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ON RELATIONS BETWEEN URBANIK AND MEHLER SEMIGROUPS*

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Abstract. It is shown that operator-selfdecomposable measures or, more precisely, their Urbanik decomposability semigroups induce generalized Mehler semigroups of bounded linear operators. Moreover, those semigroups can be represented as random integrals of operator valued functions with respect to stochastic Lévy processes. Our Banach space setting is in contrast with the Hilbert spaces on which so far and most often the generalized Mehler semigroups were studied. Furthermore, we give new proofs of the random integral representation.

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The theory of operator limiting distributions in probability theory has its origin in 70's of the last century. Its development on the Euclidean spaces was summarized in the monograph by Jurek and Mason [13]. That theory is based on the principle that normalization of random variables with values in E (or sums of those random variables) should be consistent with the structure of the state space E. Thus the linear operators are the proper normalization for the linear spaces or normalization by the group automorphisms for the case of group valued variables.

The most important new tool in that setting is the *operator decomposability semigroup* introduced by K. Urbanik in 1972. Namely, with a probability measure μ one associates a family $\mathbf{D}(\mu)$ of linear bounded operators A that "divide" μ in a sense that $\mu = A\mu * \nu_A$ for some probability measure ν_A, i.e.,

$$\mathbf{D}(\mu) = \{A : \mu = A\mu * \nu_A \text{ for some } \nu_A\}.$$

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Note that operators 0 and I are always in $\mathbf{D}(\mu)$ and that it is indeed a semigroup. If the Urbanik semigroup $\mathbf{D}(\mu)$ contains a one-parameter semigroup $T_t, t \geqslant 0$, and one defines $\rho_t := \nu_{T_t}$, then, by iteration, one arrives at the equation

$$\rho_{t+s} = \rho_t * T_t \rho_s \quad \text{for all } s, t \geqslant 0, \tag{0.1}$$

provided a cancellation is permitted; cf. Jurek [9], Jurek and Vervaat [14]. Such convolution equations were also called *measure-valued cocycles*; cf. Hofmann and Jurek [7]. [Note that the Grothendick type diagram, on p. 755 there, is still not completed!]

On the other hand, if operators $\mathcal{T}_t$, given by

$$(\mathcal{T}_t f)(x) = \int_E f(T_t x + z)\rho_t(dz), \quad f \in C_b(E),\ t \geqslant 0,$$

define a one-parameter semigroup on $C_b(E)$, then the measures ρ_t's must satisfy the cocycle relation (0.1). The above families of operators are called *generalized Mehler semigroups*, for short: *Mehler semigroups*. Cf. for instance Bogachev et al. [3] and references therein. However, one should be aware that they worked on Hilbert spaces and had a more restrictive assumption (differentiability) on the Fourier transforms of $\rho_t, t \geqslant 0$. On the other hand, cocycle equations (0.1) were considered on non-linear structures like spaces of measures; cf. Li [15].

The main result here is that on a Banach space under the continuity of the mapping $t \to \rho_t$ in many cases Mehler semigroups are of the form

$$(\mathcal{T}_t f)(x) = \mathbf{E}\big[f\big(T_t x + \int_{(0,t]} T_{t-s}\, dY(s)\big)\big], \quad f \in C_b(E),$$

for some stochastic Lévy process Y; cf. below for details.

In all papers dealing with the operator limit distributions (for instance: Jurek [9]–[11] and Jurek and Vervaat [14]) the primary goal was the random integral representation (RIR) of measures μ, whose Urbanik semigroups $\mathbf{D}(\mu)$ contain a one-parameter semigroup of operators continuous in the operator norm topology. Consequently, the solutions to the cocycle equations (0.1) were auxiliary steps in the main proofs and might have been overlooked.

These two subjects, i.e., Urbanik and Mehler semigroups, seem to be developed independently of each other, although Chojnowska-Michalik [4], [5] mentioned the theory of operator limit distributions. See also the acknowledgment at the end of this paper.

Our aim here is to show how the theory of operator limit distributions and its techniques (like the random integral method) can produce new results and proofs in the theory of Mehler semigroups on Banach spaces; cf., in particular, Proposition 3.2, Theorem 3.1, and Corollaries 3.2 and 3.3 below.

Last but not least, let us stress again that our presentation here is in the generality of Banach spaces while Hilbert space setting was often the case for the generalized Mehler semigroups.

1. BASIC NOTIONS AND NOTATION

Let E denote a *real separable* Banach space with a norm $\|\cdot\|$, let **End**(E) or, simply, **End** denote the algebra of *all* bounded linear operators on E. In **End**(E) we take the *strong operator topology*, i.e., $A_n \xrightarrow{s} A$ means that *for each* $x \in E$, $\lim_{n\to\infty} \|A_n x - Ax\| = 0$. Of a particular interest are the C_0-*one-parameter semigroups* $\mathbf{T} = (T_t, t \geqslant 0)$ in **End**; that is, we have $T_0 = I$, $T_t(T_s x) = T_{t+s}x$ for $t, s \geqslant 0$, $x \in E$, and for each x the functions $t \to T_t x$ are continuous.

Let $\mathcal{P}(E)$, or just $\mathcal{P}$, denote the family of *all Borel probability measures* on E endowed with the *convolution* operation and the *weak convergence* topology, in symbols: $*$ and $\Rightarrow$, respectively. Thus

$$\mu_n \Rightarrow \mu \;\text{ iff }\; \int_E f(x)\mu_n(dx) \to \int_E f(x)\mu(dx) \quad \text{for each } f \in C_b(E),$$

where $C_b(E)$ stands for real-valued continuous bounded functions on E (weak*-topology in $C_b(E)$). For the probability theory on Banach spaces see Araujo and Giné [2] or Linde [16].

Let E' be the topological dual Banach space and let $\langle\cdot,\cdot\rangle$ denote the bilinear form between E' and E. Recall that for a measure μ or an E-valued random variable ξ with probability distribution μ, the function

$$\widehat{\mu} : E' \to \mathbb{C} \text{ defined by } \widehat{\mu}(y) := \int_E e^{i\langle y,x\rangle}\,\mu(dx) = \mathbf{E}[e^{i\langle y,\xi\rangle}]$$

is called the *Fourier transform* (or the *characteristic function*) and that it uniquely determines the measure μ; $\mathbf{E}[\cdot]$ denotes the expectation operator.

Finally, for $A \in$ **End** and $\mu \in \mathcal{P}$ we define $A\mu \in \mathcal{P}$, the image of μ through a mapping A, as follows:

$$(A\mu)(\mathcal{E}) := \mu(\{x \in E : Ax \in \mathcal{E}\}) \qquad \text{for all Borel subsets } \mathcal{E} \text{ of } E.$$

Equivalently, in terms of integrals, it means that

$$\int_E f(x)(A\mu)(dx) = \int_E f(Ax)\mu(dx) \qquad \text{for all } f \in C_b(E).$$

In other words, if ξ is an E-valued random variable with probability distribution μ, then the random variable $A\xi$ has probability distribution $A\mu$. Having that in mind we immediately get the equalities

$$A(\mu * \nu) = A\mu * A\nu, \quad A(B\mu) = (AB)\mu, \quad \widehat{(A\mu)}(y) = \hat{\mu}(A^* y), y \in E', \tag{1.1}$$

for all linear bounded operators $A, B \in$ **End** and all measures $\mu, \nu \in \mathcal{P}$.

Finally, for the future reference let us quote here that

$$\text{if } A_n \xrightarrow{s} A \text{ and } \mu_n \Rightarrow \mu, \;\text{ then } A_n\mu_n \Rightarrow A\mu. \tag{1.2}$$

The proof can be found in Jurek [10], Proposition 1.1, or in Jurek and Mason [13], Proposition 1.7.2 on p. 24.

2. THE URBANIK DECOMPOSABILITY SEMIGROUPS

With $\mu \in \mathcal{P}$ we associate its *Urbanik decomposability semigroup* $\mathbf{D}(\mu)$ defined as follows:

$$\mathbf{D}(\mu) := \{A \in \mathbf{E}\mathrm{nd}\colon \mu = A\mu * \nu_A \text{ for some } \nu_A \in \mathcal{P}\}. \tag{2.1}$$

Obviously, the linear operators 0 (zero) and I (identity) are in all $\mathbf{D}(\mu)$ with $\nu_0 = \mu$ and $\nu_I = \delta_0$ in (2.1), and the semigroup property, under composition of operators, follows from (1.1). It is interesting that some purely probabilistic properties of μ are equivalent to some algebraic and topological properties of its Urbanik $\mathbf{D}(\mu)$ decomposability semigroup; cf. Urbanik [19], [20], Jurek and Mason [13].

In the operator-limit distribution theory the operator topology is used in $\mathbf{D}(\mu)$. However, even for the strong operator topology we also have

PROPOSITION 2.1. (i) *The Urbanik decomposability semigroups* $\mathbf{D}(\mu)$, *in* $\mathbf{E}\mathrm{nd}$, *are closed in the strong operator topology.*

(ii) *If* $\hat{\mu}(y) \neq 0$ *for all* $y \in E'$, $A_n \in \mathbf{D}(\mu)$ *and* $A_n \xrightarrow{s} A$, *then* $A \in \mathbf{D}(\mu)$ *and, in* (2.1), *we have* $\nu_{A_n} \Rightarrow \nu_A$.

(iii) *If* $\mu = A_n\mu * \nu_{A_n}$ *and* $A_n \xrightarrow{s} 0$, *then* $\nu_{A_n} \Rightarrow \mu$.

Proof. (i) For $A_n \in \mathbf{D}(\mu)$ we have

$$\mu = A_n\mu * \nu_{A_n} \qquad \text{for some } \nu_{A_n} \in \mathcal{P}. \tag{2.2}$$

Further, if $A_n \xrightarrow{s} A$, then, by (1.2), $A_n\mu \Rightarrow A\mu$. Consequently, $\{\nu_{A_n}, n = 1, 2, \ldots\} \subset \mathcal{P}$ is conditionally compact (uniformly tight); cf. Parthasarathy [17], Chapter III, Theorem 2.1, or Jurek and Mason [13], Theorem 1.7.1. Thus, passing to a subsequence in (2.2), we get $\mu = A\mu * \nu$ for some accumulation point ν of the sequence $(\nu_{A_n}, n = 1, 2, \ldots)$. Consequently, $A \in \mathbf{D}(\mu)$, which proves (i).

(ii) From (i) we get $A \in \mathbf{D}(\mu)$. Since $\hat{\mu}(A_n^* y) \neq 0$ (A^* is the conjugate bounded linear operator), from (2.2) we infer that $\lim_{n\to\infty} \widehat{\nu}_{A_n}(y)$ exists. This and the conditional compactness of $(\nu_{A_n}, n = 1, 2, \ldots)$ implies the weak convergence $\nu_{A_n} \Rightarrow \nu_A$ in (ii).

(iii) Simply note, by (1.2), that $A_n\mu \Rightarrow \delta_0$, and thus $\hat{\mu}(A_n^* y) \to 1$ for all $y \in E'$. Hence, as in the proof of (ii), we conclude $\nu_{A_n} \Rightarrow \mu$. This completes the proof of Proposition 2.1. ■

We will say that $\nu \in \mathcal{P}$ is an *operator convolution factor* of μ if there exists $A \in \mathbf{E}\mathrm{nd}$ such that $\mu = A\mu * \nu$. By $O\mathbf{F}(\mu)$ we denote the totality of the operator convolution factors of μ.

PROPOSITION 2.2. *Let* $\mu \in \mathcal{P}$ *be such that* $\hat{\mu}(y) \neq 0$ *for all* $y \in E'$. *Then* $O\mathbf{F}(\mu) = \{\nu_A : A \in \mathbf{D}(\mu)\}$ *with binary operation* $\diamond$ *defined by* $\nu_A \diamond \nu_B := \nu_A * A\nu_B$ *is a non-commutative semigroup. Moreover*, $\nu_A \diamond \nu_B = \nu_{AB}$, $\nu_I \equiv \delta_0$ *is the neutral element*, *and* $\nu_A \diamond \nu_0 = \nu_0 \diamond \nu_A = \nu_0 \equiv \mu$.

P r o o f. For $A, B \in \mathbf{D}(\mu)$ we have

$$\mu = A\mu * \nu_A = A(B\mu * \nu_B) * \nu_A = (AB)\mu * (A\nu_B * \nu_A) = (AB)\mu * \nu_{AB}$$

because of (1.1) and the fact that $AB \in \mathbf{D}(\mu)$ as well. Hence we have $\nu_A \diamond \nu_B = \nu_A * A\nu_B = \nu_{AB}$ because of $\hat{\mu}(y) \neq 0$. Furthermore,

$$(\nu_A \diamond \nu_B) \diamond \nu_C = \nu_{AB} \diamond \nu_C = \nu_{(AB)C} = \nu_A \diamond (\nu_B \diamond \nu_C),$$

which proves the associativity of the operation $\diamond$. The rest follows from the equalities $\nu_I \equiv \delta_0$ and $\nu_0 \equiv \mu$; see (2.1). ∎

REMARK 2.1. If $A_n \in \mathbf{D}(\mu)$, $\hat{\mu}(y) \neq 0$ for all $y \in E'$ and $\nu_{A_n} \Rightarrow \rho_2 \in \mathcal{P}$, then $A_n\mu \Rightarrow \rho_1$ for some $\rho_1 \in \mathcal{P}$ and $\mu = \rho_1 * \rho_2$, that is, ρ_2 is a *convolution factor* of μ. But is it an *operator convolution factor*? Can we write $\rho_1 = A\mu$ for some $A \in \mathbf{E}$nd? In the case of $E = \mathbb{R}^d$ (finite-dimensional space) and *full measures* the answer is affirmative; see Lemma 2.2.9 and Section 2.5 in Chapter II of Jurek and Mason [13].

We say that μ is *operator-selfdecomposable* on E, in symbols: $\mu \in OS$, if the Urbanik semigroup $\mathbf{D}(\mu)$ contains (at least one) C_0-semigroup $\mathbf{T} = (T_t, t \geqslant 0)$. When a semigroup $\mathbf{T}$ is fixed, we write $\mu \in OS(\mathbf{T})$ and say that μ is $\mathbf{T}$-*decomposable*.

REMARK 2.2. It is also important to realize that originally in Urbanik [20] there were operator continuous one-parameter semigroups $T_t = \exp(t\mathbf{V}), t \geqslant 0$, such that $\lim_{t\to\infty} T_t = 0$ (in the operator norm) . Furthermore, Urbanik primarily dealt with limit distributions of sequences of partial sums of E-valued variables normalized by arbitrary bounded linear operators. A similar approach was taken in Jurek [10], however with *specified* normalizing operators but with strong operator topology. For the theory of operator-selfdecomposable (and operator-stable) measures cf. Jurek and Mason [13] and references therein.

Explicitly, we have

$$\mu \in OS\big((T_t, t \geqslant 0)\big) \text{ iff } \forall\,(t \geqslant 0)\,\exists\,(\nu_{T_t} \in \mathcal{P})\;\; \mu = T_t\mu * \nu_{T_t} \tag{2.3}$$

or equivalently, by Proposition 2.2, in terms of the semigroup $O\mathbf{F}(\mu)$ of the operator convolution factors μ, we have

$$\mu \in OS\big((T_t, t \geqslant 0)\big) \text{ iff } \exists\Big(\{\rho_t, t \geqslant 0\} \subset \big(O\mathbf{F}(\mu), \diamond\big)\Big)\;\; \forall(s, t \geqslant 0)\;\; \rho_t \diamond \rho_s = \rho_{t+s}, \tag{2.4}$$

where $\rho_t := \nu_{T_t},\ t \geqslant 0$.

3. THE GENERALIZED MEHLER SEMIGROUPS

For an operator $A \in \mathbf{E}\mathrm{nd}(E)$ and a probability $\mu \in \mathcal{P}(E)$, let us define the linear operator $\mathcal{A}^{(\mu)}$ as follows:

$$(3.1)\quad \mathcal{A}^{(\mu)} : C_b(E) \to C_b(E), \quad (\mathcal{A}^{(\mu)} f)(x) := \int_E f(Ax+z)\mu(dz), \; x \in E.$$

Note that $\mathcal{A}^{(\mu)}$ can be viewed as the convolution of a function f with a measure $\delta_{Ax} * \mu$. Here are some elementary properties of those operators.

PROPOSITION 3.1. (i) *The operator* $\mathcal{A}^{(\mu)}$ *uniquely determines a measure* $\mu \in \mathcal{P}(E)$ *and an operator* $A \in \mathbf{E}\mathrm{nd}(E)$.

(ii) *For* $A, B \in \mathbf{E}\mathrm{nd}(E)$ *and* $\mu, \nu \in \mathcal{P}(E)$ *we have the equality*

$$\mathcal{B}^{(\nu)} \cdot \mathcal{A}^{(\mu)} = \mathcal{C}^{(\mu * A\nu)},$$

where $C := AB$, *and* $\cdot$ *means the composition of operators.*

(iii) *For one-parameter families of operators* $A_t \in \mathbf{E}\mathrm{nd}(E)$ *and probability measures* $\rho_t \in \mathcal{P}(E)$ $(t \geqslant 0)$,

$$[\,\mathcal{A}_s^{(\rho_s)} \cdot \mathcal{A}_t^{(\rho_t)} = \mathcal{A}_{t+s}^{(\rho_{t+s})}\,] \text{ iff } [\,A_t \cdot A_s = A_{t+s} \text{ and } \rho_{t+s} = \rho_t * A_t\rho_s\,].$$

P r o o f. (i) Suppose $\mathcal{A}^{(\mu)} = \mathcal{B}^{(\nu)}$. Putting $x = 0$ in (3.1) we have

$$(\mathcal{A}^{(\mu)} f)(0) = \int_E f(y)\mu(dy) = \int_E f(y)\nu(dy) \qquad \text{for all } f \in C_b(E),$$

which, by the Riesz theorem, implies that $\mu = \nu$. Furthermore, since

$$(\mathcal{A}^{(\mu)} f)(x) = \int_E \int_E f(u+y)\delta_{Ax}(du)\mu(dy) = \int_E f(z)(\delta_{Ax} * \mu)(dz),$$

we conclude that $\delta_{Ax} * \mu = \delta_{Bx} * \mu$ for all $x \in E$, that is, $A = B$.

(ii) For $f \in C_b(E)$ and $x \in E$, by (3.1), we have

$$\begin{aligned} &\big((\mathcal{B}^{(\nu)} \cdot \mathcal{A}^{(\mu)})f\big)(x) = \big(\mathcal{B}^{(\nu)}(\mathcal{A}^{(\mu)} f)\big)(x) = \int_E (\mathcal{A}^{(\mu)} f)(Bx+y)\nu(dy) \\ &= \int_E \Big(\int_E f\big(A(Bx+y)+z\big)\mu(dz)\Big)\nu(dy) \\ &= \int_E \int_E f\big((AB)x + Ay + z\big)\mu(dz)\nu(dy) = \int_E f\big((AB)x+u\big)(\mu * A\nu)(du) \\ &= (\mathcal{C}^{(\mu * A\nu)} f)(x), \end{aligned}$$

which proves (ii).

(iii) Since, by (ii), $\mathcal{A}_s^{(\rho_s)} \cdot \mathcal{A}_t^{(\rho_t)} = (\mathcal{A}_t\mathcal{A}_s)^{(\rho_t * A_t\rho_s)}$, in order to have the equality in (iii) it is necessary and sufficient that $A_tA_s = A_{t+s}$ and $\rho_t * A_t\rho_s = \rho_{t+s}$, because of (i). ■

For a given C_0-semigroup $(T_t, t \geqslant 0)$ on E and a family of probability measures ρ_t, one-parameter semigroups $\mathcal{T}_t \equiv \mathcal{T}_t^{(\rho_t)}$ (on $C_b(E)$) are called *one-parameter generalized Mehler semigroups*. Hence necessarily and sufficiently we have: $\rho_{t+s} = \rho_t * T_t\rho_s = \rho_s * T_s\rho_t$ for all $t, s \geqslant 0$; see Proposition 3.1 (iii). Such equations were called *cocycles* in Hofmann and Jurek [7].

Explicitly we can write

$$(3.2) \qquad (\mathcal{T}_t f)(x) = \int_E f(T_t x + y)\rho_t(dy), \qquad f \in C_b(E).$$

Firstly, let us note that we have the following relation:

COROLLARY 3.1. *Each Urbanik semigroup* $\mathbf{D}(\mu)$, $\hat{\mu}(y) \neq 0, y \in E'$, *that contains a one-parameter* C_0*-semigroup* $(T_t, t \geqslant 0)$ *induces a generalized Mehler semigroup* $(\mathcal{T}_t, t \geqslant 0)$ *by taking in* (3.2) $\rho_t = \nu_{T_t}$ *from* (2.1).

Secondly, inspired by the technique from the operator-limit distribution theory we get

PROPOSITION 3.2. *If* $t \to \rho_t$ *is continuous at zero and* $\rho_{t+s} = \rho_t * T_t\rho_s$ *for all* $t, s \geqslant 0$ (*cocycle equation*), *then there exists a càdlàg process* $Z(t), t \geqslant 0$, *with independent increments such that* $\mathcal{L}\big(Z(t)\big) = \rho_t$ *and* $Z(0) = 0$ *a.s. In particular, all* ρ_t *are infinitely divisible.*

P r o o f. By the Kolmogorov's extension theorem (on a family of consistent distributions), in order to describe (in distribution) a process Z_t, $t \geqslant 0$, starting from zero (i.e. $Z_0 = 0$ with probability 1) and with independent increments it is necessary and sufficient to give the probability distributions of all increments $(Z_t - Z_s,\ t \geqslant s \geqslant 0)$ (in particular, we get distributions of Z_t) in such a way that $Z_t \stackrel{d}{=} (Z_t - Z_s) + Z_s$ with summands independent for all $t \geqslant s \geqslant 0$.

Let us define

$$(3.3) \qquad \mathcal{L}(Z_t - Z_s) := T_s\rho_{t-s}, \ \ t \geqslant s \geqslant 0, \qquad \text{in particular,} \ \ Z_t \stackrel{d}{=} \rho_t.$$

Then, by the independence and the cocycle equation, we get

$$(Z_s - Z_0) + (Z_t - Z_s) \stackrel{d}{=} \rho_s * T_s\rho_{t-s} = \rho_t \stackrel{d}{=} Z_t.$$

Since $t \to \rho_t$ is continuous, a Banach space-valued process Z_t is continuous in probability. Consequently, for $Z_t, t \geqslant 0$, there exists its càdlàg version (in French, *càdlàg* ≡ continu à droite avec des limites à gauche, i.e., paths are right continuous with left-hand limits) $Z(t), t \geqslant 0$; cf. Jurek and Vervaat [14], Theorem A.1.1 on p. 260.

Finally, using (3.3) for each $t \geqslant 0$ and each $n \geqslant 1$ we have

$$\rho_t \stackrel{d}{=} Z(t) = \sum_{k=1}^{n} \left(Z\left(\frac{k\,t}{n}\right) - Z\left(\frac{(k-1)t}{n}\right)\right)$$

$$\stackrel{d}{=} \rho_{t/n} * T_{t/n}\, \rho_{t/n} * T_{2t/n}\, \rho_{t/n} * \ldots * T_{(n-1)t/n}\, \rho_{t/n},$$

and the triangular array is infinitesimal, i.e., for each $\epsilon > 0$

$$\lim_{n\to\infty} \max_{0\leqslant j\leqslant n-1} (T_{jt/n}\, \rho_{t/n})(\|x\| \geqslant \epsilon) = 0,$$

because of (1.1) and the fact that $\rho_s \Rightarrow \delta_0$ as $s \to 0$. This proves the infinite divisibility of ρ_t, and thus completes the proof. ∎

REMARK 3.1. Note that in Proposition 3.2 we get the infinite divisibility from the stochastic independence of increments of the process $Z(t), t \geqslant 0$, as well. The infinite divisibility in the cocycle equations was proved in Schmuland and Sun [18] by different (analytic) methods and without the continuity condition. In our approach the continuity $t \to \rho_t$ was used to get càdlàg paths of the constructed process and, consequently, the infinite divisibility property.

COROLLARY 3.2. *Each generalized Mehler semigroup $\mathcal{T}_t$, with $t \to \rho_t$ continuous at zero, is of the form*

$$(\mathcal{T}_t\, f)(x) = \mathbf{E}\left[\, f\big(T_t x + Z(t)\big)\right], \qquad f \in C_b(E),$$

for some C_0-semigroup $(T_t, t \geqslant 0)$ in $\mathbf{E}\mathrm{nd}(E)$ *and some E-valued càdlàg process $\big(Z(t), t \geqslant 0\big)$ with independent increments, $Z(0) = 0$ a.s. and $Z(t) - Z(s) \stackrel{d}{=} T_s\, Z(t-s)$ for all $t \geqslant s \geqslant 0$.*

For an $\mathbf{E}\mathrm{nd}(E)$-valued function $g(t)$ of locally bounded variation and E-valued càdlàg process with independent increments $\big(Y(t), t \geqslant 0\big)$, let us define a *random integral* by the following formula of formal integration by parts:

$$\int_{(a,b]} g(t)dY(t) := g(b)Y(b) - g(a)Y(a) - \int_{(a,b]} dg(t)Y(t), \tag{3.4}$$

where the right-hand side is defined as pathwise approximation by partial sums of the form $\sum_{j=1}^{n} \big(g(t_j) - g(t_{j-1})\big)Y(t_j)$ in a similar way as in Jurek [9] and Jurek and Vervaat [14] or Jurek and Mason [13].

Recall that by a *stochastic Lévy process* we mean a càdlàg process with stationary and independent increments, and starting from zero.

THEOREM 3.1. *For each C_0-semigroup $\mathbf{T} = (T_t, t \geqslant 0)$ and each E-valued Lévy process $\mathbf{Y} = \big(Y(t), t \geqslant 0\big)$ a semigroup of operators*

$$\mathcal{T}^{\mathbf{T},\mathbf{Y}} f \equiv (\mathcal{T}_t\, f)(x) := \mathbf{E}\big[f\big(T_t x + \int_{(0,t]} T_{t-s}\, dY(s)\big)\big], \qquad f \in C_b(E), \tag{3.5}$$

is a generalized Mehler semigroup.

Conversely, if $\mathcal{T}$ is a generalized Mehler semigroup with the mapping $t \to \rho_t$ continuous at zero and a one-parameter group $\mathbf{T} = (T_t, t \in \mathbb{R})$ *of operators, then there exists a unique (in distribution) Lévy càdlàg process $Y(t), t \geqslant 0$, such that $\mathcal{T} = \mathcal{T}^{\mathbf{T},\mathbf{Y}}$.*

P r o o f. Let V_t denote the random integral part in (3.5) and let ρ_t be its probability distribution. Then, using the standard argument of approximation by partial sums, we infer that

$$(3.6)\quad \log \hat{\rho}_t(y) := \log \mathbf{E}[\exp(i\langle y, V_t\rangle)] = \int_{(0,t]} \log \mathbf{E}\big[\exp\big(i\langle T^*_{t-s} y, Y(1)\rangle\big)\big]\, ds = \int_{(0,t]} \log \mathbf{E}\big[\exp\big(i\langle T^*_r\, y, Y(1)\rangle\big)\big]\, dr.$$

Hence by simple calculations we get $\log \hat{\rho}_t(y) + \log \hat{\rho}_s(T^*_t\, y) = \log \hat{\rho}_{t+s}(y)$, $y \in E'$. Thus the family $\rho_t, t \geqslant 0$, satisfies the cocycle equation, and so (3.5) defines a Mehler semigroup.

Conversely, let a generalized Mehler semigroup $\mathcal{T}$ be given by (3.2). Then, by Corollary 3.2, there exists a càdlàg process $\big(Z(t), t \geqslant 0\big)$ with independent increments. Furthermore, the stochastic process

$$(3.7)\qquad Y(t) := \int_{(0,t]} T_{-s}\, dZ(s), \qquad t \geqslant 0,$$

is the process with independent increments, because so is $Z(\cdot)$. And, more importantly, for $t > s$, using the fact that $T_{-s}\big(Z(t) - Z(s)\big) \stackrel{d}{=} Z(t-s)$, by (3.3), we conclude that

$$Y(t) - Y(s) = \int_{(0,t-s]} T_{-v} T_{-s} dZ(v+s) \stackrel{d}{=} \int_{(0,t-s]} T_{-v} dZ(v) = Y(t-s),$$

i.e., that Y is a Lévy process (independent and stationary increments). By (3.6) and (3.7) we have

$$\int_{(0,\,t]} T_{t-s} dY(s) \stackrel{d}{=} \int_{(0,\,t]} T_s dY(s) = Z(t) \qquad \text{for each } t > 0,$$

and this with Corollary 3.2 gives the formula (3.5). The uniqueness in distribution is a consequence of the Kolmogorov extension theorem, and therefore the proof is complete. ∎

REMARK 3.2. Note that for a C_0-semigroup T_t on a Banach space E, and for an E-valued Lévy càdlàg stochastic process Y, the processes given by random integrals

$$V(t) := \int_{(0,\,t]} T_{t-s} dY(s), \qquad Z(t) := \int_{(0,\,t]} T_s dY(s), \qquad t \geqslant 0,$$

have only identical marginal (one-dimensional) distributions. The process Z has independent increments while V is a Markov process.

COROLLARY 3.3. (a) *On Euclidean spaces* ($E = \mathbb{R}^d$) *all generalized Mehler semigroups are of the form* (3.5).

(b) *On an arbitrary separable Banach space*, *for any uniformly continuous semigroups* $T_t = \exp(tQ)$ (Q *is a bounded operator*), *all generalized Mehler semigroups are of the form* (3.5).

REMARK 3.3. (a) In the above proof of Theorem 3.1, the group property of $(T_t, t \in \mathbb{R})$ was only used to define the Lévy process Y (in (3.7)) via the additive process Z from Corollary 3.2. However, what we only need is that, for the given additive process Z (in Corollary 3.2), the stochastic differential equation

$$(*) \qquad T_t\, dX(t) = dZ(t), \qquad X(0) = 0,$$

has a solution for $X(\cdot)$ in Lévy processes. But since not all additive processes are semimartingales, the stochastic equations of the form $(*)$ may not be solvable; cf. Jacod and Shiryaev [8], Theorems 4.14 and 4.15, p. 106.

(b) The representation like the above (3.5) can be derived from Lemma 2.6 of Bogachev et al. [3], but only for Hilbert spaces H and, more importantly, under restrictive assumptions that, for $y \in H$, the functions $t \to \widehat{\rho}_t(y)$ are differentiable at zero. The same setting is also in Fuhrman and Roeckner [6].

Let us also recall here that stochastic processes from Theorem 3.1,

$$U_t := T_t x + \int_{(0,t]} T_{t-s}\, dY(s), \qquad t \geqslant 0,$$

are called *Ornstein–Uhlenbeck processes* and are well studied on Hilbert spaces. [These are solutions to so-called Langevin equations.] Furthermore, one can easily express the Lévy–Khintchine formula of U_t in terms of the corresponding parameters of $Y(1)$; cf., for instance, Jurek and Mason [13], Section 3.6 in Chapter 3.

OPEN PROBLEM. It is known that on an arbitrary separable Banach space if

$$\lim_{t\to\infty} \exp(tV) = 0$$

(in the operator topology; V is a bounded operator), then, for a càdlàg Lévy process Y, the limit $\lim_{t\to\infty} \int_{(0,t]} e^{sV}\, dY(s)$ exists (almost surely, in probability or in distribution) if and only if $\mathbf{E}\big[\log\big(1 + \|Y(1)\|\big)\big] < \infty$; cf. Jurek [9].

Is there an analogous criterion true for C_0-semigroups $\mathbf{T}$ on E? Or how does the existence of a limit depend on the infinitesimal generator J of the semigroup $\mathbf{T}$ and the variable $Y(1)$?

Recall that Applebaum [1] in Theorem 9 showed that log-integrability is sufficient for *exponentially stable contraction semigroup* $(T_t, t \geqslant 0)$ *on a Hilbert space*. His proof uses the Lévy–Itô decomposition of the process Y.

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REFERENCES

[1] D. Applebaum, *Martingale-valued measures, Ornstein–Uhlenbeck processes with jumps and operator self-decomposability in Hilbert spaces*, in: *Séminaire de Probabilités*, Vol. 39, Lecture Notes in Math. No 1874, Springer 2005, pp. 171–196.

[2] A. Araujo and E. Giné, *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, New York 1980.

[3] V. I. Bogachev, M. Roeckner and B. Schmuland, *Generalized Mehler semigroups and applications*, Probab. Theory Related Fields 105 (1996), pp. 193–225.

[4] A. Chojnowska-Michalik, *Stationary distributions for ∞-dimensional linear equations with general noise*, Lecture Notes in Control and Inform. Sci. 69 (1985), pp. 14–25.

[5] A. Chojnowska-Michalik, *On processes of Ornstein–Uhlenbeck type in Hilbert space*, Stochastics 21 (1987), pp. 252–286.

[6] M. Fuhrman and M. Roeckner, *Generalized Mehler semigroups: the non-Gaussian case*, Potential Anal. 12 (2000), pp. 1–47.

[7] K. H. Hofmann and Z. J. Jurek, *Some analytic semigroups occurring in probability theory*, J. Theoret. Probab. 9 (3) (1996), pp. 745–763.

[8] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, Berlin 1987.

[9] Z. J. Jurek, *An integral representation of operator-selfdecomposable random variables*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 30 (7–8) (1982), pp. 385–393.

[10] Z. J. Jurek, *Limit distributions and one-parameter groups of linear operators on Banach spaces*, J. Multivariate Anal. 13 (4) (1983), pp. 578–604.

[11] Z. J. Jurek, *Random integral representations for classes of limit distributions similar to Lévy class L_0*, Probab. Theory Related Fields 78 (1988), pp. 473–490.

[12] Z. J. Jurek, *Measure valued cocycles from my papers in 1982 and 1983 and Mehler semigroups*, www.math.uni.wroc.pl/~zjjurek.

[13] Z. J. Jurek and J. D. Mason, *Operator-limit Distributions in Probability Theory,* Wiley, New York 1993.

[14] Z. J. Jurek and W. Vervaat, *An integral representation for selfdecomposable Banach space valued random variables*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 62 (1983), pp. 247–262.

[15] Z. H. Li, *Skew convolution semigroups and related immigration processes*, Theory Probab. Appl. 46 (2002), pp. 274–296.

[16] W. Linde, *Probability in Banach Spaces – Stable and Infinite Divisible Distributions*, Wiley, New York 1986.

[17] K. R. Parthasarathy, *Probability Measures on Metric Spaces,* Academic Press, New York–London 1967.

[18] B. Schmuland and W. Sun, *On equation $\mu_{t+s} = \mu_s * T_s\mu_t$*, Statist. Probab. Lett. 52 (2001), pp. 183–188.

[19] K. Urbanik, *Lévy's probability measures on Euclidean spaces*, Studia Math. 44 (1972), pp. 119–148.

[20] K. Urbanik, *Lévy's probability measures on Banach spaces*, Studia Math. 63 (1978), pp. 283–308.

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