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STATIONARY SEQUENCES ASSOCIATED WITH A PERIODICALLY CORRELATED SEQUENCE

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Abstract. Arranging a periodically correlated sequence (PC) with period T into blocks of length T generates a T-dimensional stationary sequence. In this paper we discuss two other transformations that map PC sequences into T-dimensional stationary sequences and study their properties. We also indicate possible applications of these mappings in the theory of PC processes and, in particular, for study of PARMA systems. The presented construction is both a simplification and enhancement of the construction given in [20].

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1. INTRODUCTION

In what follows T is a fixed positive integer and $\langle m\rangle$ denotes the nonnegative remainder in division of m by T (for example, if $T = 12$, then $\langle -17\rangle = 7$). T-dimensional vectors $\mathbf{v} = [v^j]$ are represented as columns with coordinates labeled from 0 to $T-1$. The entries of a $T \times T$ matrix $\boldsymbol{A}$ are denoted by $A^{p,q}$, where $p = 0, 1, \ldots, T-1$ is a row, and $q = 0, 1, \ldots, T-1$ is a column index. A $T \times T$ matrix $\boldsymbol{A} = [A^{p,q}]$ is called *r-diagonal*, $0 \leqslant r < T$, if $A^{p,q} = 0$ except when $p - q = r$ *modulo* T, that is, if only the entries $A^{p,\langle p-r\rangle}$ are possibly nonzero. The identity matrix $\boldsymbol{I}$ is a matrix such that $I^{p,p} = 1$, $p = 0, \ldots, T-1$, and $I^{p,q} = 0$ otherwise.

Sequences are indexed by the set of all integers $\mathcal{Z}$, unless is stated otherwise. The dual $\hat{\mathcal{Z}}$ of $\mathcal{Z}$ is identified with the interval $[0, 2\pi)$ with addition *modulo* 2π. Similarly, the dual of $\mathcal{Z}^2$ will be identified with the square $[0, 2\pi)^2$ with addition *modulo* 2π (for example, $(\pi/4, 3\pi/4) - (\pi/2, \pi/2) = (7\pi/4, \pi/4)$).

All random variables in the paper are complex-valued with zero mean and finite variance, and we will look at them as elements of an abstract complex Hilbert space $\mathcal{H}$. A *T-dimensional stochastic sequence in $\mathcal{H}$* is, therefore, a sequence of

T-dimensional vectors $\boldsymbol{X}(n) = [X^p(n)]$, $n \in \mathcal{Z}$, whose coordinates $X^p(n)$, $p = 0, \dots, T-1$, are in $\mathcal{H}$. A one-dimensional stochastic sequence is simply called a *stochastic sequence*.

The covariance function of a sequence $\big(x(n)\big)$ is the function $R_x(m,n) = \big(x(m), x(n)\big)$, where $(\cdot,\cdot)$ denotes the scalar product in $\mathcal{H}$. The covariance function $\boldsymbol{R}_X(m,n)$ of a T-dimensional stochastic sequence $\boldsymbol{X}(n) = [X^j(n)]$ is a $T \times T$ matrix function with entries $R_X^{j,k}(m,n) = \big(X^j(m), X^k(n)\big)$. Two stochastic sequences having the same covariance function are considered identical, even if they take values in different Hilbert spaces.

A T-dimensional stochastic sequence $\big(\boldsymbol{X}(n)\big)$ is called *stationary* if for every $m, n \in \mathcal{Z}$, $\boldsymbol{R}_X(m,n) = \boldsymbol{R}_X(m-n, 0)$. If $\big(\boldsymbol{X}(n)\big)$ is stationary, then the matrix sequence $\boldsymbol{K}_X(n) = \boldsymbol{R}_X(n,0)$, $n \in \mathcal{Z}$, is called the *covariance function* of the sequence $\big(\boldsymbol{X}(n)\big)$. The spectrum $\boldsymbol{F}_X(dt) = [F_X^{j,k}(dt)]$ of a T-dimensional stationary sequence $\big(\boldsymbol{X}(n)\big)$ with the covariance function $\boldsymbol{K}_X(n)$ is a $T \times T$-matrix valued Borel measure on $[0, 2\pi)$ such that

$$\text{(1.1)} \qquad \boldsymbol{K}_X(n) = \int_0^{2\pi} e^{int} \boldsymbol{F}_X(dt), \qquad n \in \mathcal{Z}.$$

A sequence $\big(s(n)\big)$ is called T*-periodic* ($T \geqslant 1$) if $s(n) = s(n+T)$ for all $n \in \mathcal{Z}$. The *Fourier transform* of a T-periodic sequence $\big(s(n)\big)$ is a T-periodic sequence defined by

$$\text{(1.2)} \qquad \tilde{s}(n) = \frac{1}{T} \sum_{k=0}^{T-1} e^{-2\pi i n k/T} s(k), \quad n \in \mathcal{Z},$$

so that $s(n) = \sum_{k=0}^{T-1} e^{2\pi i n k/T} \tilde{s}(k)$, $n \in \mathcal{Z}$.

A one-dimensional stochastic sequence $\big(x(n)\big)$ is called *periodically correlated* (*PC*) *with period* T if for each $n \in \mathcal{Z}$ the sequence $R_x(n+r, r)$ is T-periodic in r. By (1.2) the function $R_x(n+r,r)$ has a representation

$$\text{(1.3)} \qquad R_x(n+r, r) = \sum_{j=0}^{T-1} e^{2\pi i j r/T} a_j(n),$$

where

$$\text{(1.4)} \qquad a_j(n) = (1/T) \sum_{r=0}^{T-1} e^{-2\pi i j r/T} R_x(n+r, r), \quad j \in \mathcal{Z}.$$

The beginning of the theory of PC processes is associated with the name of Gladyshev [8], [9], who first described their structure and spectra. The modern theory of PC processes was shaped mostly by Hurd (see [11]–[13] and [16]), with further contribution of several other authors, some of them being listed in the References.

2. CORRESPONDENCE THEOREMS

Let $\mathcal{PC}(T)$ and $\mathcal{S}(T)$ denote the set of one-dimensional PC sequences of period T and the set of T-dimensional stationary sequences, respectively. Further, let $\mathcal{SI}(T)$ and $\mathcal{SD}(T)$ be subsets of $\mathcal{S}(T)$ defined as follows:

1. $\mathcal{SI}(T)$ is the set of all T-dimensional stationary sequences whose covariance function $\boldsymbol{K}(n)$ has the property that for every $n \in \mathcal{Z}$, $e^{2\pi ijn/T}K^{j,k}(n)$ depends only on $\langle j-k\rangle$;
2. $\mathcal{SD}(T)$ is the set of all T-dimensional stationary sequences such that for every $n \in \mathcal{Z}$, $\boldsymbol{K}(n)$ is $\langle n\rangle$-diagonal.

Since any two stochastic sequences which have the same covariance function are identified, the members of all four sets above are, in fact, equivalence classes.

In this section we explicitly construct three bijections (i.e. mappings that are one-to-one and onto) $\mathfrak{I}$, $\mathfrak{E}$, and $\mathfrak{X}$, that subsequently transform:

$$\mathcal{PC}(T) \xrightarrow{\mathfrak{I}} \mathcal{SI}(T) \xrightarrow{\mathfrak{E}} \mathcal{SD}(T) \xrightarrow{\mathfrak{X}} \mathcal{PC}(T).$$

$\mathfrak{I}$-mapping. Let $\big(x(n)\big)$ be a stochastic sequence in $\mathcal{H}$, and let T be a fixed positive integer. For every $n \in \mathcal{Z}$ let $\boldsymbol{Z}(n)$ be a $T \times T$ matrix whose (p,q)-entry is $Z^{p,q}(n) = x(n-q)e^{-2\pi ip(n-q)/T}$. Denote

$$\text{(2.1)}\quad Z^p(n) = \\ = [x(n)e^{-2\pi ipn/T}, x(n-1)e^{-2\pi ip(n-1)/T}, \ldots, x(n-T+1)e^{-2\pi ip(n-T+1)/T}]$$

to be the p-th row of $\boldsymbol{Z}(n)$, $p = 0,1,\ldots,T-1$. We will look at $Z^p(n)$ as a function of q, more precisely, as an element of the Hilbert space $\mathcal{K} = \ell^2(\mathcal{Z}_T, \mathcal{H})$ of square-integrable $\mathcal{H}$-valued functions on $\mathcal{Z}_T = \{0,1,\ldots,T-1\}$. The set $\mathcal{Z}_T$ is treated as a group with addition *modulo* T; the inner product in $\mathcal{K}$ is defined by $(a,b)_{\mathcal{K}} = (1/T)\sum_{q=0}^{T-1}(a^q,b^q)_{\mathcal{H}}$, where $a = [a^0,\ldots,a^{T-1}]$, $b = [b^0,\ldots,b^{T-1}]$, $a, b \in \mathcal{K} = \ell^2(\mathcal{Z}_T,\mathcal{H})$. With this interpretation the sequence $\boldsymbol{Z}(n) = [Z^p(n)]$, $n \in \mathcal{Z}$, is a T-dimensional stochastic sequence in $\mathcal{K}$. We will refer to $\big(\boldsymbol{Z}(n)\big)$ as being *induced* by the sequence $\big(x(n)\big)$. The covariance function of $\big(\boldsymbol{Z}(n)\big)$ is given by the formula

$$\text{(2.2)}\quad R_Z^{j,k}(m,n) = (1/T)\sum_{t=0}^{T-1} e^{-2\pi ij(m-t)/T}e^{2\pi ik(n-t)/T}\big(x(m-t),x(n-t)\big).$$

LEMMA 2.1. *Let $\big(x(n)\big)$ be PC with period T and let $R_x(n,m)$ denote its covariance function. Then the sequence $\big(\boldsymbol{Z}(n)\big)$ is stationary and its covariance function $\boldsymbol{K}_Z(n)$ is given by*

$$\text{(2.3)}\qquad K_Z^{j,k}(n) = e^{-2\pi ijn/T}a_{j-k}(n), \qquad k,j = 0,1,\ldots,T-1,\ n \in \mathcal{Z},$$

where a_j's are given by (1.4). *Hence $\big(\boldsymbol{Z}(n)\big)$ is in the class $\mathcal{SI}(T)$.*

Moreover, the mapping

$$\mathcal{PC}(T) \ni \big(x(n)\big) = x \xrightarrow{\mathfrak{I}} \boldsymbol{Z} = \big(\boldsymbol{Z}(n)\big) \in \mathcal{SI}(T)$$

is an injection from $\mathcal{PC}(T)$ *into* $\mathcal{SI}(T)$, *that is, if* $x, y \in \mathcal{PC}(T)$ *have different covariance functions, then the sequences* $\mathfrak{I}(x)$ *and* $\mathfrak{I}(y)$ *do so.*

P r o o f. From (2.2) it follows that

(2.4)
$$R_Z^{j,k}(m,n) = (1/T)e^{-2\pi i(m-n)j/T} \sum_{r=n-T+1}^{n} e^{-2\pi i(j-k)r/T} R_x(m-n+r,r).$$

Since $\big(x(n)\big)$ is PC with period T, the function $e^{-2\pi i(j-k)r/T}R_x(m-n+r,r)$ is T-periodic in r and, consequently, the sum in (2.4) does not depend on a starting point. Therefore, $R_Z^{j,k}(m,n)$ is a function of $m-n$, and hence $\big(\boldsymbol{Z}(n)\big)$ is stationary. Formula (2.3) is now an immediate consequence of (2.4). From (2.3) it also follows that if $\boldsymbol{K}_{\mathfrak{I}(x)}(n) = \boldsymbol{K}_{\mathfrak{I}(y)}(n)$ for every $n \in \mathcal{Z}$, then for all $n \in \mathcal{Z}$ and $k = 0, \ldots, T-1$, the corresponding $a_k^x(n)$ and $a_k^y(n)$ (see (1.4)) are equal. Consequently, $R_x = R_y$. ∎

Note that $a_{-k}(n) = a_{T-k}(n)$ and that $a_{-k}(n) = e^{2\pi ikn/T}\overline{a_k(n)}$.

EXAMPLE 2.1. If $\big(x(n)\big)$ is PC of period $T = 3$, then $\boldsymbol{Z}(n)$ is given by

$$\boldsymbol{Z}(n) = \begin{bmatrix} Z^0(n) \\ Z^1(n) \\ Z^2(n) \end{bmatrix}$$
$$= \begin{bmatrix} x(n) & x(n-1) & x(n-2) \\ x(n)e^{-2\pi in/3} & x(n-1)e^{-2\pi i(n-1)/3} & x(n-2)e^{-2\pi i(n-2)/3} \\ x(n)e^{-4\pi in/3} & x(n-1)e^{-4\pi i(n-1)/3} & x(n-2)e^{-4\pi i(n-2)/3} \end{bmatrix},$$

and its covariance function $\boldsymbol{K}_Z(n)$ is

$$\boldsymbol{K}_Z(n) = \begin{bmatrix} a_0(n) & a_2(n) & a_1(n) \\ a_1(n)e^{-2\pi in/3} & a_0(n)e^{-2\pi in/3} & a_2(n)e^{-2\pi in/3} \\ a_2(n)e^{-4\pi in/3} & a_1(n)e^{-4\pi in/3} & a_0(n)e^{-4\pi in/3} \end{bmatrix}.$$

$\mathfrak{E}$-mapping. Let E be the $T \times T$ unitary matrix defined by

$$E = (1/\sqrt{T})[e^{2\pi ipq/T}].$$

Define a mapping $\mathfrak{E}$ from $\mathcal{S}(T)$ into $\mathcal{S}(T)$ by

(2.5) $$\mathfrak{E}\big(\boldsymbol{X}(n)\big) = \big(\boldsymbol{W}(n)\big), \quad \text{where } \boldsymbol{W}(n) = E\boldsymbol{X}(n).$$

Because the covariance of $\big(\boldsymbol{W}(n)\big)$ equals $\boldsymbol{R}_W(m,n) = E\boldsymbol{R}_X(m,n)E^{-1}$, the sequence $\boldsymbol{W}$ is clearly stationary provided $\boldsymbol{X}$ is so. Moreover,

$$\boldsymbol{K}_W(n) = E\boldsymbol{K}_X(n)E^{-1}. \tag{2.6}$$

Invertibility of the matrix E and (2.6) implies that the mapping $\mathfrak{E}$ is a bijection of $\mathcal{S}(T)$ onto itself.

LEMMA 2.2. *If $\big(\boldsymbol{X}(n)\big)$ is in the class $\mathcal{SI}(T)$, then $\big(\boldsymbol{W}(n)\big)$ is in the class $\mathcal{SD}(T)$.*

P r o o f. Suppose that $\big(\boldsymbol{X}(n)\big)$ is in $\mathcal{SI}(T)$, that is, that for every $n \in \mathcal{Z}$, $e^{2\pi ijn/T}K_X^{j,k}(n)$ depends on $\langle j-k\rangle$. Write $e^{2\pi ijn/T}K_X^{j,k}(n) = a_{\langle j-k\rangle}(n)$ and for convenience extend $a_m(n)$ periodically to all $m \in \mathcal{Z}$. From (2.6) we obtain

$$\begin{aligned} K_W^{p,q}(n) &= (1/T)\sum_{r=0}^{T-1}\sum_{s=0}^{T-1} e^{2\pi ipr/T}K_X^{r,s}(n)e^{-2\pi isq/T} \\ &= (1/T)\sum_{r=0}^{T-1}\sum_{s=0}^{T-1} e^{2\pi i(pr-sq-rn)/T}a_{r-s}(n) \\ &= (1/T)\sum_{u=0}^{T-1} e^{2\pi iu(p-n)/T}a_u(n)\Big(\sum_{s=0}^{T-1} e^{2\pi i(p-q-n)s/T}\Big). \end{aligned}$$

Since the sum over s is 0 except when $p-n=q$ *modulo* T, we get

$$\begin{aligned} K_W^{p,q}(n) &= \begin{cases} 0 & \text{if } q \neq \langle p-n\rangle, \\ \sum_{u=0}^{T-1} e^{2\pi iuq/T}a_u(n) & \text{if } q = \langle p-n\rangle, \end{cases} \\ &= \begin{cases} 0 & \text{if } q \neq \langle p-n\rangle, \\ R_x(n+q,q) & \text{if } q = \langle p-n\rangle. \end{cases} \end{aligned} \tag{2.7}$$

In particular, this shows that $\boldsymbol{K}_W(n)$ is $\langle n\rangle$-diagonal. ■

EXAMPLE 2.2. If $\big(x(n)\big)$ is PC of period $T=3$, then

$$\begin{aligned} \boldsymbol{W}(n) &= \begin{bmatrix} W^0(n) \\ W^1(n) \\ W^2(n) \end{bmatrix} \\ &= \frac{1}{\sqrt{3}}\begin{bmatrix} x(n)S(n) & x(n-1)S(n-1) & x(n-2)S(n-2) \\ x(n)S(n-1) & x(n-1)S(n-2) & x(n-2)S(n-3) \\ x(n)S(n-2) & x(n-1)S(n-3) & x(n-2)S(n-4) \end{bmatrix}, \end{aligned}$$

and its covariance function is

$$\boldsymbol{K}_W(n) =$$
$$= \frac{1}{3}\begin{bmatrix} R_x(n,0)S(n) & R_x(n+1,1)S(n-2) & R_x(n+2,2)S(n-1) \\ R_x(n,0)S(n-1) & R_x(n+1,1)S(n-3) & R_x(n+2,2)S(n-2) \\ R_x(n,0)S(n-2) & R_x(n+1,1)S(n-4) & R_x(n+2,2)S(n-3) \end{bmatrix},$$

where $S(m) = 3$ if m is a multiple of 3, and 0 otherwise. For example, if $n = 3k+1$, that is, if $\langle n\rangle = 1$, then

$$\boldsymbol{W}(n) = \sqrt{3}\begin{bmatrix} 0 & x(n-1) & 0 \\ x(n) & 0 & 0 \\ 0 & 0 & x(n-2) \end{bmatrix},$$

$$\boldsymbol{K}_W(n) = \begin{bmatrix} 0 & 0 & R_x(n+2,2) \\ R_x(n,0) & 0 & 0 \\ 0 & R_x(n+1,1) & 0 \end{bmatrix}.$$

$\mathfrak{X}$-mapping. Let $\big(\boldsymbol{W}(n)\big)$ be a T-dimensional stochastic sequence, and let $\big(y(n)\big)$ be a one-dimensional sequence defined by

$$y(n) = \frac{1}{\sqrt{T}}\sum_{j=0}^{T-1} W^{\langle n\rangle}(n+j), \qquad n \in \mathcal{Z}. \tag{2.8}$$

If $\big(\boldsymbol{W}(n)\big)$ is stationary, then the covariance function $R_y(m,n)$ of $\big(y(n)\big)$ is given by the formula

$$R_y(n+r,r) = \frac{1}{T}\sum_{j=0}^{T-1}\sum_{k=0}^{T-1} K_W^{\langle n+r\rangle,\langle r\rangle}(n+j-k), \tag{2.9}$$

and hence $\big(y(n)\big)$ is periodically correlated with period T. However, the mapping $\big(\boldsymbol{W}(n)\big) \xrightarrow{\mathfrak{X}} \big(y(n)\big)$ is not one-to-one in general. We will show that $\mathfrak{X}$ becomes injective if it is restricted to the set $\mathcal{SD}(T)$.

LEMMA 2.3. *Suppose that $\big(\boldsymbol{W}(n)\big)$ is in the class $\mathcal{SD}(T)$. Then the covariance function of the process $y(n) = (\mathfrak{X}\boldsymbol{W})(n)$, $n \in \mathcal{Z}$, is*

$$R_y(n+r,r) = K_W^{\langle n+r\rangle,\langle r\rangle}(n). \tag{2.10}$$

Consequently, the mapping $\mathcal{SD}(T) \xrightarrow{\mathfrak{X}} \mathcal{PC}(T)$ is one-to-one.

Proof. By assumption, $\boldsymbol{K}_W(n+j-k)$ is $\langle n+j-k\rangle$-diagonal, that is, $K_W^{\langle n+r\rangle,\langle r\rangle}(n+j-k) = 0$ except when $n+r-r = n+j-k$ *modulo* T. The latter holds iff $k = j$ because both $0 \leqslant k, j < T$. Therefore, the only possibly nonzero terms in the sum (2.9) are when $k = j$, which yields (2.10).

Recall that if $\big(\boldsymbol{W}(n)\big)$ is in $\mathcal{SD}(T)$, then $K_W^{j,k}(n) = 0$, except possibly when $j = \langle n+r\rangle$ and $k = \langle r\rangle$ for some r. Hence, if $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are in $\mathcal{SD}(T)$ and have different covariance functions, then there are n and r such that $K_{W_1}^{\langle n+r\rangle,\langle r\rangle}(n) \neq K_{W_2}^{\langle n+r\rangle,\langle r\rangle}(n)$. By (2.10), this implies that the covariance functions of $x_1 = \mathfrak{X}\boldsymbol{W}_1$ and $x_2 = \mathfrak{X}\boldsymbol{W}_2$ differ at the point $(n+r, r)$. ■

Although the mappings $\mathfrak{I}$, $\mathfrak{E}$, and $\mathfrak{X}$ were defined on concrete stochastic sequences, all three map sequences with different (the same) covariance functions onto sequences with different (the same) covariance functions, respectively. Therefore, they map injectively $\mathcal{PC}(T)$ into $\mathcal{SI}(T)$, $\mathcal{SI}(T)$ into $\mathcal{SD}(T)$, and $\mathcal{SD}(T)$ into $\mathcal{PC}(T)$ again, as shown in the following diagram:

$$\mathcal{PC}(T) \xrightarrow{\mathfrak{I}} \mathcal{SI}(T) \xrightarrow{\mathfrak{E}} \mathcal{SD}(T) \xrightarrow{\mathfrak{X}} \mathcal{PC}(T). \tag{2.11}$$

THEOREM 2.1. *Let T be a fixed natural number. Then all three mappings in* (2.11) *are bijections.*

P r o o f. We have already noted that $\mathfrak{E}$ is a bijection. Injectivity of $\mathfrak{I}$ and $\mathfrak{X}$ was proved in Lemmas 2.1 and 2.3. In order to prove that they are surjections, we will trace the path of a PC sequence under successive transformations

$$\big(x(n)\big) \xrightarrow{\mathfrak{I}} \big(\boldsymbol{Z}(n) = [Z^p(n)]\big) \xrightarrow{\mathfrak{E}} \big(\boldsymbol{W}(n) = [W^p(n)]\big) \xrightarrow{\mathfrak{X}} \big(y(n)\big). \tag{2.12}$$

Suppose that $\big(x(n)\big)$ is a PC sequence in a Hilbert space $\mathcal{H}$ with period T. By definition, $Z^p(n), W^p(n)$ and $y(n)$, $n \in \mathcal{Z}$, are elements of $\mathcal{K} = \ell^2(\mathcal{Z}_T, \mathcal{H})$. Denoting by a^q the q-coordinate of $a = [a^0, a^1, \ldots, a^{T-1}] \in \mathcal{K}$, directly from the definitions we infer that the q-th coordinates of $Z^p(n)$, $W^p(n)$, and $y(n)$ are

$$[Z^p(n)]^q = x(n-q)e^{-2\pi i p(n-q)/T}, \tag{2.13}$$

$$[W^p(n)]^q = \begin{cases} \sqrt{T}\, x(n-q) & \text{if } q = \langle n-p\rangle, \\ 0 & \text{otherwise,} \end{cases} \tag{2.14}$$

$$y^q(n) = (1/\sqrt{T}) \sum_{j=0}^{T-1} [W^{\langle n\rangle}(n+j)]^q = (1/\sqrt{T})[W^{\langle n\rangle}(n+q)]^q, \tag{2.15}$$

respectively. The formula (2.14) implies $[W^{\langle n\rangle}(n+q)]^q = \sqrt{T}x(n)$, and hence from (2.15) it follows that $y(n) = [x(n), \ldots, x(n)]$. Since, clearly, $R_y(m,n) = \big(y(m), y(n)\big)_{\mathcal{K}} = R_x(m,n)$, we conclude that $\mathfrak{X}\big(\mathfrak{E}(\mathfrak{I}x)\big) = x$, remembering that we identify processes with the same covariance function. This shows that the mapping $\mathfrak{X}$ is surjective. Therefore, both $\mathfrak{X}$ and $\mathfrak{E}$ are bijections, and hence $\mathfrak{I} = \mathfrak{E}^{-1} \circ \mathfrak{X}^{-1}$ is also so. ■

REMARK 2.1. If $\big(x(n)\big)$ is a sequence of random variables defined in a probability space Ω, then the induced sequence $\boldsymbol{Z}(n)=[Z^p(n)]$ can be simply defined by the formula

$$Z^p(n)=x(n-\tau)e^{-2\pi ip(n-\tau)/T}, \quad p=0,\ldots,T-1, \tag{2.16}$$

where τ is a random variable uniformly distributed on integers $\{0,1,\ldots,T-1\}$ and independent of the sequence $\big(x(n)\big)$. If τ is defined on Ω', then sequences $\boldsymbol{Z}(n)$, $\boldsymbol{W}(n)$, and $y(n)=\mathfrak{X}\big(\mathfrak{E}(\mathfrak{I}x)\big)(n)$ are defined on $\Omega\times\Omega'$, and by repeating the above computation we obtain

1. $Z^p(n)(\omega,\omega')=x\big(n-\tau(\omega')\big)(\omega)e^{-2\pi ip(n-\tau(\omega'))/T}$;
2. $W^p(n)(\omega,\omega')=\sqrt{T}x(n-\langle n-p\rangle)(\omega)\mathbf{1}_{\{\tau(\omega')=\langle n-p\rangle\}}(\omega')$, where $\mathbf{1}_{\{A\}}$ denotes the indicator of the set A;
3. $y(n)(\omega,\omega')=x(\omega)$.

Hence the sequences $\big(y(n)\big)$ and $\big(x(n)\big)$ have not only the same covariance function, but also have the same finite-dimensional distributions.

REMARK 2.2. The covariance matrix of $\big(\boldsymbol{Z}(n)\big)$ is up to a constant equal to the matrix that appears in [8], Theorem 1. An idea behind the definition of the induced sequence stems from the stationarizing property of a random shift discovered by Hurd [11]. As stated here, the sequence $\big(\boldsymbol{Z}(n)\big)$ first time appeared in [24] for sequences, and in [25] for continuous time PC processes. Some results presented in this note have been announced in [23].

REMARK 2.3. In the case of continuous time PC processes (i.e. PC processes indexed by the set of real numbers) the mapping $\mathfrak{I}$ was studied in [19]. It has turned out that in order to obtain bijectivity of $\mathfrak{I}$ it is necessary to extend the Gladyshev's class of continuous PC processes to include processes which are not necessarily continuous. Since in the continuous time case the process $\boldsymbol{W}$ is meaningless, the proof of the correspondence theorem given in [19] employed a different technique based on Mackey's construction of so-called *induced representation of a group*.

REMARK 2.4. It is possible to define an *induced* process for Almost Periodically Correlated (APC) processes and it was done in [22].

3. COROLLARIES

This section contains simple corollaries from the construction presented in Section 2 and some prediction related properties of the sequences $\boldsymbol{Z}(n)$ and $\boldsymbol{W}(n)$.

First we show that the two basic facts in the theory of PC processes, namely:

- if $\big(x(n)\big)$ is PC then for every j the sequence $a_j(n)$, $n\in\mathcal{Z}$, defined in (1.4), is a Fourier transform of some complex measure γ_j on $\mathcal{R}$,
- every PC sequence of period T is of the form $x(n)=\sum_{p=0}^{T-1}e^{2\pi ipn/T}X^p(n)$, where $X^p(n)$, $p=0,\ldots,T-1$ are components of some T-dimensional stationary sequence $\big(\boldsymbol{X}(n)\big)$,

are both immediate consequences of Theorem 2.1. The presented proofs provide explicit constructions of both the measure (γ_j) and the process $\big(\boldsymbol{X}(n)\big)$.

COROLLARY 3.1 (Gladyshev [8]). *Suppose that* $\big(x(n)\big)$ *is PC with period* T. *Then there are complex measures* γ_j *on* $[0,2\pi)$ *such that*

$$\int_0^{2\pi} e^{int}\gamma_j(dt) = a_j(n), \qquad j=0,\ldots,T-1,\ n\in\mathcal{Z}. \tag{3.1}$$

Proof. By substituting $j=0$ in (2.3) we obtain $a_{\langle -k\rangle(n)} = K_Z^{0,k}(n)$. Therefore, from (1.1) we conclude that

$$a_j(n) = K_Z^{0,\langle -j\rangle}(n) = \int_0^{2\pi} e^{int}F_Z^{0,\langle -j\rangle}(dt), \qquad j=0,\ldots,T-1,\ n\in\mathcal{Z},$$

and hence (3.1) holds true with $\gamma_j = F_Z^{0,\langle -j\rangle}$. ∎

COROLLARY 3.2 (cf. Gladyshev [8]). *Let* $\big(x(n)\big)$ *be a PC sequence in* $\mathcal{H}$ *with period* T. *Then there is a* T*-dimensional stationary sequence* $\boldsymbol{X}(n) = [X^p(n)]$ *in* $\mathcal{H}' \supseteq \mathcal{H}$ *such that for every* $n\in\mathcal{Z}$

$$x(n) = \sum_{p=0}^{T-1} e^{2\pi ipn/T}X^p(n). \tag{3.2}$$

Proof. Note that if $\big(\boldsymbol{Z}(n)\big)$ is the sequence induced by $\big(x(n)\big)$ as in (2.1), then the q-th coordinate of $\sum_{p=0}^{T-1} e^{2\pi ipn/T}Z^p(n)$ is

$$\sum_{p=0}^{T-1} e^{2\pi ipn/T}[Z^p(n)]^q = x(n-q)\sum_{p=0}^{T-1} e^{2\pi ipq/T} = \begin{cases} Tx(n) & \text{if } q=0,\\ 0 & \text{otherwise,}\end{cases}$$

that is, $\sum_{p=0}^{T-1} e^{2\pi ipn/T}Z^p(n) = [Tx(n),0,\ldots,0]$. The latter sequence regarded as a sequence in $\mathcal{K} = \ell^2(\mathcal{Z}_T,\mathcal{H})$ has the same covariance as $\big(x(n)\big)$, which shows that (3.2) holds true with $\boldsymbol{X}(n) = \boldsymbol{Z}(n)$ and $\mathcal{H}' = \mathcal{K} \supseteq \mathcal{H}$. ∎

The family of measures (γ_j), $j=0,\ldots,T-1$, from Corollary 3.1 is called the *spectrum* of a PC sequence $\big(x(n)\big)$. The name comes from the observation that if we transfer the measures γ_j from $[0,2\pi)$ to the square $[0,2\pi)^2$ through the mappings $\varphi_j(s) = \big(s, s-(2\pi j)/T\big)$ (recall that the subtraction is *modulo* 2π), then $\Gamma = \sum_{j=0}^{T-1}\gamma_j\circ\varphi_j^{-1}$ is the spectral measure of $x(n)$ in the harmonizable sense:

$$R_x(m,n) = \int_0^{2\pi}\int_0^{2\pi} e^{i(ms-tn)}\Gamma(ds,dt). \tag{3.3}$$

The formulas (2.3) and (3.1) yield the following relation between spectral measures of $\big(x(n)\big)$ and $\big(\boldsymbol{Z}(n)\big)$.

COROLLARY 3.3. *Let* (γ_j), $j = 0, \dots, T-1$, *be the spectrum of a PC sequence* $\big(x(n)\big)$ *and let* $\boldsymbol{F}_Z = [F_Z^{j,k}]$ *be the spectral measure of the* T*-dimensional stationary sequence* $\big(\boldsymbol{Z}(n)\big)$ *induced by* $\big(x(n)\big)$ *through the formula* (2.1). *Then for every* $j, k = 0, 1, \dots, T-1$ *and the Borel set* Δ

$$F_Z^{j,k}(\Delta) = \gamma_{\langle j-k\rangle}\big(\Delta + (2\pi j)/T\big). \tag{3.4}$$

Also please note a simple relation between covariance functions of $\big(x(n)\big)$ and the sequence $\big(\boldsymbol{W}(n)\big)$:

COROLLARY 3.4. *Let* $\big(x(n)\big)$ *be a PC sequence with period* T *and let* $\boldsymbol{W} = \mathfrak{E}(\mathfrak{I}x)$. *Then for every* $p, q = 0, 1, \dots, T-1$ *and* $n \in \mathcal{Z}$

$$K_W^{p,q}(n) = \begin{cases} 0 & \textit{if } q \neq \langle p-n\rangle, \\ R_x(n+q, q) & \textit{if } q = \langle p-n\rangle. \end{cases} \tag{3.5}$$

A sequence $\big(\boldsymbol{X}(n)\big)$ in the representation (3.2) constructed in the original Gladyshev's paper [8] assumes values in $\mathcal{H}$, while the sequence $\boldsymbol{X}(n) = \boldsymbol{Z}(n)$, $n \in \mathcal{Z}$, used in the proof of Corollary 3.2 lives in much larger space $\mathcal{K} \supset \mathcal{H}$. The advantages of our construction are, however, the simple relationship (3.4) between spectral measures of $\big(x(n)\big)$ and $\big(\boldsymbol{Z}(n)\big)$ and the fact that both sequences share common prediction properties.

For a T-dimensional stochastic sequence $\boldsymbol{X}(n) = [X^p(n)]$, $n \in \mathcal{Z}$, in $\mathcal{H}$, let $M_X(m) = \overline{\mathrm{sp}}\{X^p(k) : k \leqslant m, 0 \leqslant p < T\}$ denote the closed linear subspace of $\mathcal{H}$ spanned by $X^p(k)$, $k \leqslant m$, $0 \leqslant p < T$, and let $M_X = \overline{\mathrm{sp}}\{X^p(k) : k \in \mathcal{Z}, 0 \leqslant p < T\}$. For any element $y \in M_X$, the symbol $\big(y|M_X(m)\big)$ will denote the orthogonal projection of y onto $M_X(m)$. If $\bigcap_m M_X(m) = \{0\}$, then the sequence $\big(\boldsymbol{X}(n)\big)$ is called *regular*. On the other end, if for every $m \in \mathcal{Z}$, $M_X(m) = M_X$, then the sequence is called *deterministic*. If $\big(\boldsymbol{X}(n)\big)$ is stationary and nondeterministic, then the dimension of $M_X(1) \ominus M_X(0)$ is called the *rank of the stationary sequence* $\big(\boldsymbol{X}(n)\big)$.

If $\big(x(n)\big)$ is PC then all the above definitions apply except the last one. The *rank of a PC sequence* $\big(x(n)\big)$ of period T is defined to be the number on nonzero vectors in the set

$$\big\{\big(x(1)|M_x(0)\big), \big(x(2)|M_x(1)\big), \dots, \big(x(T)|M_x(T-1)\big)\big\}.$$

PROPOSITION 3.1. *Let* $\big(x(n)\big)$ *be a PC sequence in* $\mathcal{H}$ *with period* T, $\big(\boldsymbol{Z}(n)\big)$ *be the sequence induced by* $\big(x(n)\big)$, *and let* $\boldsymbol{W}(n) = E\boldsymbol{Z}(n)$, $n \in \mathcal{Z}$. *Then*:

(i) *For every* $n \in \mathcal{Z}$

$$M_{\boldsymbol{Z}}(n) = M_{\boldsymbol{W}}(n) = \big\{y = [y^0, \dots, y^{T-1}] \in \mathcal{K} : y^q \in M_x(n-q)\big\}, \tag{3.6}$$

where $\mathcal{K} = \ell^2(\mathcal{Z}_T, \mathcal{H})$.

(ii) *If one of the sequences* $\big(x(n)\big)$, $\big(\boldsymbol{Z}(n)\big)$, *or* $\big(\boldsymbol{W}(n)\big)$ *is regular* (*deterministic*), *then the other two are so.*

(iii) *All three sequences*: $\big(x(n)\big)$, $\big(\boldsymbol{Z}(n)\big)$, *and* $\big(\boldsymbol{W}(n)\big)$, *have the same rank.*

(iv) *If* $\big(x(n)\big)$ *and* $\big(y(n)\big)$ *are two PC sequences in* $\mathcal{H}$, *and* $\big(\boldsymbol{Z}(n)\big)$ *and* $\big(\boldsymbol{Y}(n)\big)$ *are stationary sequences induced by* $\big(x(n)\big)$ *and* $\big(y(n)\big)$, *respectively, then* $M_x(n) \subseteq M_y(n)$ *for all* $n \in \mathcal{Z}$ *iff* $M_{\boldsymbol{Z}}(n) \subseteq M_{\boldsymbol{Y}}(n)$ *for all* $n \in \mathcal{Z}$.

P r o o f. (i) Let for any $m \in \mathcal{Z}$

$$N_{\boldsymbol{X}}(m) = M_X(m) \ominus M_X(m-1) = \overline{\mathrm{sp}}\{X^p(m) : p = 0, \dots, T-1\}.$$

From (2.13) and (2.14) it follows that, for $p = 0, \dots, T-1$,

$$\begin{aligned}
Z^p(0) &= [x(0), x(-1)e^{2\pi ip/T}, \dots, x(-T+1)e^{2\pi ip(T-1)/T}],\\
W^0(0) &= [x(0), 0, \dots, 0, 0],\\
W^1(0) &= [0, 0, \dots, 0, x(-T+1)],\\
W^2(0) &= [0, 0, \dots, x(-T+2), 0],\\
&\dots\dots\dots\dots\dots\dots\\
W^{T-1}(0) &= [0, x(-1), \dots, 0, 0].
\end{aligned}$$

Hence, by stationarity, for every $m \in \mathcal{Z}$

$$N_{\boldsymbol{Z}}(m) = N_{\boldsymbol{W}}(m) = \{[c_0 x(m), c_1 x(m-1), \dots, c_{m-1} x(m-T+1)] : c_j \in \mathcal{C}\}.$$

Since $M_{\boldsymbol{X}}(n)$ is the space spanned by $\bigcup_{m \leqslant n} N_{\boldsymbol{X}}(m)$, we conclude (3.6).

Parts (ii) and (iv) of the proposition then follow immediately from part (i). In order to see (iii), note that (i) implies the following relations among projections of the corresponding sequences:

$$\begin{aligned}
\big(Z^p(0)|M_{\boldsymbol{Z}}(-1)\big) &= \big[\big(x(0)|M_x(-1)\big), \big(x(-1)|M_x(-2)\big)e^{2\pi ip/T}, \dots,\\
&\qquad \big(x(-T+1)|M_x(-T)\big)e^{2\pi ip(T-1)/T}\big],\\
\big(W^0(0)|M_{\boldsymbol{W}}(-1)\big) &= \big[\big(x(0)|M_x(-1)\big), 0, \dots, 0\big],\\
\big(W^1(0)|M_{\boldsymbol{W}}(-1)\big) &= \big[0, 0, \dots, \big(x(-T+1)|M_x(-T)\big)\big],\\
&\dots\dots\dots\dots\dots\dots\\
\big(W^{T-1}(0)|M_{\boldsymbol{W}}(-1)\big) &= \big[0, \big(x(-1)|M_x(-2)\big), \dots, 0\big].
\end{aligned}$$

Hence the rank of $\big(\boldsymbol{W}(n)\big)$ equals the number of nonzero vectors in the set

$$\big\{\big(x(1)|M_x(0)\big), \big(x(2)|M_x(1)\big), \dots, \big(x(T)|M_x(T-1)\big)\big\},$$

which is equal to the rank of $\big(x(n)\big)$. Because the matrix E is invertible, clearly, ranks of the sequences $\big(\boldsymbol{W}(n)\big)$ and $\big(\boldsymbol{Z}(n)\big)$ are the same. ∎

Proposition 3.1 allows to derive prediction properties of a PC sequence $\big(x(n)\big)$ from the corresponding properties of sequences $\big(\boldsymbol{Z}(n)\big)$ or $\big(\boldsymbol{W}(n)\big)$, to the extent they are available for T-dimensional stationary sequences. Although the same task can be achieved by splitting a PC sequence into blocks of length T, due to a simple relation between spectra of $\big(x(n)\big)$ and $\big(\boldsymbol{Z}(n)\big)$ the conditions and proofs utilizing the induced sequence are often simpler. For example, Proposition 3.1 and a characterization of regular T-dimensional stationary sequences of rank p stated in [27], Section 12, yield the following description of regular PC sequences of rank p. In what follows, $\boldsymbol{A}_I$ will denote the submatrix of a $T \times T$ matrix $\boldsymbol{A}$ made of elements $A^{i,j}$ such that $i, j \in I$, $I \subseteq \{0, \ldots, T-1\}$.

COROLLARY 3.5. *Suppose that $\big(x(n)\big)$ is a PC sequence with period T such that its spectral measures γ_j, $j = 0, \ldots, T-1$, are absolutely continuous with respect to the Lebesgue measure. Let $g_j(t)$ denote the density of γ_j, and let*

$$\boldsymbol{G}(t) =$$
$$\begin{bmatrix} g_0(t) & g_{T-1}(t) & g_{T-2}(t) & \cdots & g_1(t) \\ g_1(t+\lambda_1) & g_0(t+\lambda_1) & g_{T-1}(t+\lambda_1) & \cdots & g_2(t+\lambda_1) \\ g_2(t+\lambda_2) & g_1(t+\lambda_2) & g_0(t+\lambda_2) & \cdots & g_3(t+\lambda_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{T-1}(t+\lambda_{T-1}) & g_{T-2}(t+\lambda_{T-1}) & g_{T-3}(t+\lambda_{T-1}) & \cdots & g_0(t+\lambda_{T-1}) \end{bmatrix},$$

where $\lambda_k = (2\pi k)/T$. Then the sequence $\big(x(n)\big)$ is regular and of rank $p \geqslant 1$ if and only if

(i) $\operatorname{rank} \boldsymbol{G}(t) = p$ *dt-a.e.*,

(ii) *there is a $p \times p$ submatrix $M(t) = \boldsymbol{G}_I(t)$ of $\boldsymbol{G}(t)$ such that*

$$\int_0^{2\pi} |\log \det M(t)| dt < \infty,$$

(iii) *for every $j \notin I$ and $i \in I$,*

$$\frac{\det M_i^j(t)}{\det M(t)} = \lim_{r \to 1^-} \Psi_{j,i}(re^{it}) \ dt\text{-a.e.},$$

where $\Psi_{j,i}$ is in the Nevanlinna class N_δ and $M_i^j(t)$ denotes the matrix $M(t)$ with the i-th row replaced by the corresponding columns of the row j of $\boldsymbol{G}(t)$ (see [27] for details).

As mentioned in [27], the above theorem has only a theoretical meaning.

4. PARMA SYSTEMS

A PARMA system is an infinite system of linear equations

$$(4.1)\qquad x(n) - \sum_{k=1}^{L} \phi_k(n)x(n-k) = \sum_{k=0}^{R} \theta_k(n)\xi_{n-k}, \quad n \in \mathcal{Z},$$

where

(P.1) the scalar sequences $\big(\phi_k(n)\big)$ and $\big(\theta_k(n)\big)$ are periodic in n with the same period $T \geqslant 1$,

(P.2) $\theta_0(n) > 0$ for every $n \in \mathcal{Z}$,

(P.3) $(\xi_n)$is an orthonormal sequence in a Hilbert space$\mathcal{H}$, that is, $(\xi_n, \xi_m) = 1$ if $n = m$, and 0 otherwise.

Any stochastic sequence $\big(x(n)\big)$ in $M_\xi = \overline{\text{sp}}\{\xi_n : n \in \mathcal{Z}\}$ that satisfies (4.1) is called a *solution* of the system (4.1). A solution $\big(x(n)\big)$ is called *causal* if $M_x(n) \subseteq M_\xi(n)$ for all $n \in \mathcal{Z}$, and is called *invertible* if $M_x(n) \supseteq M_\xi(n)$ for all $n \in \mathcal{Z}$. Here and below for any $\mathcal{H}$-valued sequence $\big(y(n)\big)$, $M_y(n) = \overline{\text{sp}}\{y(k) : k \leqslant n\}$. PARMA systems were introduced by Pagano [32] and then studied by various authors: [1]–[3], [15], [18], [33], [37], [38], just to mention few.

A VARMA system is a system of the form

$$(4.2)\qquad \sum_{k=0}^{L} \boldsymbol{A}_k \boldsymbol{X}(n-k) = \sum_{k=0}^{R} \boldsymbol{B}_k \boldsymbol{\xi}_{n-k}, \quad n \in \mathcal{Z},$$

where

(A.1) $\boldsymbol{A}_k$ and $\boldsymbol{B}_k$ are $T \times T$-matrices, $\boldsymbol{A}_0$ is invertible,

(A.2) $\boldsymbol{\xi}_n = [\xi_n^p]$, $n \in \mathcal{Z}$, is a *normalized orthogonal T-dimensional* sequence, that is, $(\boldsymbol{\xi}_n)$ is a stationary T-dimensional sequence whose covariance function $\boldsymbol{K}_\xi(n) = 0$ if $n \neq 0$ and $\boldsymbol{K}_\xi(0) = \boldsymbol{I}$ (the $T \times T$ identity matrix).

Any T-dimensional sequence $\boldsymbol{X}(n) = [X^p(n)]$ that satisfies the system (4.2) and such that for all $n \in \mathcal{Z}$ and $0 \leqslant p < T$, $X^p(n) \in M_{\boldsymbol{\xi}}$, is called a *solution* of the system (4.2). As before, a solution $\big(\boldsymbol{X}(n)\big)$ is called *causal* if $M_{\boldsymbol{X}}(n) \subseteq M_{\boldsymbol{\xi}}(n)$ for all $n \in \mathcal{Z}$, and is called *invertible* if $M_{\boldsymbol{X}}(n) \supseteq M_{\boldsymbol{\xi}}(n)$ for all $n \in \mathcal{Z}$. If the system (4.2) has a unique solution, then it has to be stationary. All information about VARMA systems needed in this paper can be found in [4] or [10].

Note that the existence, uniqueness, and other covariance-related properties of a solution to (4.1) or (4.2) do not depend on the choice of the normalized uncorrelated sequence (ξ_n) or $(\boldsymbol{\xi}_n)$.

The standard way to treat a PARMA system (4.1) is to convert it into VARMA. Namely, if we arrange coefficients of the left-hand side and right-hand side of (4.1) into $T \times lT$ and $T \times rT$ matrices ($l \geqslant 1 + L/T$, $r \geqslant 1 + R/T$) as follows:

$$\begin{bmatrix} 1 & -\phi_1(0) & -\phi_2(0) & -\phi_3(0) & \dots & -\phi_L(0) & 0 & 0 & \dots & 0 \\ 0 & 1 & -\phi_1(1) & -\phi_2(1) & \dots & -\phi_{L-1}(1) & -\phi_L(1) & 0 & \dots & 0 \\ 0 & 0 & 1 & -\phi_1(2) & \dots & -\phi_{L-2}(2) & -\phi_{L-1}(2) & -\phi_L(2) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & -\phi_1(T-1) & \dots & \dots & -\phi_L(T-1) & \dots \end{bmatrix}$$

and

$$\begin{bmatrix} \theta_0(0) & \theta_1(0) & \theta_2(0) & \theta_3(0) & \dots & \theta_R(0) & 0 & 0 & \dots & 0 \\ 0 & \theta_0(1) & \theta_1(1) & \theta_2(1) & \dots & \theta_{R-1}(1) & \theta_R(1) & 0 & \dots & 0 \\ 0 & 0 & \theta_0(2) & \theta_1(2) & \dots & \theta_{R-2}(2) & \theta_{R-1}(2) & \theta_R(2) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \theta_0(T-1) & \theta_1(T-1) & \dots & \dots & \theta_R(T-1) & \dots \end{bmatrix}$$

and denote the consecutive $T \times T$ blocks of the two matrices above by $\overline{\mathbf{\Phi}}_0, \dots, \overline{\mathbf{\Phi}}_{l-1}$ and $\overline{\mathbf{\Theta}}_0, \dots, \overline{\mathbf{\Theta}}_{r-1}$, respectively, then the system (4.1) takes the form of a T-dimensional VARMA system

$$\sum_{k=0}^{l-1} \overline{\mathbf{\Phi}}_k \boldsymbol{V}(n-k) = \sum_{k=0}^{r-1} \overline{\mathbf{\Theta}}_k \boldsymbol{\xi}_{n-k}, \tag{4.3}$$

where

$$\boldsymbol{V}(n) = [x(nT-p)]_{p=T-1}^{0} \quad \text{and} \quad \boldsymbol{\xi}_n = [\xi_{nT-p}]_{p=T-1}^{0}$$

(both are column-vectors). The two systems (4.1) and (4.3) are *parallel* in the sense that (4.1) has a unique (casual or invertible) PC solution $\big(x(n)\big)$ iff (4.3) has a unique (respectively, casual or invertible) stationary solution $\big(\boldsymbol{V}(n)\big)$ (see [32] or [37]).

In this section we show that the mappings $\mathfrak{I}$ and $\mathfrak{E} \circ \mathfrak{I}$ defined in Section 2 transform a PARMA system (4.1) into two other VARMA systems which are parallel to (4.1) in the same sense.

For an arbitrary PARMA system (4.1) let $\mathbf{\Phi}_k$, $\widetilde{\mathbf{\Phi}}_k$, $k = 1, \dots, L$, and $\mathbf{\Theta}_k$, $\widetilde{\mathbf{\Theta}}_k$, $k = 0, \dots, R$, be the $T \times T$ matrices defined as follows:

$$\widetilde{\mathbf{\Phi}}_k^{p,q} = e^{-2\pi i k q/T} \tilde{\phi}_k(p-q), \qquad \widetilde{\mathbf{\Theta}}_k^{p,q} = e^{-2\pi i k q/T} \tilde{\theta}_k(p-q), \tag{4.4}$$

$$\mathbf{\Phi}_k^{p,q} = \begin{cases} \phi_k(p) & \text{if } q = \langle p-k \rangle, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{\Theta}_k^{p,q} = \begin{cases} \theta_k(p) & \text{if } q = \langle p-k \rangle, \\ 0 & \text{otherwise,} \end{cases} \tag{4.5}$$

where $0 \leqslant p, q < T$ and $\tilde{s}(n)$ denotes the discrete Fourier transform of a T-periodic sequence $\big(s(n)\big)$ as defined in (1.2). Let us note that the matrices $\mathbf{\Phi}_k$ and $\mathbf{\Theta}_k$ are $\langle k \rangle$-diagonal.

Consider the following two T-dimensional VARMA systems associated with the system (4.1):

$$\boldsymbol{Z}(n) - \sum_{k=1}^{L} \widetilde{\boldsymbol{\Phi}}_k \boldsymbol{Z}(n-k) = \sum_{k=0}^{R} \widetilde{\boldsymbol{\Theta}}_k \boldsymbol{\zeta}_{n-k} \tag{4.6}$$

and

$$\boldsymbol{W}(n) - \sum_{k=1}^{L} \boldsymbol{\Phi}_k \boldsymbol{W}(n-k) = \sum_{k=0}^{R} \boldsymbol{\Theta}_k \boldsymbol{\eta}_{n-k}, \tag{4.7}$$

where $\widetilde{\boldsymbol{\Theta}}_k$, $\boldsymbol{\Theta}_k$, $k = 0, \dots, R$, and $\widetilde{\boldsymbol{\Phi}}_k$, $\boldsymbol{\Phi}_k$, $k = 1, \dots, L$, are as above, and $\boldsymbol{\eta}_n$ and $\boldsymbol{\zeta}_n$ are uncorrelated T-dimensional normalized sequences (Example 4.1 at the end of this section illustrates the above construction in the case of a PARMA(2,1) system and $T = 3$.) We will show that

THEOREM 4.1. *For an arbitrary PARMA system* (4.1) *the following conditions are equivalent*:

(i) *The system* (4.1) *has a unique solution* $\big(x(n)\big)$.

(ii) *The VARMA system* (4.6) *has a unique solution* $\big(\boldsymbol{Z}(n)\big)$.

(iii) *The VARMA system* (4.7) *has a unique solution* $\big(\boldsymbol{W}(n)\big)$.

If one of these three conditions is satisfied, *then* $\big(x(n)\big)$ *is PC with period* T *and both* $\big(\boldsymbol{Z}(n)\big)$ *and* $\big(\boldsymbol{W}(n)\big)$ *are stationary. Moreover*, *if one of the three solutions* $\big(x(n)\big)$, $\big(\boldsymbol{Z}(n)\big)$ *or* $\big(\boldsymbol{W}(n)\big)$ *is causal or invertible*, *then the other two are so*, *respectively.*

The proof relies on the following three observations:

1. If $\big(x(n)\big)$ is a PC solution of the PARMA system (4.1), then the sequence $\boldsymbol{Z}(n) = \mathfrak{I}x(n), n \in \mathcal{Z}$, induced by $\big(x(n)\big)$ *via* the formula (2.1) is a stationary solution of the VARMA system (4.6).

2. $\big(\boldsymbol{Z}(n)\big)$ is a stationary solution of the VARMA system (4.6) if and only if the sequence $\mathfrak{E}\big(\boldsymbol{Z}(n)\big) = \big(\boldsymbol{W}(n)\big)$ defined in (2.5) is a stationary solution of the VARMA system (4.7).

3. If $\big(\boldsymbol{W}(n)\big)$ is a stationary solution of the VARMA system (4.7), then the process $x(n) = (\mathfrak{X}\boldsymbol{W})(n)$ defined in (2.8) is a PC solution of the PARMA system (4.1).

Details of the proof are in the Appendix.

Relations between characteristics of the solutions $\big(x(n)\big)$, $\big(\boldsymbol{Z}(n)\big)$, $\big(\boldsymbol{W}(n)\big)$ are summarized below. They all are obvious consequences of the formulas (2.3), (3.4), (2.7), and (2.6).

THEOREM 4.2. *Suppose that* $\big(x(n)\big)$, $\big(\boldsymbol{Z}(n)\big)$ *or* $\big(\boldsymbol{W}(n)\big)$ *are the unique solutions of* (4.1), (4.6), *and* (4.7), *respectively. Let* $R_x(m, n)$ *and* (γ_j) *be the covariance function and the spectrum of the PC sequence* $\big(x(n)\big)$, *and let* $\boldsymbol{K}_Z$, $\boldsymbol{F}_Z$ *and* $\boldsymbol{K}_W$, $\boldsymbol{F}_W$ *denote the covariance functions and spectral measures of the stationary sequences* $\big(\boldsymbol{Z}(n)\big)$ *and* $\big(\boldsymbol{W}(n)\big)$, *respectively. Then the measures* (γ_j), $\boldsymbol{F}_Z$

and $\boldsymbol{F}_W$ *are absolutely continuous with respect to Lebesgue measure*, *and*

(i) $$\frac{dF_Z^{p,q}}{dt}(t) = \frac{d\gamma_{\langle p-q\rangle}}{dt}(t+2\pi p/T),$$

(ii) $$\frac{dF_W^{p,q}}{dt}(t) = (1/T)\sum_{j=0}^{T-1} e^{2\pi ij(p-q)/T}\Bigg(\sum_{k=0}^{T-1} e^{2\pi ikq/T}\frac{d\gamma_k}{dt}(t+2\pi j/T)\Bigg).$$

Moreover,

(iii) $$K_Z^{p,q}(n) = e^{-2\pi ipn/T}a_{p-q}(n), \quad \textit{where } a_j(n) \textit{ is given in } (1.4),$$

(iv) $$K_W^{p,q}(n) = \begin{cases} 0 & \textit{if } q \neq \langle p-n\rangle, \\ R_x(n+q,q) & \textit{if } q = \langle p-n\rangle. \end{cases}$$

Theorems 4.1 and 4.2 allow us to obtain explicit conditions for existence of a unique PC solution to (4.1), and its density, in terms of Fourier transforms $\tilde{\phi}_k$ and $\tilde{\theta}_k$ of the system coefficients.

COROLLARY 4.1. *Let* $\widetilde{\mathbf{\Phi}}(z) = I - \sum_{k=1}^{L}\widetilde{\mathbf{\Phi}}_k z^k$, $\widetilde{\mathbf{\Theta}}(z) = \sum_{k=0}^{R}\widetilde{\mathbf{\Theta}}_k z^k$, *and* $\mathbf{\Phi}(z) = I - \sum_{k=1}^{L}\mathbf{\Phi}_k z^k$, $\mathbf{\Theta}(z) = \sum_{k=0}^{R}\mathbf{\Theta}_k z^k$ *be the characteristic polynomials of the left-hand and right-hand sides of the systems* (4.6) *and* (4.7), *respectively. Then*:

(i) *The system* (4.1) *has a unique PC solution* $\big(x(n)\big)$ *iff all zeros of* $\det\widetilde{\mathbf{\Phi}}(z)$ (*or* $\det\mathbf{\Phi}(z)$) *are different than* 1.

(ii) *If all zeros of* $\det\widetilde{\mathbf{\Phi}}(z)$ (*or* $\det\mathbf{\Phi}(z)$) *are outside of the unit disk*, *then the solution* $\big(x(n)\big)$ *is causal.*

(iii) *If*, *additionally*, *all zeros of* $\det\widetilde{\mathbf{\Theta}}(z)$ (*or* $\det\mathbf{\Theta}(z)$) *are outside of the unit disk*, *then the solution is invertible.*

COROLLARY 4.2. *Suppose that the system* (4.1) *has a unique solution* $\big(x(n)\big)$. *Then* $\big(x(n)\big)$ *is PC and the spectral densities of* $\big(x(n)\big)$ *are given by*

(4.8) $$\frac{d\gamma_k}{dt}(t) = f^{0,T-k}(t), \quad k = 0,\ldots,T-1,$$

where $\boldsymbol{f}(t) = 1/(2\pi)\widetilde{\mathbf{\Phi}}(e^{-it})^{-1}\widetilde{\mathbf{\Theta}}(e^{-it})\big(\widetilde{\mathbf{\Phi}}(e^{-it})^{-1}\widetilde{\mathbf{\Theta}}(e^{-it})\big)^*$ *and* $\widetilde{\mathbf{\Phi}}(z)$ *and* $\widetilde{\mathbf{\Theta}}(z)$ *are the characteristic polynomials of the system* (4.6).

We finish this section with an example that illustrates the construction of the systems (4.6) and (4.7).

EXAMPLE 4.1. If $T = 3$, $L = 2$, and $R = 1$, then (4.1) reads

(4.9) $$x(n) - \phi_1(n)x(n-1) - \phi_2(n)x(n-2) = \theta_0(n)\xi_n + \theta_1(n)\xi_{n-1}.$$

The associated induced system (4.6) is

$$\boldsymbol{Z}(n) - \widetilde{\mathbf{\Phi}}_1\boldsymbol{Z}(n-1) - \widetilde{\mathbf{\Phi}}_2\boldsymbol{Z}(n-2) = \widetilde{\mathbf{\Theta}}_0\boldsymbol{\zeta}_n + \widetilde{\mathbf{\Theta}}_1\boldsymbol{\zeta}_{n-1},$$

and its coefficients are given by

$$\widetilde{\mathbf{\Phi}}_1 = \begin{bmatrix} \tilde{\phi}_1(0) & \tilde{\phi}_1(2)e^{-2\pi i/3} & \tilde{\phi}_1(2)e^{-4\pi i/3} \\ \tilde{\phi}_1(1) & \tilde{\phi}_1(0)e^{-2\pi i/3} & \tilde{\phi}_1(2)e^{-4\pi i/3} \\ \tilde{\phi}_1(2) & \tilde{\phi}_1(1)e^{-2\pi i/3} & \tilde{\phi}_1(0)e^{-4\pi i/3} \end{bmatrix},$$

$$\widetilde{\mathbf{\Phi}}_2 = \begin{bmatrix} \tilde{\phi}_2(0) & \tilde{\phi}_2(2)e^{-4\pi i/3} & \tilde{\phi}_2(2)e^{-2\pi i/3} \\ \tilde{\phi}_2(1) & \tilde{\phi}_2(0)e^{-4\pi i/3} & \tilde{\phi}_2(2)e^{-2\pi i/3} \\ \tilde{\phi}_2(2) & \tilde{\phi}_2(1)e^{-4\pi i/3} & \tilde{\phi}_2(0)e^{-2\pi i/3} \end{bmatrix},$$

$$\widetilde{\mathbf{\Theta}}_0 = \begin{bmatrix} \tilde{\theta}_0(0) & \tilde{\theta}_0(2) & \tilde{\theta}_0(1) \\ \tilde{\theta}_0(1) & \tilde{\theta}_0(0) & \tilde{\theta}_0(2) \\ \tilde{\theta}_0(2) & \tilde{\theta}_0(1) & \tilde{\theta}_0(0) \end{bmatrix}, \quad \widetilde{\mathbf{\Theta}}_1 = \begin{bmatrix} \tilde{\theta}_1(0) & \tilde{\theta}_1(2)e^{-2\pi i/3} & \tilde{\theta}_1(1)e^{-4\pi i/3} \\ \tilde{\theta}_1(1) & \tilde{\theta}_1(0)e^{-2\pi i/3} & \tilde{\theta}_1(2)e^{-4\pi i/3} \\ \tilde{\theta}_1(2) & \tilde{\theta}_1(1)e^{-2\pi i/3} & \tilde{\theta}_1(0)e^{-4\pi i/3} \end{bmatrix}.$$

The associated diagonalized system (4.7) is

$$\boldsymbol{W}(n) - \mathbf{\Phi}_1\boldsymbol{W}(n-1) - \mathbf{\Phi}_2\boldsymbol{W}(n-2) = \mathbf{\Theta}_0\boldsymbol{\eta}_n + \mathbf{\Theta}_1\boldsymbol{\eta}_{n-1},$$

and its coefficients are given by

$$\mathbf{\Phi}_1 = \begin{bmatrix} 0 & 0 & \phi_1(0) \\ \phi_1(1) & 0 & 0 \\ 0 & \phi_1(2) & 0 \end{bmatrix}, \quad \mathbf{\Phi}_2 = \begin{bmatrix} 0 & \phi_2(0) & 0 \\ 0 & 0 & \phi_2(1) \\ \phi_2(2) & 0 & 0 \end{bmatrix},$$

$$\mathbf{\Theta}_0 = \begin{bmatrix} \theta_0(0) & 0 & 0 \\ 0 & \theta_0(1) & 0 \\ 0 & 0 & \theta_0(2) \end{bmatrix}, \quad \mathbf{\Theta}_1 = \begin{bmatrix} 0 & 0 & \theta_1(0) \\ \theta_1(1) & 0 & 0 \\ 0 & \theta_1(2) & 0 \end{bmatrix}.$$

Corollary 4.1 states in this case that if all zeros of the polynomial

$$\det \mathbf{\Phi}(z) = \det \begin{bmatrix} 1 & -\phi_2(0)z^2 & -\phi_1(0)z \\ -\phi_1(1)z & 1 & -\phi_2(1)z^2 \\ -\phi_2(2)z^2 & -\phi_1(2)z & 1 \end{bmatrix} = 1 - az^3 - bz^6$$

are outside of the unit circle, then the system (4.9) has a unique PC solution, which is additionally casual. Here $a = \phi_1(0)\phi_1(1)\phi_1(2) + \phi_1(0)\phi_2(2) + \phi_1(2)\phi_2(1) + \phi_1(1)\phi_2(0)$, and $b = \phi_2(0)\phi_2(1)\phi_2(2)$.

5. APPENDIX

Proof of Theorem 4.1. First we show that (ii) ⇔ (iii). Let $E = (1/\sqrt{T}) \times [e^{2\pi ipq/T}]$ be as in Section 2. Since the (p,q)-th entry of $E\widetilde{\mathbf{\Phi}}_k$ equals

$$[E\widetilde{\mathbf{\Phi}}_k]^{p,q} = \frac{1}{T\sqrt{T}} \sum_{r=0}^{T-1}\sum_{s=0}^{T-1} e^{2\pi is(p-r)/T} e^{2\pi iq(r-k)/T}\phi_k(r)$$
$$= \frac{1}{\sqrt{T}} e^{2\pi iq(p-k)/T}\phi_k(p),$$

we have $E\widetilde{\mathbf{\Phi}}_k = \mathbf{\Phi}_k E$. Similarly, $E\widetilde{\mathbf{\Theta}}_k = \mathbf{\Theta}_k E$, $E^{-1}\mathbf{\Phi}_k = \widetilde{\mathbf{\Phi}}_k E^{-1}$, and $E^{-1}\mathbf{\Theta}_k = \widetilde{\mathbf{\Theta}}_k E^{-1}$. If $\big(\boldsymbol{Z}(n)\big)$ satisfies (4.6), then by left-multiplying both sides of (4.6) by E, we conclude from the above that the sequence $\boldsymbol{W}(n) = E\boldsymbol{Z}(n)$ satisfies (4.7) with $\boldsymbol{\eta}_k = E\boldsymbol{\zeta}_k$. Conversely, if $\boldsymbol{W}(n)$ satisfies (4.7), then $\big(\boldsymbol{Z}(n)\big) = \big(E^{-1}\boldsymbol{W}(n)\big)$ satisfies the equation (4.6) with $(\boldsymbol{\zeta}_k) = (E^{-1}\boldsymbol{\eta}_k)$. From the relation $\boldsymbol{R}_{EX}(m,n) = E\boldsymbol{R}_X(m,n)E^{-1}$ it also follows that if $\big(\boldsymbol{Z}(n)\big)$ is a unique solution of (4.6), then $\big(\boldsymbol{W}(n)\big)$ is a unique solution of (4.7), and vice versa.

To show that (iii) $\Rightarrow$ (i) suppose that $\big(\boldsymbol{W}(n)\big)$ is a unique solution to (4.7). Then $\big(\boldsymbol{W}(n)\big)$ is stationary (see e.g. [10]). Since $\mathbf{\Phi}_k$ is $\langle k\rangle$-diagonal (see (4.5)), the p-th coordinate of $\mathbf{\Phi}_k\boldsymbol{W}(n-k)$ equals

$$[\mathbf{\Phi}_k\boldsymbol{W}(m-k)]^p = \sum_{q=0}^{T-1} \mathbf{\Phi}_k^{p,q} W^q(m-k) = \phi_k(p)W^{\langle p-k\rangle}(m-k),$$

and, similarly, $[\mathbf{\Theta}_k\boldsymbol{\eta}_{m-k}]^p = \theta_k(p)\eta_{m-k}^{\langle p-k\rangle}$. Therefore, for every $p = 0,\dots,T-1$,

$$W^p(m) - \sum_{k=1}^{L} \phi_k(p)W^{\langle p-k\rangle}(m-k) = \sum_{k=0}^{R} \theta_k(p)\eta_{m-k}^{\langle p-k\rangle}.$$

By substituting $m = n+j$ and $p = \langle n\rangle$ in the above equation, we obtain

$$W^{\langle n\rangle}(n+j) - \sum_{k=1}^{L} \phi_k(n)W^{\langle n-k\rangle}(n-k+j) = \sum_{k=0}^{R} \theta_k(n)\eta_{n-k+j}^{\langle n-k\rangle}. \tag{5.1}$$

Consequently, for every $j=0,\dots,T-1$, the scalar sequence $x^{(j)}(n) = W^{\langle n\rangle}(n+j)$ satisfies (4.1) with the innovation $\xi_n^{(j)} = \eta_{n+j}^{\langle n\rangle}$, $n \in \mathcal{Z}$, which is orthogonal and normalized because the sequences $\eta^p(n)$, $n \in \mathcal{Z}$, $0 \leqslant p < T$, are orthonormal. The latter also implies that the sequences $x^{(j)}(n) = W^{\langle n\rangle}(n+j)$, $j = 0,\dots,T-1$, lie in mutually orthogonal spaces. Hence

$$R_W^{\langle n\rangle,\langle m\rangle}(n+j, m+k) = \big(W^{\langle n\rangle}(n+j), W^{\langle m\rangle}(m+k)\big) = 0$$

if $k \neq j$, that is, $R_W^{p,q}(n,m) = 0$ whenever $p-q \neq n-m$ *modulo* T. If $\big(\boldsymbol{W}(n)\big)$ is additionally stationary, the latter means that $K_W^{p,q}(n) = 0$ except possibly when $q = \langle p-n\rangle$, so that $\big(\boldsymbol{W}(n)\big)$ belongs to the class $\mathcal{SD}(T)$. Adding equations (5.1) for $j = 0,\dots,T-1$, we conclude that $x(n) = (1/\sqrt{T})\sum_{j=0}^{T-1} W^{\langle n\rangle}(n+j)$ satisfies (4.1) with the innovation $\xi_n = (1/\sqrt{T})\sum_{j=0}^{T-1}\eta_{n+j}^{\langle n\rangle} = (1/\sqrt{T})\sum_{j=0}^{T-1}\xi_n^{(j)}$.

Suppose that $\big(\boldsymbol{W}_i(n)\big)$, $i = 1,2$, are two different solutions of (4.7). Then there is $0 \leqslant p < T$ and m such that $W_1^p(m) \neq W_2^p(m)$. Let $0 \leqslant j < T$ be such that $\langle m-j\rangle = p$. Putting $n = m-j$ we obtain $W_1^{\langle n\rangle}(n+j) \neq W_2^{\langle n\rangle}(n+j)$, and hence from (5.1) we conclude that $x_1(n) = W_1^{\langle n\rangle}(n+j)$ and $x_2(n) = W_2^{\langle n\rangle}(n+j)$ are two different solutions of (4.1).

(i) ⇒ (ii). To complete the circle suppose that $\big(x(n)\big)$ is a unique solution to (4.1). Let U be the unitary operator M_ξ defined by $U\xi_m = \xi_{m+1}$, $m \in \mathcal{Z}$. By applying U^T to both sides of (4.1) we see that the sequence $x'(n) = U^T x(n)$ satisfies (4.1) with an innovation $\xi'_n = \xi_{T+n}$, $n \in \mathcal{Z}$. If we now replace n by $n+T$ in (4.1), from uniqueness we conclude that $U^T x(n) = x(n+T)$, which shows that $\big(x(n)\big)$ is PC.

Suppose now that $\big(x(n)\big)$ satisfies (4.1). Note that if $y(n) = a(n)x(n-k)$, $n \in \mathcal{Z}$, where k is fixed and $\big(a(n)\big)$ is an arbitrary sequence of scalars, then the sequence $\boldsymbol{Y}(n) = (\mathfrak{I}y)(n)$ induced by $\big(y(n)\big)$ is given by

$$\begin{aligned} Y^p(n) &= a(n-\tau)x(n-k-\tau)e^{-2\pi i p(n-\tau)/T} \\ &= \sum_{j=0}^{T-1} \tilde{a}(j)e^{-2\pi i(p-j)(n-\tau)/T}x(n-k-\tau) \\ &= \sum_{q=0}^{T-1} \tilde{a}(p-q)e^{-2\pi i qk/T}Z^q(n-k), \end{aligned}$$

where $\boldsymbol{Z}(n) = [Z^p(n)]$ is the stationary sequence induced by $\big(x(n)\big)$. Applying the operator $\mathfrak{I}$ to both sides of (4.1) we conclude that for every $p = 0, \ldots, T-1$

$$Z^p(n) - \sum_{k=1}^{L}\sum_{q=1}^{T-1} \widetilde{\boldsymbol{\Phi}}_k^{p,q} Z^q(n-k) = \sum_{k=0}^{R}\sum_{q=1}^{T-1} \widetilde{\boldsymbol{\Theta}}_k^{p,q} [\mathfrak{I}\xi)]^q(n-k).$$

Hence $\big(\boldsymbol{Z}(n)\big)$ satisfies (4.6) with $\boldsymbol{\zeta}_n = (\mathfrak{I}\xi)_n$, which is clearly uncorrelated and normalized (see (2.2)). The solution $\big(\boldsymbol{Z}(n)\big)$ is unique because if there were two different stationary solutions $\big(\boldsymbol{Z}_1(n)\big)$ and $\big(\boldsymbol{Z}_2(n)\big)$, then from previously proved parts it would follow that $x_1 = \mathfrak{X}\big(\mathfrak{C}(\boldsymbol{Z}_1)\big)$ and $x_2 = \mathfrak{X}\big(\mathfrak{C}(\boldsymbol{Z}_2)\big)$ would be two different PC solutions to (4.1).

The "moreover" part of the theorem follows from properties of the mappings $\mathfrak{I}$, $\mathfrak{C}$, and $\mathfrak{X}$. ■

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