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COMPARISON THEOREMS FOR SMALL DEVIATIONS OF WEIGHTED SERIES*

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Abstract. We study comparison theorems for small deviation probabilities of weighted series and obtain more refined versions of the known comparison results. In particular, the following consequence is obtained immediately from Theorem 2.1 of the paper.

Let a positive random variable X belong to the domain of attraction of a stable law with an index greater than one and let its distribution function be regularly varying at zero with an exponent $\beta > 0$. If $\{X_n\}_{n\geqslant 1}$ are independent copies of X, and $\{a_n\}$ and $\{b_n\}$ are positive summable sequences such that $\sum_{n\geqslant 1} |1 - a_n/b_n| < \infty$, then as $r \to 0^+$

$$\mathbf{P}\big(\sum_{n\geqslant 1} a_n X_n < r\big) \sim \big(\prod_{n\geqslant 1} b_n/a_n\big)^\beta \mathbf{P}\big(\sum_{n\geqslant 1} b_n X_n < r\big).$$

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1. INTRODUCTION

We start with the well-known result of Li [6]:

THEOREM 1.1. *Let $\{a_n\}$ and $\{b_n\}$ be positive summable sequences such that*

$$\sum_{n\geqslant 1} |1 - a_n/b_n| < \infty. \tag{1.1}$$

If $\{Z_n\}$ is a sequence of i.i.d. standard Gaussian random variables, then as $\varepsilon \to 0$

$$\mathbf{P}\big(\sum_{n\geqslant 1} a_n Z_n^2 < \varepsilon^2\big) \sim \big(\prod_{n\geqslant 1} b_n/a_n\big)^{1/2} \mathbf{P}\big(\sum_{n\geqslant 1} b_n Z_n^2 < \varepsilon^2\big). \tag{1.2}$$

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Here and later on $x(\varepsilon) \sim y(\varepsilon)$ as $\varepsilon \to 0$ means $\lim_{\varepsilon\to 0} x(\varepsilon)/y(\varepsilon) = 1$.

The above comparison theorem is a very useful tool for investigations of the small ball probabilities of a centered Gaussian process in L_2-norm (see, for instance, [5] as an example of such an approach; other results on this subject can be found in the bibliography on small deviation probabilities compiled by Lifshits in [8]).

It looks natural to extend Li's theorem on a larger class of positive independent and identically distributed random variables, instead of Z_n^2. This question was raised in [3]. To recall the results of [3] (namely, Theorems 3 and 4), we have to use some notation and conditions.

In what follows, let $\{X_n\}_{n\geqslant 1}$ be independent copies of a positive random variable X with distribution function $F(x) = \mathbf{P}(X < x)$.

As earlier in [7], introduce the following conditions:

Condition L. There exist constants $b \in (0,1),\ c_1,\ c_2 > 1$ and $\varepsilon > 0$ such that for each $r \leqslant \varepsilon$ the relation $c_1\, F(br) \leqslant F(r) \leqslant c_2\, F(br)$ holds.

Note that **L** is obviously satisfied if $F(1/\cdot) \in \mathbf{R}_\alpha$, the class of regularly varying (at infinity) functions of order $\alpha < 0$.

Denote the Laplace transform of X by $I(s) = \mathbf{E}e^{-sX}$ and set $f(s) = \log I(s)$.

Condition I. The function $sf'(s)$ is of bounded variation on $[0,\infty)$.

This assumption played the key role in [1], where explicit forms of the small deviation probabilities of weighted series were obtained.

Note that **I** holds iff $\int_{[0,\infty)} \left|\big(sf'(s)\big)'\right| ds < \infty$, which, in turn, is equivalent to the assumption that the function $\big(sf'(s)\big)'$ is absolutely integrable at infinity. To explain this fact, observe that for any fixed positive a

$$
\begin{aligned}
(1.3)\qquad \int_0^a \left|\big(sf'(s)\big)'\right| ds &\leqslant \int_0^a \big(-f'(s) + s\,f''(s)\big)\,ds \\
&= \big(f(0) - f(a)\big) + \big(f'(a) - f(a)\big) < \infty.
\end{aligned}
$$

Also notice that **I** guarantees the existence of a finite non-positive constant

$$
(1.4)\qquad \alpha = \lim_{s\to\infty} sf'(s) = -\lim_{s\to\infty} s^2 f''(s) = \lim_{s\to\infty} s^3 f'''(s)/2.
$$

Moreover (see [1] again), if $F \in \mathbf{L} \cap \mathbf{I}$, then the assumption $(*)$ $F(1/\cdot) \in \mathbf{R}_\alpha$ holds and $\alpha < 0$.

It is worthwhile to point out that (1.4) with $\alpha \leqslant 0$ is equivalent to $(*)$. Indeed, the first part of (1.4) obviously implies $I(\cdot) \in \mathbf{R}_\alpha$, which is equivalent to $(*)$ (see Feller [2], Chapter XIII, Section 5, Theorems 2 and 3). The inverse assertion admits the direct verification.

Now recall the accurate statement of Theorem 3 from [3].

THEOREM 1.2. *Let X have a finite variance and an absolute continuous distribution F satisfying* $\mathbf{L}$ *and* $\mathbf{I}$. *Let $\{a_n\}$ and $\{b_n\}$ be positive, non-increasing, summable sequences. If* (1.1) *holds, then as $r \to 0^+$*

$$\mathbf{P}\Big(\sum_{n\geqslant 1} a_n X_n < r\Big) \sim \Big(\prod_{n\geqslant 1} a_n/b_n\Big)^{\alpha} \mathbf{P}\Big(\sum_{n\geqslant 1} b_n X_n < r\Big). \tag{1.5}$$

The next reasonable step, undertaken in [3] and [4], was to weaken the assumption (1.1).

So, in [4] (also see the corrections) it was proved that (1.2) still holds true if $\{a_n\}$ and $\{b_n\}$ are positive, non-increasing, summable sequences and $\prod_{n\geqslant 1} a_n/b_n$ converges.

Recall the revised (taking into account the corrections) version of one more result in [3].

THEOREM 1.3. *Assume that the conditions of Theorem* 1.2 *hold and that $\prod_{n\geqslant 1} b_n/a_n$ converges. Further, suppose that either* (i) $s\, I^{(j+1)}(s)/I^{(j)}(s),\ j = 0, 1, 2,$ *are bounded monotone on* $[0, \infty)$ *or* (ii) $s\, I^{(j+1)}(s)/I^{(j)}(s),\ j = 0, 1, 2,$ *are functions of bounded variation on* $[0, \infty)$, *and* $\sum_{n\geqslant 1}(1 - a_n/b_n)^2 < \infty$. *Then* (1.5) *holds.*

Our main purpose is to represent the versions of Theorems 1.2 and 1.3 with milder restrictions on F. In particular, we remove the redundant assumption of the absolute continuity and relax the moment condition $\mathbf{E}X^2 < \infty$ up to the stochastic compactness condition of Feller:

Condition F. $\limsup_{s\to\infty} s^2\big(1 - F(s)\big)/\mathbf{E}X^2\, \mathbf{1}[X < s] < \infty.$

Note that **F** is obviously satisfied when X is in the domain of attraction of any stable law. Let us also mention (see [10]) that **F** is equivalent to the assumption that $s^{-\omega}\mathbf{E}(1 \wedge s\, X^2)$ does not decrease as $s > 0$, for some $\omega \in (0, 1)$, and that implies, in particular, the existence of a positive δ such that $\mathbf{E}X^{\delta} < \infty$.

Concerning Theorem 1.2, in addition, we succeeded to replace **I**, given in terms of the Laplace transform $I(s)$, by the assumption that $F(1/\cdot) \in \mathbf{R}_{\alpha}$ for some $\alpha < 0$.

Our results, namely, Theorems 2.1–2.4, are contained in Section 2. Section 3 contains the proofs of Theorems 2.1 and 2.2. In Section 4 we prove Theorems 2.3 and 2.4, and the Remarks.

2. RESULTS

In what follows the distribution F satisfies the condition **F**, $\{a_n\}$ and $\{b_n\}$ are positive sequences such that $\prod_{n\geqslant 1} a_n/b_n$ converges, i.e.

$$0 < \prod_{n\geqslant 1} a_n/b_n < \infty; \tag{2.1}$$

$S_a = \sum_{n\geqslant 1} a_n\, X_n$, $S_b = \sum_{n\geqslant 1} b_n\, X_n$, and, moreover,

$$\mathbf{P}(S_b < \infty) = 1. \tag{2.2}$$

The condition (2.2), by the Three Series Theorem, is equivalent to

$$\sum_{n\geqslant 1} \mathbf{E}(1 \wedge b_n\, X) < \infty, \tag{2.3}$$

and if $\mathbf{E}X < \infty$, then (2.3) implies $\{b_n\}$ to be a summable sequence, and *vice versa*. Therefore, the respective series in Theorems 1.2 and 1.3 converge with probability one.

Observe also that (2.3) imposes some moment restrictions on the random variable X. For instance, if $b_n = n^{-\omega},\ \omega > 1$, then (2.3) $\Leftrightarrow \mathbf{E}X^{1/\omega} < \infty$, and if $b_n = q^n,\ 0 < q < 1$, then (2.2) $\Leftrightarrow \mathbf{E}\log(1+X) < \infty$.

The following result is a refinement of Theorem 1.2 (recall that in all theorems of this section we assume that $\mathbf{F}$, (2.1) and (2.2) are satisfied).

THEOREM 2.1. *Assume that the distribution* $F(1/\cdot) \in \mathbf{R}_\alpha$ *for some* $\alpha < 0$. *If* (1.1) *holds*, *then*

$$\mathbf{P}(r - s \leqslant S_a < r) \sim \Big(\prod_{n\geqslant 1} a_n/b_n\Big)^\alpha\, \mathbf{P}(S_b < r) \qquad \text{as } r \to 0^+ \tag{2.4}$$

uniformly in $s \geqslant \delta\, r$ *for any positive* δ.

REMARK 2.1. *If*, *under the conditions* $\mathbf{F}$ *and* $\mathbf{L}$, (1.5) *holds true for some positive sequence* $\{b_n\}$ *and any positive sequence* $\{a_n\}$, *then* $F(1/\cdot) \in \mathbf{R}_\alpha$. *In other words*, *the last condition is necessary*, *to some extent.*

Remind that the proofs of all the Remarks are given in Section 4.

Note that Theorem 2.1 implies $\mathbf{P}(r - s \leqslant S_b < r) \sim \mathbf{P}(S_b < r),\ r \to 0^+$, uniformly in $s \geqslant \delta\, r$ for any positive δ. Actually, this fact holds true under the conditions $\mathbf{L}$, $\mathbf{F}$ and (2.2) (see [10]).

Now let us formulate the refined version of Theorem 1.3 (ii).

THEOREM 2.2. *Assume that the distribution* F *satisfies* $\mathbf{I}$ *with* $\alpha < 0$ (see (1.4)). *If* $\{b_n\}$ *is non-increasing and*

$$\sum_{n\geqslant 1} (1 - a_n/b_n)^2 < \infty, \tag{2.5}$$

then (2.4) *holds.*

Let us stress that we do not require $\{a_n\}$ and $\{b_n\}$ to be non-increasing *simultaneously*.

Also note that the conditions of Theorem 2.2 implicitly include the assumption

$$\sum_{n\geqslant 1} (1 - a_n/b_n) < \infty \tag{2.6}$$

(actually, both (2.1) and (2.6) are satisfied simultaneously iff (2.5) holds true).

REMARK 2.2. *We mentioned already that* **I** *with* $\alpha \leqslant 0$ *implies* $F(1/\cdot) \in \mathbf{R}_\alpha$. *It is of special interest to point out the conditions on* F *that insure the validity of* **I**. *Some classes of such regularly varying functions* F *were described in Chapter* 5 *of* [1]. *We propose the alternative condition* **R** *under which* **I** *holds.*

Condition R. *The function* $u\,\big(\log F(u)\big)'$ *tends monotonically to* $-\alpha$ *for some* $\alpha < 0$ *as* $u \searrow 0$.

It is worthwhile to mention that **R** holds if the function $p(ut)/p(u)$, with $p(u) = F'(u)$, tends monotonically to $t^{-\alpha-1}$ for any $t \in (0,1]$ as $u \searrow 0$ or, in turn, if the following condition holds:

Condition R′. *The function* $u\,\big(\log p(u)\big)'$ *tends monotonically to* $-\alpha - 1$ *as* $u \searrow 0$.

It is obvious that the distribution F of $|Z|^p,\ p > 0$, where Z is a standard Gaussian random variable, and gamma $\Gamma(a)$ and beta $B(a,b)$ distributions satisfy **R′** (with $-\alpha = 1/p,\ a$ and a, respectively).

The results below are mainly interesting when (2.5) is violated (recall that (2.1) holds).

For positive s let us put $D_1(s) = -s\,I(s)/I(s)$, $D_2(s) = s^2\,I''(s)/I(s)$.

The first assertion refines and generalizes Theorem 1.3 (i).

THEOREM 2.3. *Assume that*

$$D_j(s),\ j = 1,2,\ \textit{are bounded monotone on } [0,\infty) \tag{2.7}$$

and also let $\{a_n\}$ *and* $\{b_n\}$ *be two non-increasing sequences. Then* (2.4) *follows provided* $\alpha = -\lim_{s\to\infty} D_1(s)$.

REMARK 2.3. *The following special case of the condition* **R** *is sufficient for* (2.7):

Condition $\mathbf{R}_\infty$. *The function* $u\big(\log F(u)\big)'$ *is bounded and monotonically decreases at* $(0,\infty)$.

Note that $\mathbf{R}_\infty$ holds if $u\big(\log p(u)\big)'$ decreases at $(0,\infty)$ and (see the notation in **R′** and below) that the distribution function of $|Z|^p,\ p > 0$, as well as gamma and beta $B(a,b),\ b \geqslant 1$, distributions have the same property.

COROLLARY 2.1. *Let* $\{Z_n\}$ *be a sequence of i.i.d. standard Gaussian random variables, and let* $\{b_n\}$ *and* $\{a_n\}$ *be non-increasing summable sequences such that* $\prod_{n\geqslant 1} a_n/b_n$ *converges. Then for any positive* p *as* $\varepsilon \to 0^+$

$$\mathbf{P}\big(\sum_{n\geqslant 1} a_n\,|Z_n|^p < \varepsilon^p\big) \sim \big(\prod_{n\geqslant 1} b_n/a_n\big)^{1/p}\,\mathbf{P}\big(\sum_{n\geqslant 1} b_n\,|Z_n|^p < \varepsilon^p\big).$$

This result was formulated in [3], Corollary 1. The correct version of the proof is available from http://www.webpages.uidaho.edu/ fuchang/.

The assertion below can be useful when both the conditions (2.5) and (2.7) do not hold.

THEOREM 2.4. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two non-increasing sequences, and assume that a constant* α *in* (1.4) *is negative*,

$$D_j'(s),\ j = 1, 2, \text{ are absolutely integrable at infinity} \tag{2.8}$$

and, furthermore,

$$\lim_{\gamma\to\infty} \sum_{n\geqslant 1} \rho_n \Big| \int\limits_{\gamma b_n}^{\gamma a_n} |D_1'(s)|\, ds \Big| = 0, \tag{2.9}$$

$$\lim_{\gamma\to\infty} \sum_{n\geqslant 1} \rho_n^3 \Big| \int\limits_{\gamma b_n}^{\gamma a_n} |D_2'(s)|\, ds \Big| = 0, \tag{2.10}$$

where $\rho_n = |1 - a_n/b_n|$. *Then* (2.4) *with* $\alpha = -\lim_{s\to\infty} D_1(s)$ *holds true.*

Note that (2.8) for $j = 1$ coincides with **I**, and also that (2.8) follows from **R**.

REMARK 2.4. *The "technical" conditions* (2.9) *and* (2.10) *are satisfied if*

$$M_{n+k} \leqslant m_n \quad \text{for all } n \geqslant 1 \text{ and some integer } k \geqslant 1, \tag{2.11}$$

where $M_n = \max(a_n, b_n)$ *and* $m_n = \min(a_n, b_n)$.

The latter criterion holds true in the case $\limsup_{n\to\infty} b_{n+1}/b_n < 1$. Thus, for such b_n Theorem 2.4 (not Theorem 2.3) works.

COROLLARY 2.2. *Let the assumptions of Theorem* 2.4 (*without* (2.9) *and* (2.10)) *be satisfied and, in addition*, $\limsup_{n\to\infty} b_{n+1}/b_n < 1$. *Then* (2.4) *holds true.*

3. PROOFS OF THEOREMS 2.1 AND 2.2

Let us define (see the notation before the condition **I**)

$$h_a(\gamma) = \sum_{n\geqslant 1} f(a_n\gamma) \quad \text{and} \quad h_b(\gamma) = \sum_{n\geqslant 1} f(b_n\gamma),$$

and set

$$m_a(\gamma) = -h_a'(\gamma) = -\sum_{n\geqslant 1} a_n\, f'(a_n\gamma), \quad m_b(\gamma) = -h_b'(\gamma) = -\sum_{n\geqslant 1} b_n\, f'(b_n\gamma),$$

$$\sigma_a^2(\gamma) = h_a''(\gamma) = \sum_{n\geqslant 1} a_n^2\, f''(a_n\gamma), \quad \sigma_b^2(\gamma) = h_b''(\gamma) = \sum_{n\geqslant 1} b_n^2\, f'(b_n\gamma),$$

$$\tau_a^2(\gamma) = \gamma^2\, \sigma_a^2(\gamma), \quad \tau_b^2(\gamma) = \gamma^2\, \sigma_b^2(\gamma).$$

LEMMA 3.1. *Let* (2.3) *hold and the distribution* F *satisfy* $\mathbf{L}$ *and* $\mathbf{F}$. *Then as* $\gamma \to \infty$

$$\begin{aligned}&\mathbf{P}(r-s \leqslant S_b < r)\\ &\qquad = \exp\big(\gamma\, r + h_b(\gamma)\big)\,\frac{1-e^{-\gamma s}}{\tau_b(\gamma)\sqrt{2\pi}}\,\big(\exp(-\beta^2/2) + \, o(1)(1+1/\gamma s)\big)\end{aligned}$$

uniformly in $r>0$ *and* $s>0$, *where* $\beta = \big(r-m_b(\gamma)\big)/\sigma_b(\gamma)$.

LEMMA 3.2. *If the condition* $\mathbf{L}$ *holds*, *then for all* $h \geqslant 1$

$$c \leqslant h\big(-f'(h)\big) \leqslant C, \quad c \leqslant h^2 f''(h) \leqslant C, \quad h^3|f'''(h)| \leqslant C;$$

if the condition $\mathbf{F}$ *holds*, *then for all* $h \in (0,1]$

$$c \leqslant h\big(-f'(h)\big)/G_1(h) \leqslant C, \quad c \leqslant h^2 f''(h)/G(h) \leqslant C, \quad h|f'''(h)|/f''(h) \leqslant C,$$

provided $G(h) = \mathbf{E}\big(1 \wedge (h\,X)^2\big)$, $G_1(h) = \mathbf{E}(1 \wedge h\,X)$, *where positive constants* c *and* C *do not depend on* h.

Lemmas 3.1 and 3.2 follow from [10], Theorems 2.3 and 3.1, and Lemma 3, respectively.

Let us start the proofs.

Henceforth we assume that (2.1) and (2.3) hold, and $\gamma = \gamma(r)$ is the solution of the equation $m_b(\gamma) = r$.

We have $a_n \sim b_n$ and, therefore, $\mathbf{P}(S_a < \infty) = 1$. Moreover, Lemma 3.2 gives

$$\gamma m_b(\gamma) \geqslant c\,\#\{n : \gamma b_n \geqslant 1\} \to \infty \qquad \text{as } \gamma \to \infty.$$

This implies for $s \geqslant \delta r$, provided a fixed $\delta > 0$,

$$\gamma\, s = \frac{s}{r}\,\gamma m_b(\gamma) \to \infty \qquad \text{as } r \to 0^+.$$

Therefore, applying Lemma 3.1 twice, we obtain as $r \to 0^+$

(3.1)

$$\frac{\mathbf{P}(r-s \leqslant S_a < r)}{\mathbf{P}(S_b < r)} = \exp\big(h_a(\gamma) - h_b(\gamma)\big)\,\frac{\tau_b(\gamma)}{\tau_a(\gamma)}\,\big(\exp(-\beta^2/2) + \, o(1)\big)$$

uniformly in $s \geqslant \delta r$, where

$$\beta(\gamma) = \frac{r-m_a(\gamma)}{\sigma_a(\gamma)} = \frac{m_b(\gamma)-m_a(\gamma)}{\sigma_a(\gamma)}. \tag{3.2}$$

Let us prove Theorems 2.1 and 2.2, and, at first, show that if (2.5) and (2.6) hold, then as $r \to 0^+$

$$\tau_a(\gamma) \sim \tau_b(\gamma) \tag{3.3}$$

and

$$\beta(\gamma) = o\,(1). \tag{3.4}$$

For positive u write $q(u) = -uf'(u)$ and $g(u) = u^2 f''(u)$. We have, with $\rho_n = (a_n - b_n)/b_n$,

$$\begin{aligned} |\tau_a^2(\gamma) - \tau_b^2(\gamma)| &\leqslant \sum_{n\geqslant 1} |g(a_n\gamma) - g(b_n\gamma)| \\ &\leqslant \sum_{n\geqslant 1} |\rho_n|\,(b_n\gamma/u_n)\,\big(2g(u_n) + u_n^3\,|f'''(u_n)|\big) \end{aligned} \tag{3.5}$$

for some u_n between γa_n and γb_n.

Taking into account Lemma 3.2, together with the simple estimate

$$g(u_n) \leqslant Cc^{-1}\big(g(a_n\gamma) + g(b_n\gamma)\big) \tag{3.6}$$

and Schwarz's inequality, we easily deduce by (3.5)

$$|\tau_a^2(\gamma) - \tau_b^2(\gamma)| \leqslant A\,\big(\tau_a^2(\gamma) + \tau_b^2(\gamma)\big)^{1/2}\,\Big(\sum_{n\geqslant 1}\rho_n^2\Big)^{1/2},$$

where $A = (2 + C)\,C\,c^{-1/2}\,\max_{n\geqslant 1} b_n/(a_n \wedge b_n)$. Since $\tau_a^2(\gamma) + \tau_b^2(\gamma) \to \infty$ as $\gamma \to \infty$, the relation (3.3) follows.

Let us check (3.4). We have

$$\gamma\big(m_a(\gamma) - m_b(\gamma)\big) = \Big(\sum_{n=1}^{N} + \sum_{n>N}\Big)\big(q(a_n\gamma) - q(b_n\gamma)\big) = I_1 + I_2. \tag{3.7}$$

Next, with some u_n between $a_n\gamma$ and $b_n\gamma$,

$$q(a_n\gamma) - q(b_n\gamma) = \rho_n\,q(b_n\gamma) - \rho_n\,g(u_n)\,(\gamma^2\,a_n b_n/u_n^2)$$

and, by Lemma 3.2, $q^2(u) \leqslant C^2c^{-1}g(u),\ u > 0$. This, together with (3.5) and Schwarz's inequality, implies

$$|I_2| \leqslant Cc^{-1/2}\Big(\sum_{n>N}\rho_n^2\Big)^{1/2}\left(\tau_b(\gamma) + \max_{n>N}\left(\frac{b_n}{a_n} \vee \frac{a_n}{b_n}\right)\big(\tau_a^2(\gamma) + \tau_b^2(\gamma)\big)^{1/2}\right). \tag{3.8}$$

Now (3.4) follows from (3.6) (see also (1.4), (3.8) and (3.3)) provided N is growing to infinity slowly enough as $r \to 0^+$.

Furthermore, we have

$$h_a(\gamma) - h_b(\gamma) = \Big(\sum_{n=1}^{N} + \sum_{n>N}\Big)\big(f(a_n\gamma) - f(b_n\gamma)\big) = J_1 + J_2. \tag{3.9}$$

If N grows to infinity slowly enough as $r \to 0^+$, then by (1.4) with $\alpha < 0$

$$J_1 \to -\alpha \sum_{n=1}^{N} \log\,(a_n/b_n) \to \log\,\Big(\prod_{n\geqslant 1} b_n/a_n\Big)^{\alpha} \qquad \text{as } r \to 0^+. \tag{3.10}$$

Now,

$$J_2 = -\sum_{n>N} \rho_n\, q(b_n\gamma) + \sum_{n>N} \frac{1}{2}\,\rho_n^2\, g(u_n) = J_{21} + J_{22} \tag{3.11}$$

with u_n between γa_n and γb_n.

Lemma 3.2 obviously implies

$$|J_{22}| \leqslant (C/2) \sum_{n>N} \rho_n^2 \tag{3.12}$$

and

$$|J_{21}| \leqslant C \sum_{n>N} |\rho_n|. \tag{3.13}$$

Thus, the assertion of Theorem 2.1 follows from the relations (3.3), (3.4), (3.9)–(3.13) and (3.1). ■

For proving Theorem 2.2 it is sufficient only to refine (3.13). To do it we use the following elementary fact.

LEMMA 3.3. *We have*

$$\Big|\sum_{n>N} \alpha_n\, \beta_n\Big| \leqslant \sup_{i\geqslant N} \Big|\sum_{n>i} \beta_n\Big| \sum_{n>N} |\alpha_n - \alpha_{n-1}|$$

provided $\alpha_N = 0$.

Lemma 3.3 (see also Lemma 5 in [3]) gives

$$|J_{21}| \leqslant A_1 \sup_{i\geqslant N} \Big|\sum_{n>i} \rho_n\Big|. \tag{3.14}$$

where $A_1 = \int_{[0,\infty)} |(sf'(s))'|\, ds < \infty$, and Theorem 2.2 follows. ■

4. PROOFS OF THEOREMS 2.3 AND 2.4 AND OF THE REMARKS

We need to change some details in the reasonings of Section 3.

At first, let us improve (3.5). We have, for $q(u) = -uf'(u)$ and $g(u) = u^2 f''(u)$ again,

$$\tau_a^2(\gamma) - \tau_b^2(\gamma) = \Big(\sum_{n=1}^{N} + \sum_{n>N}\Big) \log\,(a_n/b_n)\, R(u_n) = I_1 + I_2$$

provided $R(u) = ug'(u)$ and u_n between $a_n\gamma$ and $b_n\gamma$.

If N grows to infinity slowly enough as $r \to 0^+$, then $I_1 \to 0$ by (1.4) and, by virtue of Lemmas 3.3 and 3.2,

$$|I_2| \leqslant \sup_{i \geqslant N} \Big| \sum_{n>i} \log\left(a_n/b_n\right)\Big| \cdot 2 \sum_{n>N} R(u_n) = o\left(\tau_b^2(\gamma)\right) \quad \text{as } r \to 0^+.$$

Thus, (3.3) follows.

Now let us evaluate anew J_2 from (3.9). Setting $a_n = (1+r_n)\, b_n$, we find

$$J_2 = -\sum_{n>N} \int_{\gamma b_n}^{\gamma a_n} q(s)\, ds/s = -\sum_{n>N} \int_1^{1+r_n} q(\gamma b_n\, s)\, ds/s. \tag{4.1}$$

We have

$$\begin{aligned} \int_1^{1+r_n} q(\gamma b_n\, s)\, ds/s &= q(\gamma a_n) \log\left(1+r_n\right) - \int_1^{1+r_n} \log s\, dq(\gamma b_n\, s) \\ &= q(\gamma b_n) \log\left(1+r_n\right) + \int_1^{1+r_n} \big(q(\gamma b_n\, s) - q(\gamma b_n)\big)\, ds/s. \end{aligned} \tag{4.2}$$

Let us assume for a while that (2.7) is satisfied. Then $q(\cdot)$ grows from zero to $|\alpha|$ on $[0,\infty)$ and, therefore, (4.2) and (4.1) imply

$$\sum_{n>N} q(\gamma b_n) \log\left(1+r_n\right) \leqslant -J_2 \leqslant \sum_{n>N} q(\gamma a_n) \log\left(1+r_n\right).$$

This and Lemma 3.3 yield

$$|J_2| \leqslant |\alpha| \sup_{i \geqslant N} \Big| \sum_{n>i} \log\left(1+r_n\right)\Big|.$$

Thus (see also (3.9), (3.10) and (2.1))

$$h_a(\gamma) - h_b(\gamma) \to \log\Big(\prod_{n \geqslant 1} b_n/a_n\Big)^{\alpha} \quad \text{as } r \to 0^+. \tag{4.3}$$

In a general case, by (4.1), (4.2) and Lemma 3.3 (see (3.14)) we get

$$\begin{aligned} |J_2| &\leqslant \Big| \sum_{n>N} q(\gamma b_n) \log\left(1+r_n\right)\Big| + \sum_{n>N} \Big| \int_{\gamma b_n}^{\gamma a_n} |dq(u)| \Big| \left|\log\left(1+r_n\right)\right| \\ &\leqslant A_1 \sup_{i \geqslant N} \Big| \sum_{n>i} \log\left(1+r_n\right)\Big| + \sup_{n>N} \left|\log\left(1+r_n\right)/r_n\right| \sum_{n>N} |r_n| \Big| \int_{\gamma b_n}^{\gamma a_n} |D_1'(u)|\, du \Big| \end{aligned}$$

and (4.3) follows again now by (3.9), (3.10) and (2.9).

Finally, let us establish (3.4). We have (see (2.7) and (4.1) for the notation)

$$\gamma\left(m_b(\gamma)-m_a(\gamma)\right)=\sum_{n\geqslant 1}\int\limits_{\gamma b_n}^{\gamma b_n(1+r_n)}\left(D_2(u)-q(u)-q^2(u)\right)du/u.$$

As earlier in (4.1) and after that formula, under the condition (2.7) we have

$$\left|\gamma\left(m_b(\gamma)-m_a(\gamma)\right)\right|\leqslant 2|\alpha|(1+|\alpha|)\sup_{i\geqslant 1}\Big|\sum_{n\geqslant i}\log\left(1+r_n\right)\Big|,$$

and (see also (2.1)) (3.4) follows since $\tau_b(\gamma)\to\infty$ as $r\to 0^+$.

Now consider the general case. We get again with $g(u)=u^2 f''(u)$

$$\left|\gamma\left(m_b(\gamma)-m_a(\gamma)\right)\right|\leqslant\Big|\sum_{n\geqslant 1}\int\limits_{\gamma b_n}^{\gamma a_n}q(u)\,du/u\Big| \tag{4.4}$$

$$+\Big|\sum_{n\geqslant 1}\log\left(1+r_n\right)g(\gamma b_n)\Big|+\Big|\sum_{n\geqslant 1}\int\limits_{1}^{1+r_n}\left(g(\gamma b_n\,s)-g(\gamma b_n)\right)ds/s\Big|.$$

The first summand in (4.4) is bounded by (4.3) and the second one is bounded by Lemma 3.3 since

$$\int\limits_{[0,\infty)}|g'(s)|\,ds\leqslant\int\limits_0^a|g'(s)|\,ds+\int\limits_{[a,\infty)}\Big(|D_2'(s)|+\left|\left(D_1^2(s)\right)'\right|\Big)\,ds<\infty$$

due to (2.8), (1.3), Lemma 3.2 and the fact that for any fixed $a>0$

$$\int\limits_0^a|g'(s)|\,ds\leqslant(C+2)\int\limits_0^a s\,f''(s)\,ds<\infty.$$

Let us evaluate the last summand in (4.4). Obviously,

$$\begin{aligned}\Big|\sum_{n\geqslant 1}\int\limits_1^{1+r_n}\left(g(\gamma b_n\,s)-g(\gamma b_n)\right)ds/s\Big|&\leqslant C\sum_{n\geqslant 1}\frac{|r_n|}{\sqrt{g(\gamma b_n)}}\Big|\int\limits_{\gamma b_n}^{\gamma a_n}|g'(s)|\,ds\Big|\sqrt{g(\gamma b_n)}\\&\leqslant C\Big(\sum_{n\geqslant 1}\Big(r_n\int\limits_{\gamma b_n}^{\gamma a_n}|g'(s)|\,ds\Big)^2/g(\gamma b_n)\Big)^{1/2}\tau_b(\gamma)\end{aligned}$$

provided $C=\sup_{n\geqslant 1}|\log\left(1+r_n\right)/r_n|$.

This, along with Lemma 3.2 and (2.10), implies (3.4).

Thus, Theorems 2.3 and 2.4 are completely proved. ■

The verification of the fact that (2.8) with $j=2$ follows from **R** can be realized like the proofs of Remarks 2.2 and 2.3 ($j=2$).

Finally, let us prove the Remarks.

P r o o f o f R e m a r k 2.1. For any positive x, set $a_1 = x\,b_1$ and let $a_n = b_n,\ n \geqslant 2$. Then by (3.1), provided $m_b(\gamma) = r$,

$$\frac{\mathbf{P}(S_a < r)}{\mathbf{P}(S_b < r)} = \frac{I(x\,b_1\,\gamma)}{I(b_1\,\gamma)}\big(1 + o(1)\big) \qquad \text{as } r \to 0^+,$$

and Remark 2.1 follows from Feller [2], Chapter XIII, Section 8, the Theorem. ■

P r o o f o f R e m a r k 2.2. Observe, first of all, that **R** implies

$$F(1/\cdot) \in \mathbf{R}_\alpha. \tag{4.5}$$

Set

$$I_0(s) = \int_0^{u_0} e^{-su}\,dF(u), \quad A_0 = \int_0^{u_0} F(u)\,e^{-su}\,du, \quad \kappa(s) = 1 - I_0(s)/I(s).$$

We have

$$s\,I_0'(s)/I_0(s) - sI'(s)/I(s) = s\,\Big(\log\big(1 - \kappa(s)\big)\Big)' = -s\,\kappa'(s)\sum_{k\geqslant 0}\kappa^k(s).$$

Since (4.5) yields (1.4) and the fact that $I(s)$ and $I_0(s) \in \mathbf{R}_\alpha$, simple calculations show that the integrability at infinity of the derivative of $sI'(s)/I(s)$ is equivalent to that of $s\,I_0'(s)/I_0(s)$. Next, $I_0(s) = F(u_0)\exp(-su_0) + s\,A_0$ and, therefore, we have $s\,I_0'(s)/I_0(s) = T(s)\,\big(1 - F(u_0)\exp(-su_0)/I_0(s)\big)$, where $T(s) = I_0'(s)/A_0(s)$. Now,

$$T'(s) = \frac{I_0''(s)}{A_0(s)} - \frac{I_0'(s)\,A_0'(s)}{A_0^2(s)} = \frac{I_0''(s)}{A_0(s)} - \frac{-I_0'(s)}{A_0(s)}\cdot\frac{A_1(s)}{A_0(s)} \tag{4.6}$$

$$= \frac{\int_0^{u_0} u\,H(u)\,F(u)\,e^{-su}\,du}{A_0} - \frac{\int_0^{u_0} u\,F(u)\,e^{-su}\,du}{A_0}\,\frac{\int_0^{u_0} H(u)\,F(u)\,e^{-su}\,du}{A_0}$$

provided $H(u) = u\,\big(\log F(u)\big)'$.

Let a random variable Y, defined at $(0, u_0)$, have the distribution

$$\int_0^t F(u)\,e^{-su}\,du\Big/\int_0^{u_0} F(u)\,e^{-su}\,du, \quad 0 < t < u_0.$$

Then (4.6) together with **R** implies that for all s large enough the function $T'(s) = \mathbf{E}YH(Y) - \mathbf{E}Y\,\mathbf{E}H(Y)$ does not change the sign by the second Chebyshev inequality (Petrov [9], Chapter I, Section 2, Theorem 1). Hence (see also (4.5)) $T'(s)$ is absolutely integrable at infinity, and **I** follows. ■

P r o o f o f R e m a r k 2.3. The assertion for $j = 1$ is a special case of **R**. Let now $j = 2$. Set $\hat{F}(t) = \int_0^t F(u)\,du$, $\tilde{I}(s) = \int_0^\infty \hat{F}(u)\,e^{-su}\,du$. Then we have $I(s) = s^2\,\tilde{I}(s)$, and

$$D_2'(s) = \big(I''(s)/\tilde{I}(s)\big)' = -\big(\mathbf{E}Y\tilde{H}(Y) - \mathbf{E}Y\,\mathbf{E}\tilde{H}(Y)\big),$$

where $\tilde{H}(u) = u^2\,F'(u)/\hat{F}(u)$ and a positive random variable Y has the distribution function

$$\int\limits_0^t \hat{F}(u)\,e^{-su}\,du/\tilde{I}(s), \quad t > 0.$$

Hence $D_2'(s) \geqslant 0$ provided $\tilde{H}(\cdot)$ does not increase. Next,

$$\tilde{H}(t) = H(t)/\int\limits_0^1 \frac{F(ut)}{F(t)}\,du, \qquad \text{where } H(t) = t\,F'(t)/F(t).$$

But the function $F(ut)/F(t) = \exp\big(-\int_u^1 H(tx)\,dx/x\big)$ does not decrease on $(0,\infty)$ if $H(t)$ does not increase. Thus, $\tilde{H}(\cdot)$ does not increase, and the required result follows. ∎

P r o o f o f R e m a r k 2.4. Let us just check that (2.11) and (2.8) imply (2.9). We have for $v(s) = D_1'(s)$, setting $\bar{\rho}_N = \sup_{n\geqslant N}\rho_n$,

$$\begin{aligned}\sum_{n\geqslant N}\rho_n\Big|\int\limits_{\gamma b_n}^{\gamma a_n} v(s)\,ds\Big| &= \sum_{n\geqslant N}\rho_n\int\limits_{\gamma m_n}^{\gamma M_n} v(s)\,ds \leqslant \bar{\rho}_N\sum_{n\geqslant N}\int\limits_{\gamma M_{n+k}}^{\gamma M_n} v(s)\,ds \\ &= \bar{\rho}_N\sum_{n\geqslant N}\Big(\int\limits_{\gamma M_{n+1}}^{\gamma M_n} v(s)\,ds + \ldots + \int\limits_{\gamma M_{n+k}}^{\gamma M_{n+k-1}} v(s)\,ds\Big) \\ &= \bar{\rho}_N\Big(\int\limits_0^{\gamma M_N} v(s)\,ds + \ldots + \int\limits_0^{\gamma M_{N+k-1}} v(s)\,ds\Big) \leqslant \bar{\rho}_N\,k\int\limits_0^\infty v(s)\,ds.\end{aligned}$$

Letting N to infinity slowly enough and taking into account that by virtue of (1.4)

$$\sum_{n<N}\rho_n\Big|\int\limits_{\gamma b_n}^{\gamma a_n} v(s)\,ds\Big| \to 0 \quad \text{as } \gamma\to\infty,$$

we obtain the claim of Remark 2.4. ∎

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