathbf{s}})$ the non-symmetric field operator in 1-mode type IFS. From now on we will use the following notation: for any $\varepsilon\in\{-1,0,1\}$

$$A^{\varepsilon}(s,t):=\begin{cases}A(\chi_{[s,t]}) & \text{if } \varepsilon=-1,\\ \Lambda_\alpha(\sqrt{\mathbf{t}}-\sqrt{\mathbf{s}}) & \text{if } \varepsilon=0,\\ A^+(\chi_{[s,t]}) & \text{if } \varepsilon=1.\end{cases}$$

THEOREM 5.2. *The limit process* $\{a_{\psi}^{-1(s_j,t_j)}, a_{\psi}^{1(s_j,t_j)}, a_{\psi}^{0(s_j,t_j)}\}$ *has a representation in* $\Gamma(\mathcal{H}, \{\lambda_n\})$, *that is*

$$\psi(a_{\psi}^{\varepsilon_q(s_q,t_q)} \ldots a_{\psi}^{\varepsilon_1(s_1,t_1)}) = \langle \Phi, Q_q^{(s_q,t_q)} \ldots Q_1^{(s_1,t_1)} \Phi \rangle,$$

where $0 \leqslant s_j < t_j$ *for each* $j = 1, \ldots, q$.

Proof. Let us take $0 \leqslant s_j < t_j$ for each $j = 1, \ldots, q$. Then

$$\langle \Phi, Q_q^{(s_q,t_q)} \ldots Q_1^{(s_1,t_1)} \Phi \rangle = \sum_{\varepsilon \in \{-1,0,1\}^q} \langle \Phi, A_q^{\varepsilon_q}(s_q,t_q) \ldots A_1^{\varepsilon_1}(s_1,t_1) \Phi \rangle. \tag{5.11}$$

Make the usual partition $\{1, \ldots, q\} = \{z_1, \ldots, z_{2p}\} \cup \{z_1', \ldots, z_h'\}$. For any given ε, if ε' is its restriction to the elements of F_s, then $\varepsilon \in \{-1, 0, 1\}^q_{2p,+}$ holds if and only if $\varepsilon' \in \{-1, 1\}^{2p}_+$. Denote by $\{l_j, r_j\}_{j=1}^p$ the left-right family of indices of the unique pair partition on $\{z_1, \ldots, z_{2p}\}$ induced by ε'. Then, by the definition of 1-MT IFS and Lemma 3.3 of [3], (5.11) is equal to

$$\sum_{\varepsilon \in \{-1,0,1\}^q_{2p,+}} \sum_{z_h' < \ldots < z_1' \in \{2,\ldots,q-1\}} \prod_{d=1}^h (\sqrt{t_{z_d'}} - \sqrt{s_{z_d'}}) \alpha_{d_\varepsilon(z_d')} \times \prod_{j=1}^p \omega_{d_{\varepsilon'}(r_j)} \langle \chi_{(s_{l_j},t_{l_j})}, \chi_{(s_{r_j},t_{r_j})} \rangle. \tag{5.12}$$

Hence, by means of (5.8), the expression (5.12) can be written as follows:

$$\sum_{NC\mathcal{P}(q;2p)} \alpha(\varepsilon) \Big[\prod_{d=1}^h (\sqrt{t_{z_d'}} - \sqrt{s_{z_d'}}) \Big] \Big[\prod_{j=1}^p \omega_{d_{\varepsilon'}(r_j)} \langle \chi_{(s_{l_j},t_{l_j})}, \chi_{(s_{r_j},t_{r_j})} \rangle \Big],$$

where

$$NC\mathcal{P}(q;2p) := \big\{ \sigma \in \mathcal{P}(q) \colon \sigma = \tau \cup \gamma, \tau \cap \gamma = \emptyset, \tau \in NC\text{P.P.}(\{z_1, \ldots, z_{2p}\}) \big\}.$$

On the other hand, from (5.1), (5.2), (5.8) and (5.3) we know that

$$\psi(a_{\psi}^{\varepsilon_q(s_q,t_q)} \ldots a_{\psi}^{\varepsilon_1(s_1,t_1)}) = \sum_{NC\mathcal{P}(q;2p)} \alpha(\varepsilon)\, \omega(\varepsilon') \prod_{d=1}^h (\sqrt{t_{z_d'}} - \sqrt{s_{z_d'}}) \times \prod_{j=1}^p \langle \chi_{(s_{l_j},t_{l_j})}, \chi_{(s_{r_j},t_{r_j})} \rangle.$$

From (5.4)–(5.7) it follows that

$$\omega(\varepsilon') = \prod_{j=1}^p \omega_{d_{\varepsilon'}(r_j)}. \quad \blacksquare$$

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