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GRADIENT PERTURBATIONS OF THE SUM OF TWO FRACTIONAL LAPLACIANS

BY

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Abstract. We study the gradient perturbations of $\Delta^{\alpha/2}+\Delta^{\beta/2}$ with $0<\beta<\alpha<2$ and $\alpha>1$.

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1. INTRODUCTION

Let $d\in\mathbb{N}$ and $0<\beta<\alpha<2$. We consider the transition density $p(s,x,t,y)=p(t-s,y-x)$, where $-\infty<s<t<\infty$, $x,y\in\mathbb{R}^d$, and

$$(1.1)\qquad p(t,x)=(2\pi)^{-d}\int\limits_{\mathbb{R}^d}\exp\Big(-t\big(|\xi|^\alpha+|\xi|^\beta\big)\Big)e^{-ix\cdot\xi}d\xi,\quad x\in\mathbb{R}^d,\ t>0.$$

The infinitesimal generator of the convolution semigroup $p(t,\cdot)$ is $\Delta^{\alpha/2}+\Delta^{\beta/2}$ ([1], [12], [2]). We note that p is the transition density of a subordinate Brownian motion; see [10] and [11]. We consider $b=(b_i)_{i=1}^d\colon\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$ (the drift function) and we are interested in the existence and estimates of the transition density $\tilde{p}$ corresponding to $\Delta^{\alpha/2}+\Delta^{\beta/2}+b\cdot\nabla$. By perturbation theory we expect that

$$(1.2)\qquad \tilde{p}(s,x,t,y)=\sum_{n=0}^{\infty}p_n(s,x,t,y),$$

where $p_0(s,x,t,y)=p(s,x,t,y)$, and for $n=1,2,\ldots$

$$p_n(s,x,t,y)=\int\limits_s^t\int\limits_{\mathbb{R}^d}p_{n-1}(s,x,u,z)b(u,z)\cdot\nabla_z p(u,z,t,y)dzdu.$$

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Estimation of the series (1.2) requires proper assumptions on b. Following [9] we will assume that $b \in \mathcal{N}(\eta, Q, p)$, i.e. there are $\eta \geqslant 0$ and (a finite superadditive function) $Q\colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$ such that

$$Q(r,u) + Q(u,v) \leqslant Q(r,v), \qquad r < u < v,$$

and for all $-\infty < s < t < \infty$, $x, y \in \mathbb{R}^d$, we have

(1.3)
$$\int_s^t \int_{\mathbb{R}^d} p(s,x,u,z)|b(u,z)||\nabla_z p(u,z,t,y)|\,dz\,du \leqslant [\eta + Q(s,t)]p(s,x,t,y).$$

If $b \in \mathcal{N}(\eta, Q, p)$ with $\eta < 1/2$, Theorem 1 in [9] yields that the series in (1.2) converges absolutely and defines a transition density. Furthermore, there exists a constant $C = C(\eta) > 1$ such that for all $-\infty < s < t < \infty$, $x, y \in \mathbb{R}^d$, we have

$$C^{-(1+Q(s,t))} \leqslant \frac{\tilde{p}(s,x,t,y)}{p(s,x,t,y)} \leqslant \begin{cases} \big(1/(1-2\eta)\big)^{1+Q(s,t)/\eta} & \text{if } 0 < \eta < \frac{1}{2}, \\ e^{Q(s,t)} & \text{if } \eta = 0. \end{cases}$$

We note that so far no further restrictions on $0 < \beta < \alpha < 2$ were necessary. In fact, that result applies to very general transition densities with sufficient regularity (see [9]), and drift functions b satisfying (1.3). Here we should note that nonzero constant drift functions fail to satisfy (1.3) for the Gaussian transition density on the real line. Therefore, we see that further analysis of p is required to exhibit functions b satisfying the condition (1.3). For instance, if $1 < \beta < \alpha < 2$, then $\mathcal{N}\big(\eta, c(t-s), p\big)$ contains the Kato class $\mathcal{K}_d^{\beta-1}$, as proved in [9]. Recall that for $\gamma \in (1,2)$ we say that $b(u,z) = b(z)$ belongs to the *Kato class* $\mathcal{K}_d^{\gamma-1}$ if

$$\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^d} \int_{|z-x|<\varepsilon} |b(z)||z-x|^{\gamma-(d+1)}\,dz = 0.$$

We will strengthen and extend this result. In particular, $\mathcal{K}_d^{\alpha-1} \subset \mathcal{N}\big(\eta, c(t-s), p\big)$ provided $0 < \beta < \alpha < 2$ and $\alpha > 1$. This is proved in Theorem 1.1 below. Let us note that the extension is not obvious because $\Delta^{\beta/2}$ can be of lower differential order than $b \cdot \nabla$. The result shows the usefulness of a new approach, which will be now described. Namely, we will replace the condition (1.3) by the condition (2.5) below for a suitably chosen majorant $\hat{p}$ of the gradient of the transition density. This is obtained by the 3P inequality for p, given in Lemma 2.1, and by the following inequality given in Corollary 2.1,

(1.4)
$$p(u,x)\hat{p}(r,y) \leqslant Cp(u+r, x+y)\big(\hat{p}(u,x) + \hat{p}(r,y)\big).$$

The technique of perturbation series is then applied to p and b, with appropriate majorization provided by $\hat{p}$. For more information on the perturbation series we refer the reader to [4], [3], [8], [9] (We also refer to [7] and [5] for the results on gradient perturbations by zero-divergence drift functions.)

We now fix p as defined in the first paragraph of the Introduction where we let $0 < \beta < \alpha < 2$ and $\alpha > 1$. We observe that, for $b \in \mathcal{N}(\eta, Q, p)$ with $\eta < 1/2$ and finite superadditive Q,

$$\int_s^\infty \int_{\mathbb{R}^d} \tilde{p}(s,x,u,z)[\partial_u \phi(u,z) + (\Delta_z^{\alpha/2} + \Delta_z^{\beta/2})\phi(u,z) + b(u,z) \cdot \nabla_z \phi(u,z)]\, dz du = -\phi(s,x)$$

for all $s \in \mathbb{R}$, $x \in \mathbb{R}^d$ and $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$. We will skip the proof of this fact, which is now standard, and for more details we refer to the proof of Theorem 2 in [9], the proof of Theorem 1 in [8] and (2.1) below.

Here is our main result.

THEOREM 1.1. *Let* $0 < \beta < \alpha < 2$, $\alpha > 1$ *and let* $b(u,z) = b(z) \in \mathcal{K}_d^{\alpha-1}$. *Then for every* $\eta > 0$ *there exists a constant* $C = C(d, \alpha, \beta, b, \eta) > 0$ *such that* $b \in \mathcal{N}(\eta, Q, p)$ *with* $Q(s,t) = C(t-s)$.

For instance, the drift function $b(x) = |x|^{1-\alpha+\varepsilon}$ produces a $\tilde{p}$, which is (locally in time) comparable with p. The proof of the theorem, auxiliary estimates and the discussion of the condition (1.3) are given in the next section.

2. THE CLASS OF PERTURBATIONS

Recall that $0 < \beta < \alpha < 2$, $\alpha > 1$. By writing $f(x) \approx g(x)$ we mean that there is a number $0 < C = C(d, \alpha, \beta) < \infty$ such that for every x we have $C^{-1} f(x) \leqslant g(x) \leqslant C f(x)$. We let $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$. For $t > 0$ and $x \in \mathbb{R}^d$ we put

$$\hat{p}(t,x) = t^{-1/\alpha} p(t,x).$$

It is known that (see [6] and [9], respectively)

$$p(t,x) \approx (t^{-d/\alpha} \wedge t^{-d/\beta}) \wedge \left(\frac{t}{|x|^{d+\alpha}} + \frac{t}{|x|^{d+\beta}} \right)$$

on $(0,\infty) \times \mathbb{R}^d$, and that there exists a constant $C = C(d, \alpha, \beta) > 0$ such that, for all $t > 0$ and $x \in \mathbb{R}^d$,

$$|\nabla_x p(t,x)| \leqslant C (t^{-1/\alpha} \wedge t^{-1/\beta}) p(t,x). \tag{2.1}$$

The main tool for the study of the inequality (1.3) are various versions of 3P-type inequalities for majorants of $|\nabla p(t,x)|$. Unfortunately, the function on the right-hand side of (2.1) does not itself satisfy such an inequality when $0 < \beta < 1$. In the next lemma we will see that $\hat{p}$ is a more convenient majorant. We first note that

$$(t^{-1/\alpha} \wedge t^{-1/\beta}) p(t,x) \leqslant \hat{p}(t,x). \tag{2.2}$$

LEMMA 2.1. *There exists a constant $C = C(d,\alpha,\beta) > 0$ such that for all $0 < u, r < \infty$ and $x, y \in \mathbb{R}^d$ we have*

$$\hat{p}(u,x) \wedge \hat{p}(r,y) \leqslant C\hat{p}(u+r, x+y).$$

Proof. We have

$$\hat{p}(u,x) \wedge \hat{p}(r,y) \leqslant c(u^{-(d/\alpha+1/\alpha)} \wedge r^{-(d/\alpha+1/\alpha)}) \wedge \left(\frac{u^{1-1/\alpha}}{|x|^{d+\alpha}} + \frac{u^{1-1/\alpha}}{|x|^{d+\beta}}\right)$$
$$\wedge \left(u^{-(d/\beta+1/\alpha)} \wedge r^{-(d/\beta+1/\alpha)}\right) \wedge \left(\frac{r^{1-1/\alpha}}{|y|^{d+\alpha}} + \frac{r^{1-1/\alpha}}{|y|^{d+\beta}}\right).$$

We notice that

$$u^{-(d/\alpha+1/\alpha)} \wedge r^{-(d/\alpha+1/\alpha)} \leqslant \left(\frac{u+r}{2}\right)^{-(d/\alpha+1/\alpha)} \leqslant c\,(u+r)^{-(d/\alpha+1/\alpha)},$$

and a similar inequality holds for d/β in place of d/α. Hence

$$u^{-(d/\alpha+1/\alpha)} \wedge r^{-(d/\alpha+1/\alpha)} \leqslant c\,(u+r)^{-1/\alpha}\,(u+r)^{-d/\alpha},$$
$$u^{-(d/\beta+1/\alpha)} \wedge r^{-(d/\beta+1/\alpha)} \leqslant c\,(u+r)^{-1/\alpha}\,(u+r)^{-d/\beta}.$$

This gives

$$(2.3)\quad (u^{-(d/\alpha+1/\alpha)} \wedge r^{-(d/\alpha+1/\alpha)}) \wedge (u^{-(d/\beta+1/\alpha)} \wedge r^{-(d/\beta+1/\alpha)})$$
$$\leqslant c\,(u+r)^{-1/\alpha}\left((u+r)^{-d/\alpha} \wedge (u+r)^{-d/\beta}\right).$$

Since $u^{1-1/\alpha} \leqslant (u+r)^{1-1/\alpha}$ and $r^{1-1/\alpha} \leqslant (u+r)^{1-1/\alpha}$, we obtain

$$(2.4)\quad \left(\frac{u^{1-1/\alpha}}{|x|^{d+\alpha}} + \frac{u^{1-1/\alpha}}{|x|^{d+\beta}}\right) \wedge \left(\frac{r^{1-1/\alpha}}{|y|^{d+\alpha}} + \frac{r^{1-1/\alpha}}{|y|^{d+\beta}}\right)$$
$$\leqslant \left(\frac{(u+r)^{1-1/\alpha}}{|x|^{d+\alpha}} + \frac{(u+r)^{1-1/\alpha}}{|x|^{d+\beta}}\right) \wedge \left(\frac{(u+r)^{1-1/\alpha}}{|y|^{d+\alpha}} + \frac{(u+r)^{1-1/\alpha}}{|y|^{d+\beta}}\right)$$
$$\leqslant c\,(u+r)^{-1/\alpha}\left(\frac{u+r}{|x+y|^{d+\alpha}} + \frac{u+r}{|x+y|^{d+\beta}}\right).$$

Finally, by (2.3) and (2.4), we get

$$\hat{p}(u,x) \wedge \hat{p}(r,y) \leqslant c\,(u+r)^{-1/\alpha}\Bigg(\left((u+r)^{-d/\alpha} \wedge (u+r)^{-d/\beta}\right)$$
$$\wedge \left(\frac{u+r}{|x+y|^{d+\alpha}} + \frac{u+r}{|x+y|^{d+\beta}}\right)\Bigg) \leqslant c\,\hat{p}(u+r, x+y). \quad \blacksquare$$

COROLLARY 2.1. *There exists a constant* $C = C(d, \alpha, \beta) > 0$ *such that* (1.4) *holds for all* $0 < u, r < \infty$ *and* $x, y \in \mathbb{R}^d$.

Proof. For any $a, b \geqslant 0$ we have $ab = (a \wedge b)(a \vee b)$ and $(a \vee b) \leqslant (a + b)$. We rewrite the right-hand side of (1.4), use Lemma 2.1 and apply the inequality $u^{1/\alpha}(u + r)^{-1/\alpha} \leqslant 1$, thus obtaining

$$u^{1/\alpha}\,\hat{p}(u, x)\hat{p}(r, y) \leqslant C_3 u^{1/\alpha}\,\hat{p}(u + r, x + y)\big(\hat{p}(u, x) + \hat{p}(r, y)\big). \quad \blacksquare$$

Finally, we return to the fundamental condition (1.3). The inequalities (2.1), (2.2) and Corollary 2.1 give the existence of a constant $C = C(d, \alpha, \beta) > 0$ such that for all $-\infty < s < t < \infty$ and $x, y \in \mathbb{R}^d$ we have

$$\begin{aligned} (2.5)\qquad &\int_s^t \int_{\mathbb{R}^d} p(s, x, u, z)|b(u, z)||\nabla_z p(u, z, t, y)|\, dzdu \\ &\qquad \leqslant C\Big(\int_s^t \int_{\mathbb{R}^d} \big(\hat{p}(s, x, u, z) + \hat{p}(u, z, t, y)\big)|b(u, z)|\, dzdu\Big) p(s, x, t, y), \end{aligned}$$

where $\hat{p}(s, x, t, y) = \hat{p}(t - s, y - x)$, $s < t$, $x, y \in \mathbb{R}^d$. The inequality (2.5) deserves attention, since it will play a crucial role in verifying the conditions of the form $b \in \mathcal{N}(\eta, Q, p)$.

Proof of Theorem 1.1. Notice that

$$(2.6)\qquad \int_0^t \hat{p}(u, x)\, du \approx \Big(\frac{t^{2-1/\alpha}}{|x|^{d+\alpha}} + \frac{t^{2-1/\alpha}}{|x|^{d+\beta}}\Big) \wedge \big(|x|^{\alpha-(d+1)} \wedge |x|^{\beta-d-\beta/\alpha}\big).$$

Let us write $\beta_1 = 2 - 1/\alpha$ and $\beta_2 = (\beta + \alpha - 1)/\alpha$. By (2.6), for $t \leqslant 1$,

$$\begin{aligned} \int_0^t \hat{p}(u, x)\, du &\leqslant c\Big(\frac{t^{2-1/\alpha}}{|x|^{d+\alpha}} \wedge |x|^{\alpha-(d+1)} + \frac{t^{2-1/\alpha}}{|x|^{d+\beta}} \wedge |x|^{\alpha-(d+1)}\Big) \\ &= c\Big(\frac{t^{\beta_1}}{|x|^{d+1-\alpha+\alpha\beta_1}} \wedge |x|^{\alpha-(d+1)} + \frac{t^{\beta_2}\, t^{1-\beta/\alpha}}{|x|^{d+1-\alpha+\alpha\beta_2}} \wedge |x|^{\alpha-(d+1)}\Big) \\ &\leqslant c\Big(\frac{t^{\beta_1}}{|x|^{d+1-\alpha+\alpha\beta_1}} \wedge |x|^{\alpha-(d+1)} + \frac{t^{\beta_2}}{|x|^{d+1-\alpha+\alpha\beta_2}} \wedge |x|^{\alpha-(d+1)}\Big). \end{aligned}$$

Since $\beta_1 > (\alpha - 1)/\alpha$ and $\beta_2 > (\alpha - 1)/\alpha$, by Corollary 12 in [4], for any $\eta > 0$ there exists $h > 0$ such that for all $t - s < h$ and $x, y \in \mathbb{R}^d$ we have

$$\int_s^t \int_{\mathbb{R}^d} \big(\hat{p}(s, x, u, z) + \hat{p}(u, z, t, y)\big)|b(z)|\, dzdu \leqslant \eta/C.$$

Therefore, using (2.5), we get $b \in \mathcal{N}(\eta, Q, p)$ with $Q(s, t) = \eta(t - s)/h$ (see Definition 2 and further comments in [9]). $\blacksquare$

Summarizing, the condition (1.3), which was introduced in [9], defines a broad class of acceptable drift functions, while the present Lemma 2.1, Corollary 2.1 and the inequality (2.5) enable us to explicitly exhibit such functions. We see that the majorant $\hat{p}$, although not the least possible, simplifies the algebra of the perturbation series and yields specific results. This observation indicates further developments, in particular extensions of Theorem 1.1.

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