PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 31, Fasc. 1 (2011), pp. 79–98

A SHARP CORRELATION INEQUALITY WITH APPLICATION TO ALMOST SURE LOCAL LIMIT THEOREM

BY

MICHEL WEBER* (STRASBOURG)

Abstract. We prove a new sharp correlation inequality for sums of i.i.d. square integrable lattice distributed random variables. We also apply it to establish an almost sure version of the local limit theorem for i.i.d. square integrable random variables taking values in an arbitrary lattice. This extends a recent similar result jointly obtained with Giuliano-Antonini under a slightly stronger absolute moment assumption (of order $2+u$ with $u > 0$). The approach used to treat the case $u > 0$ breaks down when $u = 0$. MacDonald's concept of the Bernoulli part of a random variable is used in a crucial way to remedy this.

2000 AMS Mathematics Subject Classification: Primary: 60F15, 60G50; Secondary: 60F05.

Key words and phrases: Correlation inequality, i.i.d. random variables, lattice distributed, Bernoulli part, square integrable, local limit theorem, almost sure version.

1. INTRODUCTION

Throughout this work, we are concerned with i.i.d. square integrable random variables having lattice distribution. Let v_0 and $D > 0$ be some reals and let $\mathcal{L}(v_0, D)$ be the lattice defined by the sequence $v_k = v_0 + Dk$, $k \in \mathbb{Z}$. Consider a random variable X such that $\mathbb{P}\{X \in \mathcal{L}(v_0, D)\} = 1$. We assume that D (the span of X) is maximal, i.e. there is no integer multiple D' of D for which $\mathbb{P}\{X \in \mathcal{L}(v_0, D')\} = 1$. We further assume

$$\mathbb{E}X \ \text{ and } \ \mathbb{E}X^2 \ \text{ are finite.} \tag{1.1}$$

Let $\mu = \mathbb{E}X$ and $\sigma^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2$, which we assume to be positive (otherwise X is degenerated). Under these assumptions, the local limit theorem holds. Let $\{X_k, k \geqslant 1\}$ be independent copies of X, and consider their partial sums $S_n =$

---

* The author thanks an anonymous referee for a very careful reading, and for having suggested to use a finer local limit theorem for Rademacher sums. This allowed to slightly improve upon Theorem 1.

$X_1 + \ldots + X_n$, $n \geqslant 1$. To be precise, we have ([5], §43)

$$\lim_{n\to\infty} \sup_{N=v_0n+Dk} \left| \sqrt{n}\mathbb{P}\{S_n = N\} - \frac{D}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(N-n\mu)^2}{2n\sigma^2}\right)\right| = 0. \tag{1.2}$$

Now, let $\kappa_n \in \mathcal{L}(nv_0, D)$, $n = 1, 2, \ldots$, be a sequence of reals satisfying

$$\lim_{n\to\infty} \frac{\kappa_n - n\mu}{\sqrt{n}} = \kappa. \tag{1.3}$$

The central result of the paper is the following correlation inequality which we believe to be hardly improvable.

THEOREM 1.1. *Assume that*

$$\mathbb{P}\{X = k\} \wedge \mathbb{P}\{X = k+1\} > 0 \quad \textit{for some } k \in \mathbb{Z}. \tag{1.4}$$

Then there exists a constant C *depending on the sequence* $\{\kappa_n, n \geqslant 1\}$ *such that for all* $1 \leqslant m < n$

$$\sqrt{nm}\,|\mathbb{P}\{S_n = \kappa_n, S_m = \kappa_m\} - \mathbb{P}\{S_n = \kappa_n\}\mathbb{P}\{S_m = \kappa_m\}| \leqslant C\left\{\frac{1}{\sqrt{n/m}-1} + \frac{n^{1/2}}{(n-m)^{3/2}}\right\}.$$

COROLLARY 1.1. *Let* $0 < c < 1$. *Under the assumption* (1.4), *there exists a constant* C_c *such that for all* $1 \leqslant m \leqslant cn$

$$\sqrt{nm}\,|\mathbb{P}\{S_n = \kappa_n, S_m = \kappa_m\} - \mathbb{P}\{S_n = \kappa_n\}\mathbb{P}\{S_m = \kappa_m\}| \leqslant C_c\sqrt{m/n}.$$

REMARK 1.1. Condition (1.4) seems to be somehow artificial. It is, for instance, clearly not satisfied if $\mathbb{P}\{X \in \mathcal{N}\} = 1$, where $\mathcal{N} = \{\nu_j, j \geqslant 1\}$ is an increasing sequence of integers such that $\nu_{j+1} - \nu_j > 1$ for all j. This already defines a large class of examples. However, condition (1.4) is natural in our setting. By the local limit theorem (1.2), under condition (1.3),

$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}\{S_n = \ell_n\} = \frac{D}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\kappa^2}{2\sigma^2}\right) \quad (\ell_n \equiv \kappa_n \text{ or } \ell_n \equiv \kappa_n + 1).$$

Then, for some $n_\kappa < \infty$, $\mathbb{P}\{S_n = \kappa_n\} \wedge \mathbb{P}\{S_n = \kappa_n + 1\} > 0$ if $n \geqslant n_\kappa$. Changing X for $X' = S_{n_\kappa}$, we see that X' satisfies (1.4).

When X has a stronger integrability property, to be precise, if $\mathbb{E}|X|^{2+\varepsilon} < \infty$ for some positive ε, we proved in [4] (Proposition 6) a similar result:

$$\sqrt{nm}\,|\mathbb{P}\{S_n = \kappa_n, S_m = \kappa_m\} - \mathbb{P}\{S_n = \kappa_n\}\mathbb{P}\{S_m = \kappa_m\}| \leqslant C\left(\frac{1}{\sqrt{n/m}-1} + \sqrt{\frac{n}{n-m}}\,\frac{1}{(n-m)^\alpha}\right) \tag{1.5}$$

with (here and below) $\alpha = \varepsilon/2$. Condition (1.4) was not needed. The second inequality of Theorem 1.1 follows in that case directly from (1.5). The proof uses crucially a local limit theorem with remainder term.

THEOREM 1.2 ([6], Theorem 4.5.3). *Let F denote the distribution function of X. In order that the property*

$$\text{(1.6)} \qquad \sup_{N=an+dk}\left|\sqrt{n}\mathbb{P}\{S_n = N\} - \frac{d}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(N-n\mu)^2}{2n\sigma^2}\right)\right| = \mathcal{O}\big(n^{-\alpha}\big),\ 0<\alpha<1/2,$$

holds, it is necessary and sufficient that the following conditions be satisfied:

(i) $d = D$;

(ii) *as* $u \to \infty$, $\int_{|x|\geqslant u} x^2 F(dx) = \mathcal{O}(u^{-2\alpha})$.

When $\varepsilon = 0$, this can obviously no longer be applied, and another approach has to be implemented. Notice that even when $\varepsilon > 0$, our result is stronger than inequality (1.5).

An application of Theorem 1.1 is given in Section 4. We obtain an almost sure local limit theorem for i.i.d. square integrable lattice distributed random variables taking values in arbitrary lattices. By proceeding as in [3] or [1], Theorem 1.1 can be also used to prove much more, notably very general versions of the almost sure local limit theorem where the partial sums are replaced by nonlinear functional, typical examples are maxima of partial sums. This will be investigated elsewhere.

2. PRELIMINARY RESULTS

Here we follow an important approach due to MacDonald ([8], see also [9]). Let $0 < \vartheta < 1$ be fixed. Put

$$f(k) = \mathbb{P}\{X = v_k\}, \qquad k \in \mathbb{Z}.$$

We assume that there exists a sequence $\underline{\tau} = \{\tau_k, k \in \mathbb{Z}\}$ of non-negative reals such that

$$\tau_{k-1} + \tau_k \leqslant 2f(k),\ \forall k \in \mathbb{Z}, \qquad \sum_{k\in\mathbb{Z}} \tau_k = \vartheta.$$

If we choose $\vartheta = \vartheta_X = \sum_{k\in\mathbb{Z}} f(k) \wedge f(k+1)$, then this is realized with $\tau_k = f(k) \wedge f(k+1)$. Notice that $\vartheta_X < 1$. Indeed, let k_0 be some integer such that $f(k_0) > 0$. Then

$$\sum_{k=k_0}^{\infty} f(k) \wedge f(k+1) \leqslant \sum_{k=k_0}^{\infty} f(k+1) = \sum_{k=k_0+1}^{\infty} f(k)$$

and, consequently,

$$\vartheta_X \leqslant \sum_{k<k_0} f(k) + \sum_{k=k_0+1}^{\infty} f(k) < 1.$$

Notice also that $\vartheta_X > 0$. This follows from assumption (1.4), and is further necessary in order to make this approach efficient. MacDonald's construction applies to the slightly more general case we consider, and is even easier to present. We define a pair of random variables (V,ε) as follows. For $k \in \mathbb{Z}$,

$$\begin{aligned} &\mathbb{P}\{(V,\varepsilon) = (v_k,1)\} = \tau_k, \\ &\mathbb{P}\{(V,\varepsilon) = (v_k,0)\} = f(k) - \frac{\tau_{k-1}+\tau_k}{2}. \end{aligned} \tag{2.1}$$

This is well defined by assumption. Observe that

$$\sum_{k\in\mathbb{Z}} [\mathbb{P}\{(V,\varepsilon) = (v_k,1)\} + \mathbb{P}\{(V,\varepsilon) = (v_k,0)\}] = \sum_{k\in\mathbb{Z}} f(k) + \frac{1}{2}\sum_{k\in\mathbb{Z}}[\tau_k - \tau_{k-1}] = 1.$$

LEMMA 2.1. *We have for* $k \in \mathbb{Z}$

$$\mathbb{P}\{V = v_k\} = f(k) + \frac{\tau_k - \tau_{k-1}}{2},$$

and $\mathbb{P}\{\varepsilon = 1\} = 1 - \mathbb{P}\{\varepsilon = 0\} = \vartheta$.

Proof. Plainly,

$$\begin{aligned} \mathbb{P}\{V = v_k\} &= \mathbb{P}\{(V,\varepsilon) = (v_k,1)\} + \mathbb{P}\{(V,\varepsilon) = (v_k,0)\} \\ &= f(k) + \frac{1}{2}[\tau_k - \tau_{k-1}]. \end{aligned}$$

Further

$$\mathbb{P}\{\varepsilon = 1\} = \sum_{k\in\mathbb{Z}} \mathbb{P}\{(V,\varepsilon) = (v_k,1)\} = \sum_{k\in\mathbb{Z}} \tau_k = \vartheta. \quad \blacksquare$$

LEMMA 2.2. *Assume that* L *is a Bernoulli random variable* ($\mathbb{P}\{L = 0\} = \mathbb{P}\{L = 1\} = 1/2$) *which is independent of* (V,ε), *and put* $Z = V + \varepsilon DL$. *We have* $Z \stackrel{\mathcal{D}}{=} X$.

Proof. Indeed,

$$\begin{aligned} \mathbb{P}\{Z = v_k\} &= \mathbb{P}\{V + \varepsilon DL = v_k, \varepsilon = 1\} + \mathbb{P}\{V + \varepsilon DL = v_k, \varepsilon = 0\} \\ &= \frac{\mathbb{P}\{V = v_{k-1}, \varepsilon = 1\} + \mathbb{P}\{V = v_k, \varepsilon = 1\}}{2} + \mathbb{P}\{V = v_k, \varepsilon = 0\} \\ &= \frac{\tau_{k-1}+\tau_k}{2} + f(k) - \frac{\tau_{k-1}+\tau_k}{2} \\ &= f(k). \quad \blacksquare \end{aligned}$$

Now let $\{X_j, j \geqslant 1\}$ be independent copies of X. According to the previous construction, we may associate with them a sequence $\{(V_j, \varepsilon_j, L_j), j \geqslant 1\}$ of independent copies of (V, ε, L) such that

$$\{V_j + \varepsilon_j D L_j, j \geqslant 1\} \stackrel{\mathcal{D}}{=} \{X_j, j \geqslant 1\}.$$

Further $\{(V_j, \varepsilon_j), j \geqslant 1\}$ and $\{L_j, j \geqslant 1\}$ are independent sequences. Moreover, $\{L_j, j \geqslant 1\}$ is a sequence of independent Bernoulli random variables. Set

$$(2.2) \qquad S_n = \sum_{j=1}^{n} X_j, \quad W_n = \sum_{j=1}^{n} V_j, \quad M_n = \sum_{j=1}^{n} \varepsilon_j L_j, \quad B_n = \sum_{j=1}^{n} \varepsilon_j.$$

We notice that M_n is a sum of exactly B_n Bernoulli random variables. The following lemma is now immediate.

LEMMA 2.3. *We have the representation*

$$\{S_n, n \geqslant 1\} \stackrel{\mathcal{D}}{=} \{W_n + D M_n, n \geqslant 1\}.$$

Moreover, $M_n \stackrel{\mathcal{D}}{=} \sum_{j=1}^{B_n} L_j$.

We need an extra lemma.

LEMMA 2.4. *Let* $0 < \theta \leqslant \vartheta$. *For any positive integer* n, *we have*

$$\mathbb{P}\{B_n \leqslant \theta n\} \leqslant \left(\frac{1-\vartheta}{1-\theta}\right)^{n(1-\theta)} \left(\frac{\vartheta}{\theta}\right)^{n\theta}.$$

Let $1 - \vartheta < \rho < 1$. *There exists* $0 < \theta < \vartheta$, $\theta = \theta(\rho, \vartheta)$, *such that for any positive integer* n

$$\mathbb{P}\{B_n \leqslant \theta n\} \leqslant \rho^n.$$

The proof is a simple exercise in large deviation bounds of Cramér–Chernoff type, so we omit it.

We choose

$$\rho = 1 - (\vartheta/2),$$

and let $0 < \theta < \vartheta$ be such that, by the preceding lemma, $\mathbb{P}\{B_n \leqslant \theta n\} \leqslant \rho^n$ and $\mathbb{P}\{B_n - B_m \leqslant \theta(n-m)\} \leqslant \rho^{n-m}$ for all integers $n > m \geqslant 1$.

3. PROOF OF THEOREM 1.1

Let us put

$$(3.1) \qquad Y_n = \sqrt{n}(\mathbf{1}_{\{S_n = \kappa_n\}} - \mathbb{P}\{S_n = \kappa_n\}).$$

We have to establish that there exists a constant C such that for all $1 \leqslant m < n$

$$|\mathbb{E}Y_nY_m| \leqslant C\Big\{\frac{1}{\sqrt{n/m}-1} + \frac{n^{1/2}}{(n-m)^{3/2}}\Big\} \tag{3.2}$$

and, given $0 < c < 1$, that there exists a constant C_c such that for all $1 \leqslant m \leqslant cn$

$$|\mathbb{E}Y_nY_m| \leqslant C_c\sqrt{\frac{m}{n}}. \tag{3.3}$$

We denote by $\mathbb{E}_{(V,\varepsilon)}$, $\mathbb{P}_{(V,\varepsilon)}$ (resp. $\mathbb{E}_L$, $\mathbb{P}_L$) the expectation and probability symbols relatively to the σ-algebra generated by the sequence $\{(V_j,\varepsilon_j), j=1,\dots,n\}$ (resp. $\{L_j, j=1,\dots,n\}$). We know that these algebras are independent. Let $n > m \geqslant 1$. Then

$$\mathbb{E}Y_nY_m = \sqrt{m}\mathbb{P}\{S_m=\kappa_m\}\,\sqrt{n}(\mathbb{P}\{S_{n-m}=\kappa_n-\kappa_m\} - \mathbb{P}\{S_n=\kappa_n\}). \tag{3.4}$$

Further, when $n = m$, by (1.2) we have

$$\mathbb{E}Y_n^2 = n\mathbb{P}\{S_n=\kappa_n\}(1-\mathbb{P}\{S_n=\kappa_n\}) = \mathcal{O}(\sqrt{n}). \tag{3.5}$$

Let

$$\begin{aligned} A &:= \sqrt{n}(\mathbb{P}\{S_n-S_m=\kappa_n-\kappa_m\} - \mathbb{P}\{S_n=\kappa_n\}) \\ &= \sqrt{n}\mathbb{E}(\mathbf{1}_{\{B_n\leqslant n\theta\}} + \mathbf{1}_{\{B_n>n\theta\}})(\mathbf{1}_{\{S_n-S_m=\kappa_n-\kappa_m\}} - \mathbf{1}_{\{S_n=\kappa_n\}}). \end{aligned}$$

By Lemma 2.4 we obtain

$$\sqrt{n}\,\mathbb{E}\mathbf{1}_{\{B_n\leqslant n\theta\}}|\mathbf{1}_{\{S_n-S_m=\kappa_n-\kappa_m\}} - \mathbf{1}_{\{S_n=\kappa_n\}}| \leqslant \sqrt{n}\,\rho^n.$$

Thus

$$|A - \sqrt{n}\,\mathbb{E}\mathbf{1}_{\{B_n>n\theta\}}(\mathbf{1}_{\{S_n-S_m=\kappa_n-\kappa_m\}} - \mathbf{1}_{\{S_n=\kappa_n\}})| \leqslant \sqrt{n}\,\rho^n. \tag{3.6}$$

In view of Lemma 2.3 we can write

$$\begin{aligned} &\sqrt{n}\,\mathbb{E}\mathbf{1}_{\{B_n>n\theta\}}(\mathbf{1}_{\{S_n-S_m=\kappa_n-\kappa_m\}} - \mathbf{1}_{\{S_n=\kappa_n\}}) \\ &\quad= \sqrt{n}\mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta\}}\big(\mathbb{P}_L\big\{D\sum_{j=m+1}^{n}\varepsilon_jL_j = \kappa_n-\kappa_m-(W_n-W_m)\big\} \\ &\qquad\qquad - \mathbb{P}_L\big\{D\sum_{j=1}^{n}\varepsilon_jL_j = \kappa_n - W_n\big\}\big). \end{aligned} \tag{3.7}$$

Observe that if $B_n = B_m$, then $\sum_{j=1}^n \varepsilon_jL_j = \sum_{j=1}^m \varepsilon_jL_j$. Thus

$$\big\{D\sum_{j=m+1}^{n}\varepsilon_jL_j = \kappa_n-\kappa_m-(W_n-W_m)\big\} = \big\{W_n-W_m=\kappa_n-\kappa_m\big\}.$$

Consequently, (3.7) may be written as

$$
\begin{aligned}
(3.8)\quad & \sqrt{n}\Big\{\mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta,B_n=B_m\}}\big(\mathbf{1}_{\{W_n-W_m=\kappa_n-\kappa_m\}} \\
&\qquad\qquad\qquad - \mathbb{P}_L\big\{D\sum_{j=1}^{n}\varepsilon_j L_j=\kappa_n-W_n\big\}\big)\Big\} \\
&+\sqrt{n}\Big\{\mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta,B_n>B_m\}}\big[\mathbb{P}_L\big\{D\sum_{j=m+1}^{n}\varepsilon_j L_j=\kappa_n-\kappa_m- \\
&\qquad\qquad (W_n-W_m)\big\}-\mathbb{P}_L\big\{D\sum_{j=1}^{n}\varepsilon_j L_j=\kappa_n-W_n\big\}\big]\Big\} \\
&:= A'+A''.
\end{aligned}
$$

We bound A' as follows:

$$
(3.9)\qquad |A'|\leqslant \sqrt{n}\mathbb{P}\{B_n=B_m\}=\sqrt{n}\,2^{-(n-m)}.
$$

As to A'', we have $\sum_{j=1}^{n}\varepsilon_j L_j \overset{\mathcal{D}}{=} \sum_{j=1}^{B_n} L_j$, $\sum_{j=m+1}^{n}\varepsilon_j L_j \overset{\mathcal{D}}{=} \sum_{j=B_m+1}^{B_n} L_j$. We now need a local limit theorem for Bernoulli sums. By applying Theorem 13 in Chapter 7 of [10], we obtain

$$
(3.10)\quad \sup_z\left|\sqrt{N}\,\mathbb{P}\big\{\sum_{j=1}^{N}L_j=z\big\}-\frac{2}{\sqrt{2\pi}}\exp\Big(-\frac{\big(z-(N/2)\big)^2}{N/2}\Big)\right|=o\Big(\frac{1}{N}\Big).
$$

Therefore

$$
\begin{aligned}
&\bigg|\mathbb{P}_L\big\{D\sum_{j=1}^{B_n}L_j=\kappa_n-W_n\big\} \\
&\qquad -\frac{2\exp\big\{\big[-\big(\kappa_n-W_n-(B_n/2)\big)^2\big]/D^2(B_n/2)\big\}}{\sqrt{2\pi B_n}}\bigg| = o\Big(\frac{1}{B_n^{3/2}}\Big).
\end{aligned}
$$

Moreover, on the set $\{B_n>B_m\}$ we have

$$
\begin{aligned}
&\bigg|\mathbb{P}_L\big\{D\sum_{j=1}^{B_n-B_m}L_j=\kappa_n-\kappa_m-(W_n-W_m)\big\}- \\
&\frac{2\exp\big\{\big[-\big(\kappa_n-\kappa_m-(W_n-W_m)-(B_n-B_m)/2\big)^2\big]/[D^2(B_n-B_m)/2]\big\}}{\sqrt{2\pi(B_n-B_m)}}\bigg| \\
&\qquad\qquad = o\Big(\frac{1}{(B_n-B_m)^{3/2}}\Big).
\end{aligned}
$$

It follows that

(3.11) $|A''| \leqslant$

$$\leqslant \sqrt{n}\,\Big|\mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta,B_n>B_m\}}\Big\{2\exp\Big(-\frac{\big(\kappa_n-W_n-(B_n/2)\big)^2}{D^2(B_n/2)}\Big)/\sqrt{2\pi B_n}$$
$$-2\exp\Big(-\frac{\big(\kappa_n-\kappa_m-(W_n-W_m)-(B_n-B_m)/2\big)^2}{D^2(B_n-B_m)/2}\Big)/\sqrt{2\pi(B_n-B_m)}\Big\}\Big|$$
$$+C_0\sqrt{n}\,\mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta,B_n>B_m\}}\Big\{\frac{1}{B_n^{3/2}}+\Big(\frac{1}{(B_n-B_m)^{3/2}}\Big)\Big\}$$
$$:= A_1''+C_0A_2''.$$

Moreover, the constant C_0 comes from the Landau symbol o in (3.10). The second term is easily estimated. Indeed,

$$A_2''=\sqrt{n}\,\mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta,B_n>B_m\}}\Big(\frac{1}{B_n^{3/2}}+\frac{1}{(B_n-B_m)^{3/2}}\Big) \tag{3.12}$$
$$\leqslant 2\sqrt{n}\,\mathbb{P}\{B_n-B_m\leqslant(n-m)\theta\}$$
$$+\sqrt{n}\,\mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n-B_m>(n-m)\theta\}}\Big(\frac{1}{(n\theta)^{3/2}}+\frac{1}{(B_n-B_m)^{3/2}}\Big)$$
$$\leqslant C\,\sqrt{n}\Big\{\rho^{n-m}+\frac{1}{(n\theta)^{3/2}}+\frac{1}{\big((n-m)\theta\big)^{3/2}}\Big\}.$$

We now estimate A_1'', which we bound as follows:

(3.13)

$$A_1''\leqslant C\,\mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta,B_n>B_m\}}\Big\{\Big(\frac{n}{B_n}\Big)^{1/2}\Big[\sqrt{\frac{B_n}{B_n-B_m}}-1\Big]$$
$$\times\exp\Big(-\frac{\big(\kappa_n-\kappa_m-(W_n-W_m)-(B_n-B_m)/2\big)^2}{D^2(B_n-B_m)/2}\Big)\Big\}$$
$$+C\,\mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta,B_n>B_m\}}\Big\{\Big(\frac{n}{B_n}\Big)^{1/2}\Big|\exp\Big(-\frac{\big(\kappa_n-W_n-(B_n/2)\big)^2}{D^2(B_n/2)}\Big)$$
$$-\exp\Big(-\frac{\big(\kappa_n-\kappa_m-(W_n-W_m)-(B_n-B_m)/2\big)^2}{D^2(B_n-B_m)/2}\Big)\Big|\Big\}$$
$$\leqslant C_\theta\Big\{\,\mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta,B_n>B_m\}}\Big[\sqrt{\frac{B_n}{B_n-B_m}}-1\Big]$$
$$+\mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta,B_n>B_m\}}\Big|\exp\Big(-\frac{\big(\kappa_n-W_n-(B_n/2)\big)^2}{D^2(B_n/2)}\Big)$$
$$-\exp\Big(-\frac{\big(\kappa_n-\kappa_m-(W_n-W_m)-(B_n-B_m)/2\big)^2}{D^2(B_n-B_m)/2}\Big)\Big|\Big\}$$
$$:= C_\theta\big\{A_{11}''+A_{12}''\big\}.$$

On the one hand, on the set $\{B_n > B_m\}$ we have

$$\sqrt{\frac{B_n}{B_n - B_m}} - 1 = \frac{\sqrt{B_n} - \sqrt{B_n - B_m}}{\sqrt{B_n - B_m}} \leqslant \frac{\sqrt{B_m}}{\sqrt{B_n - B_m}}.$$

Thus

$$\begin{aligned}
(3.14)\quad & A''_{11} = \\
&= \mathbb{E}_{(V,\varepsilon)} \mathbf{1}_{\{B_n > n\theta, B_n > B_m\}} \left[\sqrt{\frac{B_n}{B_n - B_m}} - 1\right] \leqslant \mathbb{E}_{(V,\varepsilon)} \mathbf{1}_{\{B_n > B_m\}} \frac{\sqrt{B_m}}{\sqrt{B_n - B_m}} \\
&= \mathbb{E}_{(V,\varepsilon)} (\mathbf{1}_{\{B_n - B_m \leqslant (n-m)\theta\}} + \mathbf{1}_{\{B_n - B_m > (n-m)\theta\}}) \mathbf{1}_{\{B_n > B_m\}} \frac{\sqrt{B_m}}{\sqrt{B_n - B_m}} \\
&\leqslant C_\theta \left\{\rho^{n-m} + \frac{1}{\sqrt{n/m - 1}}\right\} \leqslant C_\theta \left\{\rho^{n-m} + \frac{1}{\sqrt{n/m} - 1}\right\},
\end{aligned}$$

since $\sqrt{x} - \sqrt{y} \leqslant \sqrt{x - y}$ if $x \geqslant y \geqslant 0$.

Now we turn to A''_{12}. Put $\kappa'_n = \kappa_n - W_n - (B_n/2)$. Then

$$\begin{aligned}
A''_{12} = \mathbb{E}_{(V,\varepsilon)} \mathbf{1}_{\{B_n > n\theta, B_n > B_m\}} \bigg| & \exp\left(-\frac{\big(\kappa_n - W_n - (B_n/2)\big)^2}{D^2(B_n/2)}\right) \\
& - \exp\left(-\frac{\big(\kappa_n - \kappa_m - (W_n - W_m) - (B_n - B_m)/2\big)^2}{D^2(B_n - B_m)/2}\right)\bigg| \\
= \mathbb{E}_{(V,\varepsilon)} \mathbf{1}_{\{B_n > n\theta, B_n > B_m\}} & \left| \exp\left(-\frac{{\kappa'_n}^2}{D^2(B_n/2)}\right) - \exp\left(-\frac{(\kappa'_n - \kappa'_m)^2}{D^2(B_n - B_m)/2}\right)\right|.
\end{aligned}$$

We have

$$\begin{aligned}
(3.15)\quad \mathbb{E}_{(V,\varepsilon)} \bigg\{ & \mathbf{1}_{\{B_n > n\theta, 0 < B_n - B_m \leqslant \theta(n-m)\}} \\
& \times \left| \exp\left(-\frac{{\kappa'_n}^2}{D^2(B_n/2)}\right) - \exp\left(-\frac{(\kappa'_n - \kappa'_m)^2}{D^2(B_n - B_m)/2}\right)\right| \bigg\} \\
& \leqslant 2\mathbb{P}\{B_n - B_m \leqslant \theta(n-m)\} \leqslant 2\rho^{n-m}.
\end{aligned}$$

It remains to bound

$$\mathbb{E}_{(V,\varepsilon)} \mathbf{1}_{\{B_n > n\theta, B_n - B_m > \theta(n-m)\}} \left| \exp\left(-\frac{{\kappa'_n}^2}{D^2(B_n/2)}\right) - \exp\left(-\frac{(\kappa'_n - \kappa'_m)^2}{D^2(B_n - B_m)/2}\right)\right|.$$

Let $b_n = \kappa'_n/\sqrt{B_n}$. Using the inequality $|e^{-u} - e^{-v}| \leqslant |u - v|$ valid for all reals $u \geqslant 0, v \geqslant 0$, we have

$$(3.16)\quad \frac{D^2}{2}\left|\exp\Big(-\frac{{\kappa'_n}^2}{D^2(B_n/2)}\Big) - \exp\Big(-\frac{(\kappa'_n-\kappa'_m)^2}{D^2(B_n-B_m)/2}\Big)\right|$$
$$\leqslant \left| -\frac{(\kappa'_n-\kappa'_m)^2}{B_n-B_m} + \frac{{\kappa'_n}^2}{B_n}\right| = \left| -\frac{(\sqrt{B_n}b_n - \sqrt{B_m}b_m)^2}{B_n-B_m} + b_n^2\right|$$
$$= \left|\frac{-B_nb_n^2 - B_mb_m^2 + 2\sqrt{B_nB_m}b_nb_m + B_nb_n^2 - B_mb_n^2}{B_n-B_m}\right|$$
$$= \left|\frac{-(b_n-b_m)^2 + 2b_mb_n(\sqrt{B_n/B_m}-1)}{B_n/B_m - 1}\right|$$
$$\leqslant \frac{2(b_n^2+b_m^2) + 2|b_m||b_n|(\sqrt{B_n/B_m}-1)}{B_n/B_m-1}.$$

Hence

$$(3.17)\quad \mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta, B_n-B_m>\theta(n-m)\}}$$
$$\times\left|\exp\Big(-\frac{{\kappa'_n}^2}{D^2(B_n/2)}\Big) - \exp\Big(-\frac{(\kappa'_n-\kappa'_m)^2}{D^2(B_n-B_m)/2}\Big)\right|$$
$$\leqslant C\mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta, B_n-B_m>\theta(n-m)\}}\left\{\frac{b_n^2+b_m^2}{B_n/B_m-1} + \frac{|b_m||b_n|}{\sqrt{B_n/B_m}-1}\right\}.$$

We notice that on the set $\{B_n > n\theta, B_n - B_m > \theta(n-m)\}$ we have

$$\frac{1}{\sqrt{B_n/B_m}-1} = \frac{\sqrt{B_m}}{\sqrt{B_n}-\sqrt{B_m}} \leqslant \frac{\sqrt{B_m}(\sqrt{B_n}+\sqrt{B_m})}{\theta(n-m)}.$$

We also observe that

$$\mathbb{E}S_m = \mathbb{E}_{(V,\varepsilon)}\mathbb{E}_L\big(W_m + D\sum_{j=1}^m \varepsilon_j L_j\big) = \mathbb{E}_{(V,\varepsilon)}\Big(W_m + \frac{DB_m}{2}\Big) = m\mu.$$

Thus $W_m + (DB_m/2) - m\mu = W_m + (DB_m/2) - \mathbb{E}_{(V,\varepsilon)}(W_m + DB_m/2)$.

Besides,

$$|b_j| = \frac{\big|\kappa_j - j\mu - \big(W_j + (B_j/2) - j\mu\big)\big|}{\sqrt{B_j}}$$
$$\leqslant \frac{C}{\sqrt{B_j}}[\sqrt{j} + |S'_j - \mathbb{E}_{(V,\varepsilon)}S'_j|],$$

where we have put $S'_n = W_m + (B_m/2)$. We have

$$\begin{aligned}
\frac{|b_n||b_m|}{\sqrt{B_n/B_m}-1} &\leqslant \frac{C}{\sqrt{B_nB_m}}\frac{\sqrt{B_m}(\sqrt{B_n}+\sqrt{B_m})}{\theta(n-m)} \\
&\quad \times [\sqrt{n}+|S'_n-\mathbb{E}_{(V,\varepsilon)}S'_n|][\sqrt{m}+|S'_m-\mathbb{E}_{(V,\varepsilon)}S'_m|] \\
&\leqslant \frac{C\sqrt{m}(\sqrt{n}+\sqrt{m})}{\theta(n-m)} \\
&\quad \times \left[1+\frac{|S'_n-\mathbb{E}_{(V,\varepsilon)}S'_n|}{\sqrt{n}}\right]\left[1+\frac{|S'_m-\mathbb{E}_{(V,\varepsilon)}S'_m|}{\sqrt{m}}\right] \\
&= \frac{C}{\theta(\sqrt{n/m}-1)}\left[1+\frac{|S'_n-\mathbb{E}_{(V,\varepsilon)}S'_n|}{\sqrt{n}}\right]\left[1+\frac{|S'_m-\mathbb{E}_{(V,\varepsilon)}S'_m|}{\sqrt{m}}\right].
\end{aligned}$$

By the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\mathbb{E}_{(V,\varepsilon)}\frac{|S'_n-\mathbb{E}_{(V,\varepsilon)}S'_n|}{\sqrt{n}}\,\frac{|S'_m-\mathbb{E}_{(V,\varepsilon)}S'_m|}{\sqrt{m}} \\
&\qquad \leqslant \left[\mathbb{E}_{(V,\varepsilon)}\frac{|S'_n-\mathbb{E}_{(V,\varepsilon)}S'_n|^2}{n}\right]^{1/2}\left[\mathbb{E}_{(V,\varepsilon)}\frac{|S'_m-\mathbb{E}_{(V,\varepsilon)}S'_m|^2}{m}\right]^{1/2} \leqslant C.
\end{aligned}$$

Moreover, we also have

$$\mathbb{E}_{(V,\varepsilon)}\frac{|S'_j-\mathbb{E}_{(V,\varepsilon)}S'_j|}{\sqrt{j}} \leqslant \left[\mathbb{E}_{(V,\varepsilon)}\frac{|S'_j-\mathbb{E}_{(V,\varepsilon)}S'_j|^2}{j}\right]^{1/2} \leqslant C.$$

Since

$$\begin{aligned}
&\mathbb{E}_{(V,\varepsilon)}\frac{|b_n||b_m|}{\sqrt{B_n/B_m}-1} \\
&\qquad \leqslant \frac{C_\theta}{\sqrt{n/m}-1}\mathbb{E}_{(V,\varepsilon)}\left[1+\frac{|S'_n-\mathbb{E}_{(V,\varepsilon)}S'_n|}{\sqrt{n}}\right]\left[1+\frac{|S'_m-\mathbb{E}_{(V,\varepsilon)}S'_m|}{\sqrt{m}}\right],
\end{aligned}$$

we deduce

$$\mathbb{E}_{(V,\varepsilon)}\ \frac{|b_n||b_m|}{\sqrt{B_n/B_m}-1} \leqslant \frac{C_\theta}{\sqrt{n/m}-1}. \tag{3.18}$$

Now

$$\begin{aligned}
&\frac{|b_n|^2}{B_n/B_m-1} \leqslant \frac{C}{B_n}[\sqrt{n}+|S'_n-\mathbb{E}_{(V,\varepsilon)}S'_n|]^2\frac{B_m}{B_n-B_m} \\
&\leqslant C\frac{nB_m}{B_n(B_n-B_m)}\left[1+\frac{|S'_n-\mathbb{E}_{(V,\varepsilon)}S'_n|}{\sqrt{n}}\right]^2 \\
&\leqslant \frac{C}{\theta^2(n/m-1)}\left[1+\frac{|S'_n-\mathbb{E}_{(V,\varepsilon)}S'_n|}{\sqrt{n}}\right]^2 \leqslant \frac{C_\theta}{\sqrt{n/m}-1}\left[1+\frac{|S'_n-\mathbb{E}_{(V,\varepsilon)}S'_n|}{\sqrt{n}}\right]^2.
\end{aligned}$$

Therefore

$$(3.19)\quad \mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta, B_n-B_m>\theta(n-m)\}}\frac{|b_n|^2}{B_n/B_m-1}$$
$$\leqslant \frac{C_\theta}{\sqrt{n/m}-1}\mathbb{E}_{(V,\varepsilon)}\Big[1+\frac{|S'_n-\mathbb{E}_{(V,\varepsilon)}S'_n|}{\sqrt{n}}\Big]^2$$
$$\leqslant \frac{C_\theta}{\sqrt{n/m}-1}\Big[1+\frac{1}{n}\mathbb{E}_{(V,\varepsilon)}|S'_n-\mathbb{E}_{(V,\varepsilon)}S'_n|^2\Big] \leqslant \frac{C_\theta}{\sqrt{n/m}-1}.$$

Next $|b_m| \leqslant (C/\sqrt{B_m})[\sqrt{m}+|S'_m-\mathbb{E}_{(V,\varepsilon)}S'_m|]$. Thus

$$\frac{|b_m|^2}{B_n/B_m-1} \leqslant \frac{C}{B_m}[\sqrt{m}+|S'_m-\mathbb{E}_{(V,\varepsilon)}S'_m|]^2\frac{B_m}{B_n-B_m}$$
$$\leqslant C\frac{m}{\theta(n-m)}\Big[1+\frac{|S'_m-\mathbb{E}_{(V,\varepsilon)}S'_m|}{\sqrt{m}}\Big]^2 \leqslant \frac{C_\theta}{\sqrt{n/m}-1}\Big[1+\frac{|S'_m-\mathbb{E}_{(V,\varepsilon)}S'_m|}{\sqrt{m}}\Big]^2.$$

Therefore

$$(3.20)\quad \mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta, B_n-B_m>\theta(n-m)\}}\frac{|b_m|^2}{B_n/B_m-1}$$
$$\leqslant \frac{C_\theta}{\sqrt{n/m}-1}\Big[1+\frac{1}{m}\mathbb{E}_{(V,\varepsilon)}|S'_m-\mathbb{E}_{(V,\varepsilon)}S'_m|^2\Big] \leqslant \frac{C_\theta}{\sqrt{n/m}-1}.$$

Inserting the estimates (3.18), (3.19), (3.20) into (3.17), we get

$$(3.21)\quad \mathbb{E}_{(V,\varepsilon)}\mathbf{1}_{\{B_n>n\theta, B_n-B_m>\theta(n-m)\}}$$
$$\times\left|\exp\Big(-\frac{{\kappa'_n}^2}{D^2(B_n/2)}\Big)-\exp\Big(-\frac{(\kappa'_n-\kappa'_m)^2}{D^2(B_n-B_m)/2}\Big)\right| \leqslant \frac{C_\theta}{\sqrt{n/m}-1}.$$

This estimate along with (3.15) yields, in view of (3.13),

$$(3.22)\qquad A''_1 \leqslant 2\rho^{n-m}+\frac{C_\theta}{\sqrt{n/m}-1}.$$

Moreover, using (3.11)–(3.14), we obtain

$$(3.23)\quad |A''| \leqslant C_\theta\Big\{\rho^{n-m}+\frac{1}{\sqrt{n/m}-1}$$
$$+\sqrt{n}\Big(\frac{1}{n^{3/2}}+\rho^{n-m}+\frac{1}{(n-m)^{3/2}}\Big)\Big\}.$$

Consequently, by (3.9) we have

$$\text{(3.24)}\quad |A'| + |A''| \leqslant C_\theta \Big\{ \rho^{n-m} + \frac{1}{\sqrt{n/m}-1} + \sqrt{n}\Big(\frac{1}{n^{3/2}} + \rho^{n-m} + 2^{-(n-m)} + \frac{1}{(n-m)^{3/2}} \Big) \Big\}.$$

Finally, by (3.6),

$$\text{(3.25)}\quad |A| \leqslant C_\theta \Big\{ \rho^{n-m} + \frac{1}{\sqrt{n/m}-1} + \sqrt{n}\Big(\frac{1}{n^{3/2}} + \rho^{n-m} + 2^{-(n-m)} + \frac{1}{(n-m)^{3/2}} \Big) \Big\}.$$

Moreover, using (3.4), we obtain

$$\text{(3.26)}\quad |\mathbb{E}Y_nY_m| \leqslant C_\theta \Big\{ \rho^{n-m} + \frac{1}{\sqrt{n/m}-1} + \sqrt{n}\Big(\frac{1}{n^{3/2}} + \rho^{n-m} + 2^{-(n-m)} + \frac{1}{(n-m)^{3/2}} \Big) \Big\} \leqslant C_\theta \Big\{ \frac{1}{\sqrt{n/m}-1} + \frac{n^{1/2}}{(n-m)^{3/2}} \Big\}.$$

This proves (3.2).

Now, let $0 < c < 1$. Let $m \leqslant cn$. Then

$$\frac{1}{\sqrt{n/m}-1} \leqslant \frac{1}{1-\sqrt{c}} \sqrt{\frac{m}{n}}.$$

Further

$$\frac{\sqrt{n}}{(n-m)^{3/2}} = \sqrt{\frac{n}{n-m}}\,\frac{1}{n-m} = \frac{1}{(1-m/n)^{3/2}}\,\frac{1}{n} \leqslant \frac{1}{(1-c)^{3/2}}\,\frac{1}{n}.$$

Incorporating these estimates into (3.20), we get

$$\text{(3.27)}\quad |\mathbb{E}Y_nY_m| \leqslant C_\theta \sqrt{\frac{m}{n}}.$$

This establishes (3.3). The proof is now complete. ■

4. APPLICATION

In this section, we deduce from Theorem 1.1 an almost sure local limit theorem for i.i.d. square integrable random variables taking values in an arbitrary lattice $\mathcal{L}(v_0, D)$. In [2] (Sections 1 and 2), the notion of almost sure local limit theorem is introduced by analogy with the usual almost sure central limit theorem:

"A stationary sequence of random variables $\{X_n, n \geqslant 1\}$ taking values in $\mathbb{R}$ or $\mathbb{Z}$ with partial sums $S_n = X_1 + \ldots + X_n$ satisfies an *almost sure local limit theorem* if there exist sequences $\{a_n, n \geqslant 1\}$ in $\mathbb{R}$ and $\{b_n, n \geqslant 1\}$ in $\mathbb{R}^+$ satisfying $b_n \to \infty$, such that

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{b_n}{n} \chi\{S_n \in k_n + I\} \overset{\text{a.s.}}{=} g(\kappa)|I| \quad \text{as } \frac{k_n - a_n}{b_n} \to \kappa, \tag{4.1}$$

where g denotes some density and $I \subset \mathbb{R}$ is some bounded interval. Further $|I|$ denotes the length of the interval I in the case where X_1 is real valued and the counting measure of I otherwise."

In what follows, we restrict our consideration to the i.i.d. case. We assume that $\mathbb{P}\{X_1 \in \mathcal{L}(v_0, D)\} = 1$, $\mathcal{L}(v_0, D) \subset \mathbb{Z}$. We also assume that $\sigma^2 = \mathbb{E}X_1^2 < \infty$ and let $\mu = \mathbb{E}X_1$.

Let us first observe that even in this restricted case the above definition is incomplete, since the span of X_1 is missing. Indeed, let $v_0 = 0$ for simplicity. As g is a density, there are reals κ such that $g(\kappa) \neq 0$. Clearly, if $\{k_n, n \geqslant 1\}$ is such that

$$\frac{k_n - a_n}{b_n} \to \kappa,$$

then any sequence $\{\kappa_n, n \geqslant 1\}$, $\kappa_n = k_n + u_n$, where u_n are uniformly bounded, also has this property. But we can arrange the u_n so that $\kappa_n \notin \mathcal{L}(0, D)$ for all n. Therefore $\mathbb{P}\{S_n = \kappa_n\} \equiv 0$. If $I = [-\delta, \delta]$ with $\delta < 1/2$, then $|I| = 1$ and we see that, *no matter* the sequences $\{a_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$ are, property (4.1) cannot hold for the sequence $\{\kappa_n, n \geqslant 1\}$, since

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{b_n}{n} \chi\{S_n \in k_n + I\} \overset{\text{a.s.}}{=} 0 \neq g(\kappa)|I|.$$

It thus appears necessary (also when v_0 is arbitrary) to complete the above definition by introducing the additional requirement:

$$\kappa_n \in \mathcal{L}(nv_0, D), \qquad n = 1, 2, \ldots \tag{4.2}$$

Then $k_n + I \subset L(nv_0, D)$ if and only if $I \subset L(0, D)$. It is also necessary to change $|I|$ for $\#\{I \cap \mathcal{L}(0, D)\}$. Then (4.1) is modified as follows:

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{b_n}{n} \chi\{S_n \in k_n + I\} \overset{\text{a.s.}}{=} g(\kappa)\#\{I \cap \mathcal{L}(0, D)\} \text{ as } \frac{k_n - a_n}{b_n} \to \kappa, \tag{4.3}$$

where I is a bounded interval. This is coherent with the local limit theorem which relies upon the three parameters: μ, σ and the (maximal) span of X_1. It is obvious, by invoking a simple additivity argument, that (4.3) holds for any bounded interval I if and only if

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{b_n}{n} \chi\{S_n = k_n\} \overset{\text{a.s.}}{=} g(\kappa) \quad \text{as } \frac{k_n - a_n}{b_n} \to \kappa. \tag{4.4}$$

As mentioned by the authors in [2], p. 146, the existence of almost sure local limit theorems is of fundamental interest. A recent application to a problem of representation of integers is given in [12]. It seems reasonable to expect many other applications. By (1.2), the local limit theorem holds, and if $\kappa_n \in \mathcal{L}(nv_0, D)$ is a sequence which satisfies condition (1.3), namely

$$\lim_{n\to\infty} \frac{\kappa_n - n\mu}{\sqrt{n}} = \kappa,$$

then

$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}\{S_n = \kappa_n\} = \frac{D}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\kappa^2}{2\sigma^2}\right). \tag{4.5}$$

We deduce from Theorem 1.1 an almost sure local limit theorem for i.i.d. square integrable random variables taking values in an arbitrary lattice $\mathcal{L}(v_0, D)$.

THEOREM 4.1. *Let X be a square integrable lattice distributed random variable with maximal span D. Let $\mu = \mathbb{E}X$, $\sigma^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2$. Let also $\{X_k, k \geqslant 1\}$ be independent copies of X, and put $S_n = X_1 + \ldots + X_n$, $n \geqslant 1$. Then*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n\leqslant N} \frac{1}{\sqrt{n}} \mathbf{1}_{\{S_n=\kappa_n\}} \overset{a.s.}{=} \frac{D}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\kappa^2}{2\sigma^2}\right)$$

for any sequence of integers $\{\kappa_n, n \geqslant 1\}$ such that (1.3) *holds.*

REMARK 4.1. In [2], Corollary 2 (see also pp. 148–149), the authors show that "the almost sure local limit theorem holds for i.i.d. sequences of square integrable $\mathbb{Z}$-valued random variables, that is:

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n\leqslant N} \frac{1}{\sqrt{n}} \mathbf{1}_{\{S_n=\kappa_n\}} \overset{\text{a.s.}}{=} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\kappa^2}{2\sigma^2}\right) \quad \text{if } \frac{k_n - n\mu}{\sqrt{n}} \to \kappa."$$

By the remarks made before (4.1), this statement needs a correction. The proof is sketched as follows. Let ϕ denote the characteristic function of X. By the Fourier inversion formula, $\mathbb{P}\{S_n = k\} = \int_0^1 e^{-2i\pi kt}\phi^n(t)dt$. Thus

$$\begin{aligned}
&\sqrt{nm}\mathbb{P}\{S_n = k_n, S_m = k_m\} = \sqrt{nm}\mathbb{P}\{S_m = k_m\}\mathbb{P}\{S_n - S_m = k_n - k_m\} \\
&= \sqrt{nm}\int_0^1 \exp(-2i\pi k_m t)\phi^m(t)dt \int_0^1 \exp\big(-2i\pi(k_n - k_m)t\big)\phi^{n-m}(t)dt \\
&= \sqrt{\frac{n}{n-m}}\int_0^{\sqrt{m}} \exp\Big(-2i\pi\frac{k_m}{\sqrt{m}}u\Big)\phi^m\Big(\frac{u}{\sqrt{m}}\Big)du \\
&\times \int_0^{\sqrt{n-m}} \exp\Big(-2i\pi\Big(\frac{k_n-k_m}{\sqrt{n-m}}\Big)u\Big)\phi^{n-m}\Big(\frac{v}{\sqrt{n-m}}\Big)dv.
\end{aligned}$$

By the central limit theorem,

$$\lim_{m\to\infty}\phi^m\Big(\frac{u}{\sqrt{m}}\Big) = \frac{\exp(-\kappa^2/2)}{\sqrt{2\pi}}, \qquad \lim_{n-m\to\infty}\phi^{n-m}\Big(\frac{v}{\sqrt{n-m}}\Big) = \frac{\exp(-\kappa^2/2)}{\sqrt{2\pi}}.$$

Next, it is claimed that it implies that

$$\sqrt{nm}\mathbb{P}\{S_n = k_n, S_m = k_m\} \to \sqrt{n/(n-m)}\frac{1}{2\pi}\exp(-\kappa^2).$$

We presume that this should rather be

$$\sqrt{nm}\mathbb{P}\{S_n = k_n, S_m = k_m\} \to \frac{1}{2\pi}\exp(-\kappa^2).$$

However, we have not been able to check this. From our main result, we only get

$$\lim_{\substack{n,m\to\infty \\ n/m\to\infty}} \sqrt{nm}\,|\mathbb{P}\{S_n = \kappa_n, S_m = \kappa_m\} - \mathbb{P}\{S_n = \kappa_n\}\mathbb{P}\{S_m = \kappa_m\}| = 0.$$

The authors argue that the proof could be continued as in the Bernoulli case where a theorem of Mori is invoked. This requires to have at disposal a correlation bound. For having tried to apply Mori's result with our correlation inequality in Theorem 1.1, we could only treat subsequences $n = n_k$ with $n_{k=1}/n_k \geqslant \sqrt{2}$. We believe that the proof needs some complementary explanations.

The notion of quasi-orthogonal system is used in the proof of Theorem 4.1. We recall it briefly. A sequence $\underline{f} = \{f_n, n \geqslant 1\}$ in a Hilbert space H is called (see [7] or [11], p. 22) a *quasi-orthogonal system* if the quadratic form on ℓ_2 defined by $\{x_h, h \geqslant 1\} \mapsto \|\sum_h x_h f_h\|^2$ is bounded. A necessary and sufficient condition for $\underline{f}$ to be quasi-orthogonal is that the series $\sum c_n f_n$ converges in H for any sequence $\{c_n, n \geqslant 1\}$ such that $\sum c_n^2 < \infty$. This follows from the fact that $\underline{f}$ is quasi-orthogonal if and only if there exists a constant L depending on $\underline{f}$ only, such that

$$\Big\|\sum_{i\leqslant n} x_i f_i\Big\| \leqslant L\Big(\sum_{i\leqslant n}|x_i|^2\Big)^{1/2}.$$

Further, as observed in [7]: “Every theorem on orthogonal systems whose proof depends only on Bessel’s inequality holds for quasi-orthogonal systems.” In particular, for $H = L^2(X, \mathcal{A}, \mu)$, $(X, \mathcal{A}, \mu)$ a probability space, Rademacher–Menchov’s theorem applies. We recall it (see [11], p. 363, for instance).

LEMMA 4.1. *Let* $\{f_n, n \geqslant 1\} \subset H$ *be an orthogonal sequence. The series* $\sum c_n f_n$ *converges almost everywhere provided that* $\sum c_n^2 \log^2 n < \infty$.

Proof. We first give the proof under the additional assumption (1.4). Next we establish the result without this one. Assume thus, at first, that assumption (1.4) is fulfilled; the proof is then identical to the one of Theorem 1 in [4]. Put for any positive integer j

$$Z_j = \sum_{2^j \leqslant n < 2^{j+1}} \frac{Y_n}{n}.$$

By (1.2), we have

$$\mathbb{E}Y_n^2 = n\mathbb{P}\{S_n = \kappa_n\}(1 - \mathbb{P}\{S_n = \kappa_n\}) = \mathcal{O}(\sqrt{n}).$$

This and the second inequality of Theorem 1.1 imply that $\{Z_j, j \geqslant 1\}$ is a quasi-orthogonal system. As the Rademacher–Menchov theorem applies to quasi-orthogonal systems, the series

$$\sum_j \frac{Z_j}{j^{1/2}(\log j)^b}$$

thus converges almost surely if $b > 3/2$. By Kronecker’s lemma we have

$$\frac{1}{N^{1/2}(\log N)^b} \sum_{j=1}^{N} Z_j = \frac{1}{N^{1/2}(\log N)^b} \sum_{1 \leqslant n < 2^{N+1}} \frac{Y_n}{n} \to 0$$

as N tends to infinity, almost surely. It is then a routine calculation to derive from this that

$$\lim_{t\to\infty} \frac{1}{\log t} \sum_{n \leqslant t} \frac{1}{\sqrt{n}} \mathbf{1}_{\{S_n = \kappa_n\}} \overset{\text{a.s.}}{=} g(\kappa).$$

Now we pass to the general case. By Remark 1.1, we may “change” X for $X' = S_{n_\kappa}$. But this is not so simple as it looks, some extra work is necessary in order to make this step precise. Let $X_1', X_2', \dots$ be independent copies of X', which we assume to be also independent of the sequence $X_1, X_2, \dots$, and put similarly $S_m' = X_1' + \dots + X_m'$, $m \geqslant 1$. Let us observe first that, given $0 \leqslant a < n_\kappa$, the sequence $\{S_{a+mn_\kappa}, m \geqslant 1\}$ has the same law as the sequence $\{S_a + S_m', m \geqslant 1\}$. It is indeed immediate if we write that $S_{a+mn_\kappa} = S_a + (S_{a+n_\kappa} - S_a) + \dots + (S_{a+mn_\kappa} - S_{a+(m-1)n_\kappa})$ (and not $S_{a+mn_\kappa} = S_{n_\kappa} + \dots + (S_{mn_\kappa} - S_{(m-1)n_\kappa}) +$

$(S_{a+mn_\kappa} - S_{mn_\kappa})$!). Like this, we are thus preliminarily led to consider the sequence $\{S_a + S'_m, m \geqslant 1\}$. But this one is a bit outside from our framework and we have to understand more the role played by the additional independent term S_a. Let $\kappa_n \in L(nv_0, D)$, $n = 1, 2, \ldots$, be a sequence of integers such that

$$\lim_{n\to\infty} \frac{\kappa_n - n\mu}{\sqrt{n}} = \kappa.$$

Then, for any $0 \leqslant a < n_\kappa$,

$$\lim_{m\to\infty} \frac{\kappa_{a+mn_\kappa} - (a + mn_\kappa)\mu}{\sqrt{m}} = \kappa\sqrt{n_\kappa}.$$

Further, not only

$$\lim_{m\to\infty} \frac{\kappa_{a+mn_\kappa} - mn_\kappa\mu}{\sqrt{m}} = \kappa\sqrt{n_\kappa},$$

but also, for any $0 \leqslant a < n_\kappa$,

$$\lim_{m\to\infty} \frac{\kappa_{a+mn_\kappa} - S_a - mn_\kappa\mu}{\sqrt{m}} \overset{\text{a.s.}}{=} \kappa\sqrt{n_\kappa}.$$

Noticing that $\{S'_m, m \geqslant 1\}$ has the same law as $\{S_{mn_\kappa}, m \geqslant 1\}$, next applying (1.2) to X and specifying it for the subsequence $\{mn_\kappa, m \geqslant 1\}$, we get

(4.6)
$$\lim_{m\to\infty} \sup_{N=v_0mn_\kappa+Dk} \left| \sqrt{m}\mathbb{P}\{S'_m = N\} - \frac{D}{\sigma\sqrt{2\pi n_\kappa}} \exp\left(-\frac{(N - mn_\kappa\mu)^2}{2mn_\kappa\sigma^2}\right) \right| = 0.$$

Let $\tilde{\kappa}_m = \kappa_{a+mn_\kappa} - S_a$, $0 \leqslant a < n_\kappa$ being fixed. Since

$$\kappa_{a+mn_\kappa} \in L\big((a + mn_\kappa)v_0, D\big), \qquad S_a \in L(av_0, D),$$

we have $\tilde{\kappa}_m \in L(mn_\kappa v_0, D)$. Thus,

$$\begin{aligned}\lim_{m\to\infty} \sqrt{mn_\kappa}\, \mathbb{P}\{S'_m = \tilde{\kappa}_m\} &\overset{\text{a.s.}}{=} \lim_{m\to\infty} \frac{D}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(\tilde{\kappa}_m - mn_\kappa\mu)^2}{2m\sigma^2}\right) \\ &\overset{\text{a.s.}}{=} \frac{D}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\kappa^2}{2\sigma^2}\right).\end{aligned}$$

Instead of considering $Y_m = \sqrt{m}(\mathbf{1}_{\{S'_m=\kappa_m\}} - \mathbb{P}\{S'_m = \kappa_m\})$, we rather work with

$$Y'_m = \sqrt{mn_\kappa}(\mathbf{1}_{\{S'_m=\kappa_m\}} - \mathbb{P}\{S'_m = \kappa_m\}).$$

This amounts to the same, apart from the constant factor $\sqrt{n_\kappa}$. By the first step,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{m=1}^{N} \frac{Y'_m}{m} \overset{\text{a.s.}}{=} 0. \tag{4.7}$$

Thus

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{m=1}^{N}\left(\frac{\sqrt{mn_\kappa}}{m}\mathbf{1}_{\{S'_m=\kappa_m-S_a\}}-\frac{\sqrt{mn_\kappa}\,\mathbb{P}\{S'_m=\kappa_m-S_a\}}{m}\right)\overset{\text{a.s.}}{=}0.$$

But

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{m=1}^{N}\frac{\sqrt{mn_\kappa}\,\mathbb{P}\{S'_m=\kappa_m-S_a\}}{m}\overset{\text{a.s.}}{=}\frac{D}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\kappa^2}{2\sigma^2}\right).$$

Therefore,

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{m=1}^{N}\frac{\sqrt{mn_\kappa}}{m}\mathbf{1}_{\{S_a+S'_m=\kappa_m\}}\overset{\text{a.s.}}{=}\frac{D}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\kappa^2}{2\sigma^2}\right).$$

We deduce that

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{m\leqslant N}\frac{\sqrt{mn_\kappa}}{m}\mathbf{1}_{\{S_{a+mn_\kappa}=\kappa_{a+mn_\kappa}\}}\overset{\text{a.s.}}{=}\frac{D}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\kappa^2}{2\sigma^2}\right).$$

Now divide both sides by n_κ. We get

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{m\leqslant N}\frac{1}{\sqrt{mn_\kappa}}\mathbf{1}_{\{S_{a+mn_\kappa}=\kappa_{a+mn_\kappa}\}}\overset{\text{a.s.}}{=}\frac{D}{n_\kappa\sigma\sqrt{2\pi}}\exp\left(-\frac{\kappa^2}{2\sigma^2}\right).$$

But this in turn also implies

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{m\leqslant N}\frac{1}{\sqrt{a+mn_\kappa}}\mathbf{1}_{\{S_{a+mn_\kappa}=\kappa_{a+mn_\kappa}\}}\overset{\text{a.s.}}{=}\frac{D}{n_\kappa\sigma\sqrt{2\pi}}\exp\left(-\frac{\kappa^2}{2\sigma^2}\right).$$

Now $\log N\sim\log(N+1)n_\kappa$, and by summing the latter over $0\leqslant a<n_\kappa$, we get

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{\log(N+1)n_\kappa}\sum_{0\leqslant a<n_\kappa}\sum_{m\leqslant N}\frac{1}{\sqrt{a+mn_\kappa}}\mathbf{1}_{\{S_{a+mn_\kappa}=\kappa_m\}}\\&=\lim_{N\to\infty}\frac{1}{\log(N+1)n_\kappa}\sum_{n\leqslant(N+1)n_\kappa}\frac{1}{\sqrt{n}}\mathbf{1}_{\{S_n=\kappa_n\}}\overset{\text{a.s.}}{=}\frac{D}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\kappa^2}{2\sigma^2}\right).\end{aligned}$$

Hence Theorem 4.1 is proved. ■

REFERENCES

[1] I. Berkes and E. Csaki, *A universal result in almost sure central limit theory*, Stoch. Process. Appl. 94 (2001), pp. 105–134.

[2] M. Denker and S. Koch, *Almost sure local limit theorems*, Statist. Neerlandica 56 (2) (2002), pp. 143–151.

[3] R. Giuliano-Antonini and M. Weber, *The intersective ASCLT*, Stochastic Anal. Appl. 22 (4) (2004), pp. 1009–1025.

[4] R. Giuliano-Antonini and M. Weber, *Almost sure local limit theorems with rate*, in revision to Stochastic Anal. Appl.

[5] B. V. Gnedenko, *Course in the Theory of Probability*, 5th edition, Nauka, Moscow 1967. English translation of 4th edition: Chelsea, New York.

[6] I. A. Ibragimov and Y. V. Linnik, *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff Publishing Groningen, The Netherlands, 1971.

[7] M. Kac, R. Salem and A. Zygmund, *A gap theorem*, Trans. Amer. Math. Soc. 63 (1948), pp. 235–243.

[8] D. MacDonald, *A local limit theorem for large deviations of sums of independent, non-identically distributed random variables*, Ann. Probab. 7 (3) (1979), pp. 526–531.

[9] D. MacDonald and B. Davis, *An elementary proof of the local central limit theorem*, J. Theoret. Probab. 8 (3) (1995), pp. 695–701.

[10] V. V. Petrov, *Sums of Independent Random Variables*, Ergeb. Math. Grenzgeb. 82 (1975).

[11] M. Weber, *Dynamical Systems and Processes*, IRMA Lect. Math. Theor. Phys. 14 (2009), xiii+761pp.

[12] M. Weber, *On representation of integers*, submitted to publication.

C.N.R.S.
IRMA, Université Louis-Pasteur
7 rue René Descartes
67084-Strasbourg Cedex, France
E-mail: michel.weber@math.unistra.fr

Received on 27.5.2010;
revised version on 19.11.2010