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ON A GENERAL ZERO-SUM STOCHASTIC GAME WITH OPTIMAL STOPPING

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Abstract. In the paper a general zero-sum stochastic game with stopping is considered. Using the so-called penalty method the author shows the existence of the value of the game under fairly general assumptions.

1. Introduction. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(\mathscr{F}_t)_{t\geqslant 0}$ an increasing, right continuous family of complete sub-σ-fields of $\mathscr{F}$. Let us suppose we have two right continuous, $(\mathscr{F}_t)_{t\geqslant 0}$ adapted, bounded processes $(f_t)_{t\geqslant 0}$ and $(g_t)_{t\geqslant 0}$ such that $f_t \geqslant g_t$ P-a.e. for each $t \geqslant 0$. We shall consider the following game. There are two players and each of them is choosing, as his strategy, stopping time relative to $(\mathscr{F}_t)_{t\geqslant 0}$. If τ and σ are stopping times chosen by the first and the second players, respectively, then the first one pays to the second the amount equal to $e^{-\alpha\tau} f_\tau$ or $e^{-\alpha\sigma} g_\sigma$ according as $\tau < \sigma$ or $\sigma \leqslant \tau$. The aim of the first (second) player is to minimize (maximize) the expectation

$$\mathscr{J}(\tau, \sigma) \stackrel{\mathrm{df}}{=} \mathrm{E}\{\chi_{\tau<\sigma} e^{-\alpha\tau} f_\tau + \chi_{\sigma\leqslant\tau} e^{-\alpha\sigma} g_\sigma\}.$$

For a fixed stopping time τ of the first player, the second one is interested in choosing a time σ which achieves, or at least approximates, the supremum $\sup_\sigma \mathscr{J}(\tau, \sigma)$. Thus, if the first player is cautious, he will choose a time giving (or at least approximating) the infimum

$$\bar{x} = \inf_{\tau\in\Lambda_0} \sup_{\sigma\in\Lambda_0} \mathscr{J}(\tau, \sigma),$$

where Λ_s, $s \geqslant 0$, denotes a family of all Markov times almost surely greater than s.

Reversing the roles of the two players, we also see that an expected gain of the second player is at least equal to

$$\underline{x} = \sup_{\sigma\in\Lambda_0} \inf_{\tau\in\Lambda_0} \mathscr{J}(\tau, \sigma)$$

regardless of the strategy adopted by the first player. It is always true that $\underline{x} \leqslant \bar{x}$. The identity $\underline{x} = \bar{x}$ holds if there exists a saddle point for the game. In our situation a saddle point is a pair of Markov times $\hat{\tau}$, $\hat{\sigma}$ such that $\underline{x} = \mathscr{J}(\hat{\tau}, \hat{\sigma}) = \bar{x}$. Sufficient conditions for the existence of a saddle point for the game are given by Bismut [3].

The main result of the paper is Theorem 3 which shows the existence of the value of the game, equivalently the identity $\underline{x} = \bar{x}$ under fairly general conditions. Our method of the proof is new and, for instance, different from that of [3]. Namely, we use the penalization method in a general setting similar to the one considered in [10]. This method can be described as follows.

First we prove (Theorem 1) that for each $\beta > 0$ and $\gamma > 0$ there exists a unique pair of right continuous, $(\mathscr{F}_s)_{s\geqslant 0}$ adapted processes $b_s = b_s^{\beta,\gamma}$, $c_s = c_s^{\beta,\gamma}$ which satisfy the equations

$$(1)\qquad \begin{aligned} b_s &= \gamma \mathrm{E}\Big\{\int_s^\infty e^{-(\alpha+\gamma)(t-s)}[(c_t+g_t-b_t)^+ + b_t]\,dt \,|\, \mathscr{F}_s\Big\},\\ c_s &= \beta \mathrm{E}\Big\{\int_s^\infty e^{-(\alpha+\beta)(t-s)}[(b_t-f_t-c_t)^+ + c_t]\,dt \,|\, \mathscr{F}_s\Big\} \end{aligned}$$

P-a.s. for each $s \geqslant 0$.

It is possible to prove (Theorem 2) that the limits

$$\hat{b}_s = \lim_{\beta,\gamma\to\infty} b_s^{\beta,\gamma} \quad \text{and} \quad \hat{c}_s = \lim_{\beta,\gamma\to\infty} c_s^{\beta,\gamma}$$

are well defined and under some additional assumptions one can show (Theorem 3) that

$$\underline{x} = \mathrm{E}\{\hat{b}_0 - \hat{c}_0\} = \bar{x}.$$

The problem of zero-sum stochastic game with optimal stopping was introduced by Dynkin [5] for discrete time case. The continuous time game was considered first by Krylov [6], [7] for diffusion processes and a class of standard Markov processes. Later this game was investigated from the point of view of variational inequalities by Bensoussan and Friedman [1], and by using the convex duality by Bismut [2]-[4]. An extension of results due to Krylov [6], [7] for standard Markov processes will appear in [9].

2. Penalized systems of equations and their interpretation. In this section we consider the problem of existence and uniqueness of the solution of equations (1). We also give a stochastic control interpretation of this solution, i.e. a problem which consists in finding right continuous, $(\mathscr{F}_s)_{s\geq 0}$ adapted processes $(b_s)_{s\geq 0}$ and $(c_s)_{s\geq 0}$ such that the equations

$$
\begin{aligned}
b_s &= \operatorname*{sup\,ess}_{u^2\in M_\gamma} \mathrm{E}\Big\{\int_s^\infty \exp\Big[-\int_s^t(\alpha+u_r^2)\,dr\Big]u_t^2(c_t+g_t)\,dt \,\Big|\, \mathscr{F}_s\Big\},\\
c_s &= \operatorname*{sup\,ess}_{u^1\in M_\beta} \mathrm{E}\Big\{\int_s^\infty \exp\Big[-\int_s^t(\alpha+u_r^1)\,dr\Big]u_t^1(b_t-f_t)\,dt \,\Big|\, \mathscr{F}_s\Big\}
\end{aligned}
\tag{2}
$$

are fulfilled P-a.s. for each $s \geq 0$. In the system (2) the symbols M_β and M_γ denote the sets of all adapted processes with values from the intervals $[0, \beta]$ and $[0, \gamma]$, respectively. The solutions $(b_s)_{s\geq 0}$ and $(c_s)_{s\geq 0}$ depend on β and γ, so to be more precise one should write $(b_s^{\beta,\gamma})_{s\geq 0}$ and $(c_s^{\beta,\gamma})_{s\geq 0}$, and when it is necessary to emphasize the dependence of the solution on β, γ, we shall use this more cumbersome notation. The systems (1) and (2) will play the main role in our paper. We have

THEOREM 1. *The systems* (1) *and* (2) *are equivalent and have, as a unique solution, the pair* $(b_s, c_s)_{s\geq 0}$ *of right continuous,* $(\mathscr{F}_s)_{s\geq 0}$ *adapted processes for each positive* β *and* γ.

Proof. Similarly as in [10] we introduce a certain Banach space. For every right continuous, $(\mathscr{F}_s)_{s\geq 0}$ adapted process f we define the norm

$$\|f\| \overset{\mathrm{df}}{=} \operatorname*{ess\,sup}_{\Omega} \sup_{s\geq 0} |f(s, \omega)|.$$

It can be verified that the space $\mathscr{W}$ of all right continuous, $(\mathscr{F}_s)_{s\geq 0}$ adapted processes f such that $\|f\| < \infty$ with the norm $\|\cdot\|$ is a Banach space. Now, let us note that if $(\mathscr{W}_1, \|\cdot\|_1)$ and $(\mathscr{W}_2, \|\cdot\|_2)$ are $\mathscr{W}$-spaces, then their Cartesian product $\mathscr{W}_1\times\mathscr{W}_2$ is the Banach space with the norm

$$|||f||| = \max\{\|f_1\|_1, \|f_2\|_2\}, \quad \text{where } f = (f_1, f_2)\in\mathscr{W}_1\times\mathscr{W}_2.$$

We define the following transformation Ψ in the space $\mathscr{W}_1\times\mathscr{W}_2$:

$$
\Psi\colon \left(\begin{bmatrix} z_s^1 \\ z_s^2 \end{bmatrix}\right)_{s\geq 0} \mapsto \begin{bmatrix} \Psi^1\big((z_s^1, z_s^2)_{s\geq 0}\big) \\ \Psi^2\big((z_s^1, z_s^2)_{s\geq 0}\big)\end{bmatrix}
$$

$$
= \left(\begin{bmatrix} \gamma\mathrm{E}\Big\{\int_s^\infty e^{-(\alpha+\gamma)(t-s)}[(z_t^2+g_t-z_t^1)^+ + z_t^1]\,dt \,\Big|\, \mathscr{F}_s\Big\} \\ \beta\mathrm{E}\Big\{\int_s^\infty e^{-(\alpha+\beta)(t-s)}[(z_t^1-f_t-z_t^2)^+ + z_t^2]\,dt \,\Big|\, \mathscr{F}_s\Big\} \end{bmatrix}\right)_{s\geq 0}.
$$

The transformation Ψ works from $\mathscr{W}_1 \times \mathscr{W}_2$ into $\mathscr{W}_1 \times \mathscr{W}_2$ since as the processes we can take their right continuous modifications. We want to check that Ψ is a contraction.

If $z = (z^1, z^2)$, $w = (w^1, w^2) \in \mathscr{W}_1 \times \mathscr{W}_2$, then

$$|||\Psi(z) - \Psi(w)||| = \max\{||\Psi^1(z^1, z^2) - \Psi^1(w^1, w^2)||, ||\Psi^2(z^1, z^2) - \Psi^2(w^1, w^2)||\}$$

and

$$||\Psi^1(z^1, z^2) - \Psi^1(w^1, w^2)|| = \sup_{\Omega} \operatorname{ess} \sup_{s \geqslant 0} \Big|\gamma \mathrm{E}\Big\{\int_s^\infty e^{-(\alpha+\gamma)(t-s)} [(z_t^2 + g_t - z_t^1)^+ + z_t^1 - (w_t^2 + g_t - w_t^1)^+ - w_t^1]\, dt \mid \mathscr{F}_s\Big\}\Big|.$$

To continue the proof we need the following

LEMMA 1. *If* $F(x, y) = \beta(x-y)^+ + \beta y$, *then*

$$|F(z^1, z^2) - F(w^1, w^2)| \leqslant \beta \max\{|z^1 - w^1|, |z^2 - w^2|\} \quad (\beta > 0).$$

The proof of this lemma is not difficult, and therefore can be omitted. From Lemma 1 we obtain

$$||\Psi^1(z^1, z^2) - \Psi^1(w^1, w^2)|| \leqslant \sup_{\Omega} \operatorname{ess} \sup_{s \geqslant 0} \mathrm{E}\Big\{\int_s^\infty e^{-(\alpha+\gamma)(t-s)} \gamma \max\{|z_t^1 - w_t^1|, |z_t^2 - w_t^2|\}\, dt \mid \mathscr{F}_s\Big\} \leqslant \frac{\gamma}{\alpha+\gamma} |||z - w|||$$

and, analogously,

$$||\Psi^2(z^1, z^2) - \Psi^2(w^1, w^2)|| \leqslant \frac{\beta}{\alpha+\beta} |||z - w|||.$$

Thus, finally,

$$|||\Psi(z) - \Psi(w)||| \leqslant \frac{\max\{\gamma, \beta\}}{\alpha + \max\{\gamma, \beta\}} |||z - w|||.$$

The last inequality insures the existence of the unique solution of the system (1). Using similar considerations as in [10] we can show that this solution satisfies also the system (2). Now, by the Banach principle we can show the uniqueness of the solution of (2).

Let us define the transformation Φ: $\mathscr{W}_1 \times \mathscr{W}_2 \to \mathscr{W}_1 \times \mathscr{W}_2$ by

$$\Phi(z^1, z^2) = \begin{bmatrix} \Phi^1(z^1, z^2) \\ \Phi^2(z^1, z^2) \end{bmatrix}$$

$$= \left(\begin{bmatrix} \operatorname*{sup\,ess}_{u^2 \in M_\gamma} \mathrm{E}\{\int_s^\infty \exp[-\int_s^t (\alpha + u_r^2)\, dr]\, u_t^2 (z_t^2 + g_t)\, dt \mid \mathscr{F}_s\} \\ \operatorname*{sup\,ess}_{u^1 \in M_\beta} \mathrm{E}\{\int_s^\infty \exp[-\int_s^t (\alpha + u_r^1)\, dr]\, u_t^1 (z_t^1 - f_t)\, dt \mid \mathscr{F}_s\} \end{bmatrix} \right)_{s \geqslant 0}.$$

We obtain easily

$$|||\Phi(z^1, z^2) - \Phi(w^1, w^2)||| \leqslant \frac{\max\{\gamma, \beta\}}{\alpha + \max\{\gamma, \beta\}} |||z - w|||.$$

Consequently, Φ is a contraction, and thus we have established the theorem.

Remark 1. Applying Lemma 1 of [10] to the system (1) one can obtain the third equivalent system of penalized equations:

$$(3) \qquad \begin{aligned} b_s &= \gamma \mathrm{E}\{\int_s^\infty e^{-\alpha(t-s)} (c_t + g_t - b_t)^+\, dt \mid \mathscr{F}_s\}, \\ c_s &= \beta \mathrm{E}\{\int_s^\infty e^{-\alpha(t-s)} (b_t - f_t - c_t)^+\, dt \mid \mathscr{F}_s\}. \end{aligned}$$

COROLLARY 1. *There exists a unique, right continuous, $(\mathscr{F}_s)_{s \geqslant 0}$ adapted process $(a_s^{\beta,\gamma})_{s \geqslant 0}$ satisfying the equation*

$$(4) \qquad a_s^{\beta,\gamma} = \mathrm{E}\{\int_s^\infty e^{-\alpha(t-s)} [-\beta (a_t^{\beta,\gamma} - f_t)^+ + \gamma (a_t^{\beta,\gamma} - g_t)^-]\, dt \mid \mathscr{F}_s\}$$

P-a.s. for each $s \geqslant 0$ and, furthermore, $a_s^{\beta,\gamma} = b_s^{\beta,\gamma} - c_s^{\beta,\gamma}$.

Proof. Obviously, $(b_s - c_s)_{s \geqslant 0}$ from (3) satisfies (4). By Lemma 1 of [10], equation (4) is equivalent to

$$a_s^{\beta,\gamma} = \mathrm{E}\{\int_s^\infty e^{-(\alpha+\beta+\gamma)(t-s)} [-\beta (a_t - f_t)^+ + \gamma (a_t - g_t)^- + (\beta + \gamma) a_t]\, dt \mid \mathscr{F}_s\}$$

and the transformation

$$\mathscr{W} \ni (z_s)_{s \geqslant 0} \mapsto \Big(\mathrm{E}\{\int_s^\infty e^{-(\alpha+\beta+\gamma)(t-s)} [-\beta (z_t - f_t)^+ + \gamma (z_t - g_t)^- + (\beta + \gamma) z_t]\, dt \mid \mathscr{F}_s\}\Big)_{s \geqslant 0}$$

is a contraction with the parameter $(\beta + \gamma)/(\alpha + \beta + \gamma) < 1$.

From [10] we obtain without difficulty

COROLLARY 2. *The solution of equation* (4) *is of the form*

$$a_s^{\beta,\gamma} = \inf_{u^1\in M_\beta}\operatorname{ess}\sup_{u^2\in M_\gamma}\operatorname{ess} \mathrm{E}\Big\{\int_s^\infty \exp\Big[-\int_s^t (\alpha+u_r^1+u_r^2)\,dr\Big](u_t^1 f_t+u_t^2 g_t)\,dt \,\Big|\, \mathscr{F}_s\Big\}$$
$$= \sup_{u^2\in M_\gamma}\operatorname{ess}\inf_{u^1\in M_\beta}\operatorname{ess} \mathrm{E}\Big\{\int_s^\infty \exp\Big[-\int_s^t (\alpha+u_r^1+u_r^2)\,dr\Big](u_t^1 f_t+u_t^2 g_t)\,dt \,\Big|\, \mathscr{F}_s\Big\}$$

P-a.e. for each $s \geqslant 0$.

This corollary explains a probabilistic idea of the penalized method. The process a_s denotes the value of the game in which the first and the second players stop with densities

$$u_t^1 \exp\Big[-\int_s^t u_r^1\,dr\Big] \quad \text{and} \quad u_t^2 \exp\Big[-\int_s^t u_r^2\,dr\Big],$$

respectively.

3. Identification of the solution of the system (1) and a limit theorem. The system (1) is a counterpart of a penalized equation studied in [10] in connection with the optimal stopping. Moreover, it turns out that solutions are also α-supermartingales. Let us recall that a process $(z_s)_{s\geqslant 0}$ is an α-*supermartingale* if $(e^{-\alpha s} z_s)_{s\geqslant 0}$ is a supermartingale. Namely, we have

PROPOSITION 1. *The solutions of the system* (1), *i.e. processes* $(b_s^{\beta,\gamma})_{s\geqslant 0}$ *and* $(c_s^{\beta,\gamma})_{s\geqslant 0}$, *are right continuous* α-*supermartingales.*

This proposition follows easily from the form of equations (1) and (3).

Now we can prove the following important convergence result:

THEOREM 2. *If*

$$\sup_{\beta\geqslant 0,\gamma\geqslant 0}\operatorname{ess} b_s^{\beta,\gamma} \overset{\mathrm{df}}{=} \hat{b}_s \quad \textit{and} \quad \sup_{\beta\geqslant 0,\gamma\geqslant 0}\operatorname{ess} c_s^{\beta,\gamma} \overset{\mathrm{df}}{=} \hat{c}_s$$

are finite, then $(\hat{b}_s)_{s\geqslant 0}$ *and* $(\hat{c}_s)_{s\geqslant 0}$ *are right continuous* α-*supermartingales.*

Proof. Let us introduce some additional notation

$$(5)\qquad \begin{aligned} b^{1,\beta,\gamma} &= \Phi^1(0,0), \quad \ldots \quad b^{n+1,\beta,\gamma} = \Phi^1(b^{n,\beta,\gamma}, c^{n,\beta,\gamma}),\\ c^{1,\beta,\gamma} &= \Phi^2(0,0), \quad \ldots \quad c^{n+1,\beta,\gamma} = \Phi^2(b^{n,\beta,\gamma}, c^{n,\beta,\gamma}). \end{aligned}$$

It is well known that

$$\lim_{n\to\infty} b^{n,\beta,\gamma} = b^{\beta,\gamma} \quad \text{and} \quad \lim_{n\to\infty} c^{n,\beta,\gamma} = c^{\beta,\gamma}.$$

If $\gamma_1 \leqslant \gamma_2$ and $\beta_1 \leqslant \beta_2$, then

$$b_s^{1,\beta_1,\gamma_1} \leqslant b_s^{1,\beta_2,\gamma_2}, \quad c_s^{1,\beta_1,\gamma_1} \leqslant c_s^{1,\beta_2,\gamma_2}$$

and, inductively,

$$b_s^{n,\beta_1,\gamma_1} \leqslant b_s^{n,\beta_2,\gamma_2}, \quad c_s^{n,\beta_1,\gamma_1} \leqslant c_s^{n,\beta_2,\gamma_2}$$

P-a.e. for each $s \geqslant 0$ and $n \in N$. Taking the limits with $\beta \to \infty$ and $\gamma \to \infty$ in the first identities of (5) we obtain

$$b_s^{1,\infty,\infty} = \lim_{\beta,\gamma\to\infty} b_s^{1,\beta,\gamma} = \operatorname*{sup\,ess}_{u^2\in M_\infty} \mathrm{E}\{\int_s^\infty \exp[-\int_s^t (\alpha+u_r^2)\,dr]\,u_t^2 g_t\,dt \mid \mathscr{F}_s\},$$

$$c_s^{1,\infty,\infty} = \lim_{\beta,\gamma\to\infty} c_s^{1,\beta,\gamma} = \operatorname*{sup\,ess}_{u^1\in M_\infty} \mathrm{E}\{\int_s^\infty \exp[-\int_s^t (\alpha+u_r^1)\,dr]\,u_t^1 (-f_t)\,dt \mid \mathscr{F}_s\}$$

P-a.e. for each $s \geqslant 0$. Now, by [10] we notice easily that these processes are the α-Snell envelopes of the processes $(g_s)_{s\geqslant 0}$ and $(-f_s)_{s\geqslant 0}$. This means that $(b_s^{1,\infty,\infty})_{s\geqslant 0}$ and $(c_s^{1,\infty,\infty})_{s\geqslant 0}$ are the smallest right continuous α-supermartingales majorizing $(g_s)_{s\geqslant 0}$ and $(-f_s)_{s\geqslant 0}$, respectively. Analogously,

$$b_s^{n+1,\infty,\infty} = \operatorname*{sup\,ess}_{u^2\in M_\infty} \mathrm{E}\{\int_s^\infty \exp[-\int_s^t (\alpha+u_r^2)\,dr]\,u_t^2 (c_t^{n,\infty,\infty}+g_t)\,dt \mid \mathscr{F}_s\},$$

$$c_s^{n+1,\infty,\infty} = \operatorname*{sup\,ess}_{u^1\in M_\infty} \mathrm{E}\{\int_s^\infty \exp[-\int_s^t (\alpha+u_r^1)\,dr]\,u_t^1 (b_t^{n,\infty,\infty}-f_t)\,dt \mid \mathscr{F}_s\}$$

are the α-Snell envelopes of the processes $(c_s^{n,\infty,\infty}+g_s)_{s\geqslant 0}$ and $(b_s^{n,\infty,\infty}-f_s)_{s\geqslant 0}$. Now it is easy to see that $[(b_s^{n,\infty,\infty})_{s\geqslant 0}]_{n\in N}$ and $[(c_s^{n,\infty,\infty})_{s\geqslant 0}]_{n\in N}$ are increasing sequences of right continuous α-supermartingales, and $b_s^{n,\infty,\infty} \uparrow \hat{b}_s$, $c_s^{n,\infty,\infty} \uparrow \hat{c}_s$ P-a.e. for each s as $n \to \infty$. This completes our proof.

Remark 2. Theorem 2 was proved under the assumption that $(\hat{b}_s)_{s\geqslant 0}$ and $(\hat{c}_s)_{s\geqslant 0}$ are finite. This demand will be satisfied if we impose the following assumption similar to that introduced by Mokobodzki [8] in the case of Markov games:

ASSUMPTION. *There exist two right continuous positive α-supermartingales $(x_s)_{s\geqslant 0}$ and $(y_s)_{s\geqslant 0}$ such that for each s*

(6) $$g_s \leqslant x_s - y_s \leqslant f_s \quad P\text{-a.e.}$$

We find out immediately that $(\hat{b}_s)_{s\geqslant 0}$ and $(\hat{c}_s)_{s\geqslant 0}$ are finite since for each n we have $b_s^{n,\infty,\infty} \leqslant x_s$ and $c_s^{n,\infty,\infty} \leqslant y_s$ P-a.e. for each $s \geqslant 0$.

4. The main result. In this section we prove the main result of the paper.

THEOREM 3. *Assume that the right continuous, $(\mathscr{F}_s)_{s\geqslant 0}$ adapted, bounded processes $(f_s)_{s\geqslant 0}$ and $(g_s)_{s\geqslant 0}$ are such that*

(i) $f_s \geqslant g_s$ *P-a.e. for each* $s \geqslant 0$,

(ii) *the assumption* (6) *holds.*

Then $\bar{x} = \underline{x} = \mathrm{E}\hat{a}_0$, *where* $\hat{a} \overset{\mathrm{df}}{=} \hat{b} - \hat{c}$. *Moreover,*

$$\hat{a}_r = \operatorname*{inf\,ess}_{\tau\in\Lambda_r} \operatorname*{sup\,ess}_{\sigma\in\Lambda_r} \mathrm{E}\{\chi_{\tau<\sigma} e^{-\alpha(\tau-r)} f_\tau + \chi_{\sigma\leqslant\tau} e^{-\alpha(\sigma-r)} g_\sigma \mid \mathscr{F}_r\}$$

$$= \operatorname*{sup\,ess}_{\sigma\in\Lambda_r} \operatorname*{inf\,ess}_{\tau\in\Lambda_r} \mathrm{E}\{\chi_{\tau<\sigma} e^{-\alpha(\tau-r)} f_\tau + \chi_{\sigma\leqslant\tau} e^{-\alpha(\sigma-r)} g_\sigma \mid \mathscr{F}_r\}$$

P-a.e. for each $r \geqslant 0$.

Proof. The proof consists of three steps.

1. First we establish a new representation of $(a_s^{\beta,\gamma})_{s\geq 0}$. We need the following obvious lemma:

LEMMA 2. *If $(d_s)_{s\geq 0}$ is right continuous and for each $s \geq 0$*

$$d_s = \mathrm{E}\{\int_s^{\infty} e^{-\alpha(t-s)} h_t\, dt \mid \mathscr{F}_s\} \qquad P\text{-a.e.},$$

where $(h_s)_{s\geq 0}$ is a right continuous, $(\mathscr{F}_s)_{s\geq 0}$ adapted, bounded process, then

$$\varrho_r(s) \overset{\mathrm{df}}{=} d_s e^{-\alpha(s-r)} + \int_r^s e^{-\alpha(t-r)} h_t\, dt$$

for $s \geq r$ is a right continuous, bounded martingale.

From (4) and Lemma 2 we infer that for each $r \geq 0$

$$\varrho_r(s) = a_s^{\beta,\gamma} e^{-\alpha(s-r)} + \int_r^s e^{-\alpha(t-r)} [-\beta(a_t^{\beta,\gamma} - f_t)^+ + \gamma(a_t^{\beta,\gamma} - g_t)^-]\, dt$$

is an $(\mathscr{F}_s)_{s\geq r}$ right continuous, bounded martingale. Thus for $\tau, \sigma \in \Lambda_r$ we obtain the representation

$$\begin{aligned}(7)\qquad a_r^{\beta,\gamma} &= \varrho_r(r) = \mathrm{E}(\varrho_r(\tau\wedge\sigma) \mid \mathscr{F}_r)\\ &= \mathrm{E}\{a_{\tau\wedge\sigma}^{\beta,\gamma} e^{-\alpha(\tau\wedge\sigma - r)} \mid \mathscr{F}_r\} +\\ &\quad + \mathrm{E}\{\int_r^{\tau\wedge\sigma} e^{-\alpha(t-r)} [-\beta(a_t^{\beta,\gamma} - f_t)^+ + \gamma(a_t^{\beta,\gamma} - g_t)^-]\, dt \mid \mathscr{F}_r\}.\end{aligned}$$

2. Equation (7) can be transformed in the following way. First we have

$$\begin{aligned}a_r^{\beta,\gamma} &\geq \mathrm{E}\{\int_r^{\tau\wedge\sigma} e^{-\alpha(t-r)} [-\beta(a_t^{\beta,\gamma} - f_t)^+]\, dt \mid \mathscr{F}_r\} +\\ &\quad + \mathrm{E}\{\chi_{\sigma\leq\tau} a_\sigma^{\beta,\gamma} e^{-\alpha(\sigma-r)} \mid \mathscr{F}_r\} + \mathrm{E}\{\chi_{\tau<\sigma} a_\tau^{\beta,\gamma} e^{-\alpha(\tau-r)} \mid \mathscr{F}_r\}\\ &\geq \mathrm{E}\{\int_r^{\tau\wedge\sigma} e^{-\alpha(t-r)} [-\beta(a_t^{\beta,\gamma} - f_t)^+\, dt] \mid \mathscr{F}_r\} +\\ &\quad + \mathrm{E}\{\chi_{\sigma\leq\tau} e^{-\alpha(\sigma-r)} [g_\sigma - (a_\sigma^{\beta,\gamma} - g_\sigma)^-] \mid \mathscr{F}_r\} + \mathrm{E}\{\chi_{\tau<\sigma} a_\tau^{\beta,\gamma} e^{-\alpha(\tau-r)} \mid \mathscr{F}_r\}\end{aligned}$$

with the identity for $\sigma = \inf\{t \geq r\colon a_t^{\beta,\gamma} \leq g_t\}$. Then we obtain

$$\begin{aligned}a_r^{\beta,\gamma} &= \operatorname*{sup\,ess}_{\sigma\in\Lambda_r} \mathrm{E}\{\int_r^{\tau\wedge\sigma} e^{-\alpha(t-r)} [-\beta(a_t^{\beta,\gamma} - f_t)^+]\, dt +\\ &\quad + \chi_{\sigma\leq\tau} e^{-\alpha(\sigma-r)} [g_\sigma - (a_\sigma^{\beta,\gamma} - g_\sigma)^-] + \chi_{\tau<\sigma} a_\tau^{\beta,\gamma} e^{-\alpha(\tau-r)} \mid \mathscr{F}_r\}\end{aligned}$$

P-a.e. for each $r \geq 0$.

Similarly,

$$(8)\quad a_r^{\beta,\gamma} \leqslant \operatorname*{sup\,ess}_{\sigma\in\Lambda_r} \mathrm{E}\{\chi_{\sigma\leqslant\tau} e^{-\alpha(\sigma-r)}[g_\sigma-(a_\sigma^{\beta,\gamma}-g_\sigma)^-]+ \\ +\chi_{\tau<\sigma} e^{-\alpha(\tau-r)}[f_\tau+(a_\tau^{\beta,\gamma}-f_\tau)^+]\,|\,\mathscr{F}_r\},$$

and since for $\tau = \inf\{t \geqslant r\colon a_t^{\beta,\gamma} \geqslant f_t\}$ we have the equality in (8), we finally obtain

$$a_r^{\beta,\gamma} = \operatorname*{inf\,ess}_{\tau\in\Lambda_r} \operatorname*{sup\,ess}_{\sigma\in\Lambda_r} \mathrm{E}\{\chi_{\sigma\leqslant\tau} e^{-\alpha(\sigma-r)} g_\sigma + \chi_{\tau<\sigma} e^{-\alpha(\tau-r)} f_\tau - \\ -\chi_{\sigma\leqslant\tau} e^{-\alpha(\sigma-r)}(a_\sigma^{\beta,\gamma}-g_\sigma)^- + \chi_{\tau<\sigma} e^{-\alpha(\tau-r)}(a_\tau^{\beta,\gamma}-f_\tau)^+\,|\,\mathscr{F}_r\}.$$

In a similar way, changing the role of inf and sup operations, we get

$$a_r^{\beta,\gamma} = \operatorname*{sup\,ess}_{\sigma\in\Lambda_r} \operatorname*{inf\,ess}_{\tau\in\Lambda_r} \mathrm{E}\{\chi_{\sigma\leqslant\tau} e^{-\alpha(\sigma-r)} g_\sigma + \chi_{\tau<\sigma} e^{-\alpha(\tau-r)} f_\tau - \\ -\chi_{\sigma\leqslant\tau} e^{-\alpha(\sigma-r)}(a_\sigma^{\beta,\gamma}-g_\sigma)^- + \chi_{\tau<\sigma} e^{-\alpha(\tau-r)}(a_\tau^{\beta,\gamma}-f_\tau)^+\,|\,\mathscr{F}_r\}.$$

3. Our aim is now to estimate the processes $((a_s^{\beta,\gamma}-g_s)^-)_{s\geqslant 0}$ and $((a_s^{\beta,\gamma}-f_s)^+)_{s\geqslant 0}$ as $\beta, \gamma \to +\infty$. For this purpose we need the following lemma:

LEMMA 3. *For each $s \geqslant 0$ we have $g_s \leqslant \hat{a}_s \leqslant f_s$ P-a.e.*

Proof. From the system (3) we can obtain

$$(9)\quad \begin{aligned} b_s^{\infty,\gamma} &= \gamma \mathrm{E}\{\int_s^\infty e^{-\alpha(t-s)}(g_t-a_t^{\infty,\gamma})^+\,dt\,|\,\mathscr{F}_s\},\\ c_s^{\beta,\infty} &= \beta \mathrm{E}\{\int_s^\infty e^{-\alpha(t-s)}(a_t^{\beta,\infty}-f_t)^+\,dt\,|\,\mathscr{F}_s\}\end{aligned}$$

P-a.e. for each $s \geqslant 0$. Let us write

$$K^{\varepsilon,\gamma} = \{(t,\omega)\colon g_t(\omega)-a_t^{\infty,\gamma}(\omega) \geqslant \varepsilon\},\quad K^\varepsilon = \{(t,\omega)\colon g_t(\omega)-\hat{a}_t(\omega) \geqslant \varepsilon\}.$$

Thus $K^{\varepsilon,\gamma} \supset K^\varepsilon$ (see Section 2, Corollary 2) and

$$b_s^{\infty,\gamma} \geqslant \gamma \mathrm{E}\{\int_s^\infty e^{-\alpha(t-s)}\varepsilon\chi_{K^{\varepsilon,\gamma}}\,dt\,|\,\mathscr{F}_s\} \geqslant \gamma\varepsilon \mathrm{E}\{\int_s^\infty e^{-\alpha(t-s)}\chi_{K^\varepsilon}\,dt\,|\,\mathscr{F}_s\}.$$

Consequently,

$$\frac{b_s^{\infty,\gamma}}{\gamma\varepsilon} \geqslant \mathrm{E}\{\int_s^\infty e^{-\alpha(t-s)}\chi_{K^\varepsilon}\,dt\,|\,\mathscr{F}_s\} \geqslant 0,$$

and as $\gamma \to \infty$ we infer that K^ε has $dt\otimes dP$ measure zero. Since $(g_s)_{s\geqslant 0}$ and $(\hat{a}_s)_{s\geqslant 0}$ are the right continuous processes and ε is arbitrary, we have $\hat{a}_s \geqslant g_s$ P-a.e. for each $s \geqslant 0$. Similarly, from the second equation of (9) we obtain $\hat{a}_s \leqslant f_s$ P-a.e. for each $s \geqslant 0$.

Summarizing the results of steps 2 and 3 of our proof we establish the required assertion of Theorem 3.

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