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COMPARISON OF HARMONIC KERNELS ASSOCIATED WITH A CLASS OF SEMILINEAR ELLIPTIC EQUATIONS*

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Abstract. Let D be a smooth domain in $\mathbb{R}^N$, $N \geqslant 3$, and let f be a positive continuous function on ∂D. Under some assumptions on φ, it is shown that the problem $\Delta u = 2\varphi(u)$ in D and $u = f$ on ∂D admits a unique solution which will be denoted by $H_D^\varphi f$. Given two functions φ and ψ, our main goal in this paper is to investigate the existence of a constant $c > 0$ such that

$$\frac{1}{c} H_D^\varphi f \leqslant H_D^\psi f \leqslant c H_D^\varphi f.$$

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1. INTRODUCTION

Let D be a bounded smooth domain in $\mathbb{R}^N$, $N \geqslant 3$. We consider the following semilinear problem:

$$\begin{cases} \Delta u = 2\varphi(u) & \text{in } D, \\ u = f & \text{on } \partial D, \end{cases} \tag{1.1}$$

where f is a positive continuous function on ∂D. Under some conditions on φ, it will be shown that problem (1.1) admits a unique solution which will be denoted by $H_D^\varphi f$. In the particular case where $\varphi \equiv 0$, (1.1) reduces to the classical Dirichlet problem whose unique solution will be denoted by $H_D f$.

Given two functions φ and ψ, we say that $H_D^\varphi f$ and $H_D^\psi f$ are *proportional* and we write $H_D^\varphi f \approx H_D^\psi f$ if there exists $c > 0$ such that, for every $x \in D$,

$$\frac{1}{c} H_D^\psi f(x) \leqslant H_D^\varphi f(x) \leqslant c H_D^\psi f(x).$$

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The operators H_D^φ and H_D^ψ are said to be *proportional* (we write $H_D^\varphi \approx H_D^\psi$) if $H_D^\varphi f$ and $H_D^\psi f$ are proportional for every positive continuous function f on ∂D.

The main goal of this paper is to study the proportionality between H_D^φ and H_D^ψ. To this end, we shall rather compare H_D^φ to H_D. Since f is positive on ∂D, it is very simple to observe that $H_D^\varphi f \leqslant H_D f$. However, the question whether $H_D^\varphi f \geqslant c H_D f$ for some constant $c > 0$ seems to be more difficult.

There are several papers dealing with the existence of solutions to semilinear problems which are bounded below by a harmonic function (see [3], [4], [9], [14], and their references). The second-named author studied in [14] the problem

$$\begin{cases} \Delta u + \xi(x)\Psi(u) = 0 & \text{in } D, \\ u > h & \text{in } D, \\ u - h = 0 & \text{on } \partial D, \end{cases} \tag{1.2}$$

where $h \geqslant 0$ is harmonic in D, $\xi \geqslant 0$ is locally bounded, and $\Psi > 0$ is a nonincreasing continuous function on $]0, \infty[$. He proved that (1.2) admits a unique solution provided the function

$$x \mapsto \int_D G_D(x, y)\xi(y)\, dy$$

is continuous on D and vanishes on the boundary of D, where $G_D(\cdot, \cdot)$ denotes the Green function of Δ on D (see (2.3) below).

Athreya [4] considered the problem (1.1) where $\varphi \colon]0, \infty[\to]0, \infty[$ is a locally Hölder continuous function such that $\varphi(t)$ tends to ∞ as t tends to 0 at the rate t^{-p} with $p \in]0, 1[$. Given a function h_0 which is continuous on $\overline{D}$ and harmonic in D, under some additional conditions he showed that problem (1.1) has a unique solution which is bounded below by h_0. By probabilistic techniques, Chen et al. investigated in [9] the same problem where $-t \leqslant \varphi(t) \leqslant t$, $t \in]0, b[$, for some $b > 0$. They proved the existence of a solution bounded below by a positive harmonic function provided the nontrivial function f admits a sufficiently small norm (see Theorem 1.2 in [9] and its proof).

The problem (1.1), with $\varphi(t) = \varphi_p(t) = t^p$, was already studied by Atar et al. in [3] where they showed that the proportionality of $H_D^{\varphi_p}$ and H_D holds true for every $p \geqslant 1$. In this paper we shall prove that $H_D^\varphi \approx H_D$ for a large class of functions φ containing $\varphi_p, p \geqslant 1$. Furthermore, we shall prove a conjecture stated in [3] and claiming that $H_D^{\varphi_p}$ and H_D are not proportional for $0 \leqslant p < 1$. More precisely, we give sufficient conditions on φ under which the operators H_D^φ and H_D are not proportional.

We briefly recall in Section 2 some basic facts on Brownian motion and then establish in Section 3 the existence of a unique solution to problem (1.1) where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and nondecreasing, and $\varphi(0) = 0$. In Section 4, we are concerned with the proportionality between H_D^φ and the harmonic kernel H_D.

We prove that the proportionality holds true provided

$$\limsup_{t\to 0} \frac{\varphi(t)}{t} < \infty, \tag{1.3}$$

and does not hold if, for some $\varepsilon > 0$,

$$\int_0^\varepsilon \Big(\int_0^t \varphi(s)ds\Big)^{-1/2} dt < \infty. \tag{1.4}$$

In particular, the fact that condition (1.4) is valid for $\varphi = \varphi_p$ with $0 \leqslant p < 1$ yields an immediate proof of the conjecture mentioned above. The last section will be devoted to investigating the problem (1.1) in the case where the function φ is non-increasing.

2. PRELIMINARIES

For every subset F of $\mathbb{R}^N$, let $\mathcal{B}(F)$ be the set of all Borel measurable functions on F and let $\mathcal{C}(F)$ be the set of all continuous real-valued functions on F. If $\mathcal{G}$ is a set of numerical functions, then $\mathcal{G}^+$ (respectively, $\mathcal{G}_b$) will denote the class of all functions in $\mathcal{G}$ which are nonnegative (respectively, bounded). The uniform convergence norm will be denoted by $\|\cdot\|$.

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P^x)$ be the canonical Brownian motion on the Euclidean space $\mathbb{R}^N$, $N \geqslant 3$. Here Ω is the set of all continuous functions from $[0, \infty[$ to $\mathbb{R}^N$endowed with its Borel σ-algebra $\mathcal{F}$. For every $t \geqslant 0$ and $\omega \in \Omega$,

$$X_t(\omega) = \omega(t) \quad \text{and} \quad \mathcal{F}_t := \sigma(X_s; 0 \leqslant s \leqslant t).$$

For every $x \in \mathbb{R}^N$, P^x is the probability measure on $(\Omega, \mathcal{F})$ under which the Brownian motion starts at x (i.e., $P^x(X_0 = x) = 1$) and $E^x[\cdot]$ denotes the corresponding expectation. Let D be a bounded domain in $\mathbb{R}^N$and let τ_D be the first exit time from D by X, i.e.,

$$\tau_D = \inf\{t > 0; X_t \notin D\}.$$

We denote by (X_t^D) the Brownian motion killed upon exiting D. It is well known that its transition density is given by

$$p^D(t, x, y) = p(t, x, y) - r^D(t, x, y), \quad t > 0, \ x, y \in D,$$

where

$$p(t, x, y) = \frac{1}{(2\pi t)^{N/2}} \exp\Big(-\frac{|x-y|^2}{2t}\Big),$$

and

$$r^D(t, x, y) = E^x\left[p(t - \tau_D, X_{\tau_D}, y), \tau_D < t\right].$$

The corresponding semigroup is then defined by

$$P_t^D f(x) = E^x\left[f(X_t), t < \tau_D\right] = \int_D p^D(t,x,y) f(y)dy, \quad x \in D,$$

for every Borel measurable function f for which this integral makes sense.

Let h be a positive harmonic function in D and define for $x, y \in D, t > 0$,

$$p_h^D(t,x,y) = p^D(t,x,y)\frac{h(y)}{h(x)}.$$

Then there exists a Markov process, called the *h-conditioned Brownian motion*, with state space D and having p_h^D as transition density (see [6], [10], [11]). The corresponding probability measures are denoted by $(P_h^x)_{x\in D}$: for every Borel subset B of D we have

$$\begin{aligned} P_h^x(X_t \in B) &= \frac{1}{h(x)} \int_B p^D(t,x,y)h(y)\,dy \\ &= \frac{1}{h(x)} E^x[h(X_t), X_t \in B, t < \tau_D]. \end{aligned}$$

Besides, using the monotone class theorem, it is easily seen that, for every $t > 0$ and every $\mathcal{F}_t$-measurable random variable $Z \geqslant 0$,

$$E_h^x\left[Z, t < \tau_D\right] = \frac{1}{h(x)} E^x\left[Z\, h(X_t), t < \tau_D\right]. \tag{2.1}$$

The open bounded subset D is called *regular* (for Δ) if each function $f \in \mathcal{C}(\partial D)$ admits a continuous extension $H_D f$ on $\overline{D}$ such that $H_D f$ is harmonic in D. In other words, the function $h = H_D f$ is the unique solution to the classical Dirichlet problem

$$\begin{cases} \Delta h = 0 & \text{in } D, \\ h = f & \text{on } \partial D. \end{cases}$$

For every $x \in D$, the mapping $f \mapsto H_D f(x)$ defines a probability measure on ∂D which will be denoted by $H_D(x,\cdot)$ and called the *harmonic measure* relative to x and D.

In the sequel, let $x_0 \in D$ be a fixed point and assume that D is a bounded Lipschitz domain of $\mathbb{R}^N$ (in fact, we impose that the Martin boundary of D coincides with its Euclidean one; see [2], Section 8.7). Hence, there exists a unique function $K_D : D \times \partial D \to \mathbb{R}_+$ satisfying:

For every $z \in \partial D$, $K_D(x_0, z) = 1$.

For every $x \in D$, $K_D(x,\cdot)$ is continuous in ∂D.

For every $z \in \partial D$, $K_D(\cdot, z)$ is a positive harmonic function in D.

For every $z, w \in \partial D$ such that $z \neq w$, $\lim_{x\to w} K_D(x,z) = 0$.

We extend the function $K_D(\cdot, z)$ to $\overline{D}\setminus\{z\}$ by letting $K_D(w, z) = 0$ for every $w \in \partial D\setminus\{z\}$. The function K_D is called the *Martin kernel* on D. It is well known that the formula

$$h(x) = \int_{\partial D} K_D(x, z)\, d\nu(z) \tag{2.2}$$

realizes a one-to-one correspondence between nonnegative harmonic functions on D and (positive) Radon measures on ∂D.

The *Green function* $G_D(\cdot, \cdot)$ is defined on $D \times D$ by

$$G_D(x, y) = \int_0^\infty p^D(t, x, y)dt. \tag{2.3}$$

It follows that G_D is continuous (in the extended sense) on $D \times D$,

$$G_D(x, y) \leqslant G_{\mathbb{R}^N}(x, y) = \frac{\Gamma(N/2 + 1)}{2\pi^{N/2}|x - y|^{N-2}},$$

and $\lim_{x\to z} G_D(x, y) = 0$ for every $z \in \partial D$ (see [15], Chapter 4). Moreover,

$$K_D(x, z) = \frac{dH_D(x, \cdot)}{dH_D(x_0, \cdot)}(z) = \lim_{y\in D, y\to z} \frac{G_D(x, y)}{G_D(x_0, y)}, \qquad x \in D, z \in \partial D. \tag{2.4}$$

For $h = K_D(\cdot, z)$, where $z \in \partial D$, the h-conditioned Brownian motion will be simply called the *z-Brownian motion* and its transition density is given by

$$p_z^D(t, x, y) = \frac{1}{K_D(x, z)} p^D(t, x, y)K_D(y, z), \qquad t > 0,\ x, y \in D.$$

The corresponding family of probability measures will be denoted by $(P_z^x)_{x\in D}$.

3. SEMILINEAR PROBLEM

We assume that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function such that $\varphi(0) = 0$. The following comparison principle will be useful to prove not only the uniqueness but also the existence of a solution to problem (1.1). A more general comparison principle can be found in [14].

LEMMA 3.1. *Let $\Psi \in \mathcal{B}(\mathbb{R})$ be a nondecreasing function and let $u, v \in \mathcal{C}(\overline{D})$ such that $\Delta u \leqslant \Psi(u)$, $\Delta v \geqslant \Psi(v)$ in the distributional sense in D, and $u(z) \geqslant v(z)$ for every $z \in \partial D$. Then $u(x) \geqslant v(x)$ for every $x \in D$.*

P r o o f. Define $w = u - v$ and suppose that the open set

$$\Omega = \{x \in D; w(x) < 0\}$$

is not empty. Since Ψ is nondecreasing, it is obvious that $\Delta w \leqslant \Psi(u) - \Psi(v) \leqslant 0$ in Ω, which means that w is superharmonic in Ω. Moreover, for every $z \in \partial\Omega \cap D$ we have $w(z) = 0$ (because w is continuous in D), and for every $z \in \partial\Omega \cap \partial D$ we have $\lim_{x\in\Omega, x\to z} w(x) \geqslant 0$ (by assumption). Then $w \geqslant 0$ in Ω by the classical minimum principle for superharmonic functions. This yields a contradiction, and therefore Ω is empty. Hence $u \geqslant v$ in D. ∎

The *Green operator* in D is defined by

$$G_D f(x) = \int_D G_D(x,y) f(y) dy, \quad x \in D, \tag{3.1}$$

for every Borel measurable function f for which the integral exists. In other words,

$$G_D f(x) = E^x \Big[\int_0^{\tau_D} f(X_t) dt \Big] = \int_0^{\infty} P_t^D f(x) dt, \quad x \in D.$$

We recall that, for every $f \in \mathcal{B}_b(D)$, $G_D f$ is a bounded continuous function on D satisfying $\lim_{x\to z} G_D f(x) = 0$ for every $z \in \partial D$. Moreover, it is simple to check that

$$\Delta G_D f = -2f$$

in the distributional sense (see [10], [11]).

LEMMA 3.2. *For every $M > 0$, the family $\{G_D u; \|u\| \leqslant M\}$ is relatively compact with respect to the uniform convergence norm.*

Proof. Since the family $\{G_D u; \|u\| \leqslant M\}$ is uniformly bounded, it suffices in virtue of the Arzelà–Ascoli theorem to show that the family is equicontinuous. We first claim that the family $\{G_D(x,\cdot); x \in D\}$ is uniformly integrable. Indeed, let $\varepsilon > 0$ and $\eta_0 > 0$. There exist $c_1 > 0$ and $c_2 > 0$ such that, for every Borel subset A of D,

$$\begin{aligned} \int_A G_D(x,y) dy &\leqslant c_1 \int_A \frac{dy}{|x-y|^{N-2}} \\ &\leqslant c_1 \int_{B(x,\eta_0)} \frac{dy}{|x-y|^{N-2}} + c_1 \int_{A\backslash B(x,\eta_0)} \frac{dy}{\eta_0^{N-2}} \\ &\leqslant c_2 \eta_0^2 + c_2 \frac{m(A)}{\eta_0^{N-2}}. \end{aligned}$$

Here and in all the following, m denotes the Lebesgue measure in $\mathbb{R}^N$. Take $\eta_0 = \sqrt{\varepsilon/(2c_2)}$ and $\eta = \varepsilon \eta_0^{N-2}/(2c_2)$. Then for every Borel subset A of D such that $m(A) < \eta$ we have

$$\int_A G_D(x,y)\, dy \leqslant \varepsilon.$$

Hence, the uniform integrability of the family $\{G_D(x,\cdot); x \in D\}$ is shown. Therefore, in virtue of Vitali's convergence theorem (see, e.g., [16]), we conclude that, for every $z \in D$,

$$\lim_{x\to z} \sup_{\|u\|\leqslant M} \Big| \int_D G_D(x,y)u(y)dy - \int_D G_D(z,y)u(y)dy \Big| \leqslant M \lim_{x\to z} \int_D |G_D(x,y) - G_D(z,y)|\, dy = 0.$$

This means that the family $\{G_D u; \|u\| \leqslant M\}$ is equicontinuous, which completes the proof of the lemma. ■

The existence of solutions to semilinear Dirichlet problems of kind (1.1) was widely studied in the literature with various assumptions on the function φ (see, e.g., [5], [12]–[14]). In our setting, we get the following:

THEOREM 3.1. *For every $f \in \mathcal{C}^+(\partial D)$, there exits one and only one function $u \in \mathcal{C}^+(\overline{D})$ satisfying the problem* (1.1). *Furthermore, a bounded Borel function u on D is a solution to* (1.1) *if and only if $u + G_D\varphi(u) = H_D f$.*

P r o o f. By a classical computation, it is not hard to establish the second part of the theorem. We also observe that, by the comparison principle (Lemma 3.1), problem (1.1) has at most one solution. So, it remains to prove the existence of a solution to (1.1). Take

$$f \in \mathcal{C}^+(\partial D), \qquad a = \|f\|, \qquad M = a + \varphi(a) \sup_{x\in D} E^x[\tau_D]$$

and define $\Lambda = \{u \in \mathcal{C}(\overline{D}); \|u\| \leqslant M\}$. Let $h = H_D f$ and consider the operator $T : \Lambda \to \mathcal{C}(\overline{D})$ defined by

$$Tu(x) = h(x) - E^x\Big[\int_0^{\tau_D} g\big(u(X_s)\big)ds\Big], \qquad x \in D,$$

where g is a real-valued odd function given by $g(t) = \inf\big(\varphi(t), \varphi(a)\big)$ for every $t \geqslant 0$. Since $|g(t)| \leqslant \varphi(a)$ for every $t \in \mathbb{R}$, we get

$$|Tu(x)| \leqslant M$$

for every $x \in D$ and every $u \in \Lambda$. This implies that $T(\Lambda) \subset \Lambda$. Now, let $(u_n)_{n\geqslant 0}$ be a sequence in Λ converging uniformly to $u \in \Lambda$. Let $\varepsilon > 0$. Since g is uniformly continuous in $[-M, M]$, we deduce that there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ and $s \in [0, \tau_D]$

$$\big|g\big(u_n(X_s)\big) - g\big(u(X_s)\big)\big| < \varepsilon.$$

It follows that, for every $n \geqslant n_0$ and $x \in D$,

$$\begin{aligned}|Tu_n(x) - Tu(x)| &= \Big|E^x\Big[\int_0^{\tau_D} g\big(u_n(X_s)\big)ds\Big] - E^x\Big[\int_0^{\tau_D} g\big(u(X_s)\big)ds\Big]\Big| \\ &\leqslant E^x\Big[\int_0^{\tau_D} \big|g\big(u_n(X_s)\big) - g\big(u(X_s)\big)\big|ds\Big] \\ &\leqslant \varepsilon \sup_{x\in D} E^x[\tau_D].\end{aligned}$$

This shows that $(Tu_n)_{n\geqslant 0}$ converges uniformly to Tu. We then conclude that T is a continuous operator. On the other hand, Λ is a closed bounded convex subset of $\mathcal{C}(\overline{D})$. Moreover, in virtue of Lemma 3.2, $T(\Lambda)$ is relatively compact. Thus, the Schauder's fixed point theorem ensures the existence of a function $u \in \Lambda$ such that $u = h - G_D g(u)$. Applying the comparison principle, we obtain $0 \leqslant u \leqslant a$, and so $g(u) = \varphi(u)$. Hence, the proof is completed. ■

The unique solution to problem (1.1) will always be denoted by $H_D^\varphi f$. However, in the particular case where $\varphi = \varphi_p$ we may write $H_D^p f$ instead of $H_D^\varphi f$.

4. PROPORTIONALITY OF $H_D^\varphi f$ AND $H_D f$

The Feynman–Kac theorem (see [10], Theorem 4.7) states that, for every $f \in \mathcal{C}^+(\partial D)$ and $q \in \mathcal{B}_b^+(D)$, the function $v \in \mathcal{C}(\overline{D})$ given by

$$v(x) = E^x\Big[f(X_{\tau_D})\exp\Big(-\int_0^{\tau_D} q(X_s)\,ds\Big)\Big], \quad x \in D, \tag{4.1}$$

is the unique solution of the problem

$$\begin{cases} \Delta v = 2qv & \text{in } D, \\ v = f & \text{on } \partial D. \end{cases}$$

Let us notice that v given by (4.1) satisfies the integral equation:

$$v(x) = h(x) - \int_D G_D(x,y)q(y)v(y)dy, \quad x \in D.$$

Our first result in this section is the following:

THEOREM 4.1. *Assume that*

$$\limsup_{t\to 0} \frac{\varphi(t)}{t} < \infty. \tag{4.2}$$

Then $H_D^\varphi f \approx H_D f$ *for every function* $f \in \mathcal{C}^+(\partial D)$.

P r o o f. Let $f \in \mathcal{C}^+(\partial D)$ be nontrivial, that is, $h = H_D f > 0$ in D. Let $u = H_D^{\varphi} f$ and define

$$q := \frac{\varphi(u)}{u} 1_{\{u>0\}}.$$

Then q is a positive bounded function in D by (4.2), and u satisfies the problem

$$\begin{cases} \Delta u = 2qu & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases} \tag{4.3}$$

We define

$$w(x,z) = E_z^x\big[\exp\big(-\int_0^{\tau_D} q(X_t)dt\big)\big], \qquad x \in D, z \in \partial D.$$

By the Feynman–Kac theorem and Proposition 5.12 in [10], we have

$$\begin{aligned} u(x) &= E^x\big[f(X_{\tau_D})\exp\big(-\int_0^{\tau_D} q(X_s)\,ds\big)\big] \\ &= \int_{\partial D} w(x,z)f(z)H_D(x,dz). \end{aligned} \tag{4.4}$$

Since for every $x \in D$

$$E^x\big[\exp\big(-\int_0^{\tau_D} q(X_s)\,ds\big)\big] < \infty,$$

by Theorem 7.6 in [10] there exists $c > 0$ such that

$$\frac{1}{c} \leqslant w(x,z) \leqslant c, \qquad x \in D, z \in \partial D. \tag{4.5}$$

Combining (4.4) and (4.5) we conclude that, for every $x \in D$,

$$\begin{aligned} \frac{1}{c}H_D f(x) &= \frac{1}{c}\int_{\partial D} f(z)H_D\,(x,dz) \\ &\leqslant u(x) \\ &\leqslant c\int_{\partial D} f(z)H_D\,(x,dz) = cH_D f(x). \end{aligned}$$

Hence $H_D f \approx H_D^{\varphi} f$. ∎

Let us notice that the assumption mentioned in the previous theorem will be trivially satisfied provided the function $t \mapsto \varphi(t)/t$ is nondecreasing or if it is bounded and nonincreasing on $]0,\infty[$. In particular, it follows that $H_D^{\varphi} f \approx H_D f$ for every function $f \in \mathcal{C}^+(\partial D)$ if the function φ is given by

$$\varphi(t) = t^p \text{ with } p \geqslant 1 \quad \text{or} \quad \varphi(t) = \log(1+t).$$

We shall write $H_D^\varphi \approx H_D$ if $H_D^\varphi f \approx H_D f$ for every function $f \in \mathcal{C}^+(\partial D)$. Hence, by Theorem 4.1, $H_D^p \approx H_D$ for every $p \geqslant 1$. This was established by Atar et al. in [3]. In the same paper, the authors conjectured that $H_D^p \not\approx H_D$ for every $0 < p < 1$. In the following, we shall prove this conjecture. More precisely, we give a sufficient condition on φ under which $H_D^\varphi \not\approx H_D$.

From now on, D is a bounded $C^{1,1}$-domain of $\mathbb{R}^N$, $N \geqslant 3$. As usual, the diameter of D is $\operatorname{diam}(D) = \sup_{x,y\in D} |x-y|$ and $\delta_D(x) = \inf_{z\notin D} |x-z|$ denotes the Euclidean distance from $x \in D$ to the complement of D. The following property of Green function G_D is established by Zhao [17]:

$$G_D(x,y) \approx \min\left\{ \frac{1}{|x-y|^{N-2}}, \frac{\delta_D(x)\delta_D(y)}{|x-y|^N} \right\}, \qquad x, y \in D. \tag{4.6}$$

THEOREM 4.2. *Assume that there exists $\varepsilon > 0$ such that*

$$\int_0^\varepsilon \Big(\int_0^s \varphi(r)\, dr \Big)^{-1/2} ds < \infty. \tag{4.7}$$

Then there exists $f \in \mathcal{C}^+(\partial D)$ such that $H_D^\varphi f \not\approx H_D f$.

P r o o f. It is well known (see Lemma 2.2 in [1]) that D satisfies the ball condition with some radius $r > 0$, which in turn means that D is $C^{1,1}$ at scale r in the sense of Bogdan–Jakubowski [7]. Let $z \in \partial D$ be a fixed point. Applying Lemma 1 of [7] for the complement of $\overline{D}$, we find a bounded $C^{1,1}$-domain F such that $F \cap \overline{D} = \emptyset$ and

$$F \cap B(z, r/4) = B(z, r/4) \setminus \overline{D}.$$

Let $d = \operatorname{diam}(D \cup F)$, $R > 0$, and $x_0 \in \mathbb{R}^N$ such that $2d < |x_0 - z| < R$. It is easily verified that $U = B(z, R) \setminus \overline{F}$ is a $C^{1,1}$-open set, $D \subset U$, and

$$B(z, r/4) \cap \partial D \subset \partial U.$$

The function $g = G_U(\cdot, x_0)$ is positive harmonic in $V := U \setminus \{x_0\}$ and vanishes on $B(z, r/4) \cap \partial D$. By arguments similar to those used in [8], Lemma 3.2, there exists a constant $c_1 > 0$ such that, for every $x \in V$,

$$|\nabla g(x)| \leqslant c_1 \frac{g(x)}{\delta_V(x)}.$$

On the other hand, by (4.6) there exists $c_2 > 0$ such that $g(x) \leqslant c_2 \delta_U(x)$ for every $x \in V$. Since $\delta_V(x) = \delta_U(x)$ for every $x \in D$, it follows that

$$M := \sup_{x\in D} |\nabla g(x)| \leqslant c_1 c_2.$$

In virtue of condition (4.7), we easily observe that the function

$$Q : t \mapsto \frac{1}{2}\int_0^t \Big(\int_0^s \varphi(r)\,dr\Big)^{-1/2} ds$$

is increasing and continuous on $[0,\infty[$, twice differentiable on $]0,\infty[$, and invertible from $[0,\infty[$ to $[0,\bar\rho[$, where $\bar\rho := \lim_{t\to\infty} Q(t)$ (notice that $\bar\rho$ may be infinite). Let R denote the inverse function of Q and let $0 < \lambda < \min(1/M, \bar\rho/M')$, where $M' = \sup_D g$ and define

$$v = R(\lambda g), \quad f = v|_{\partial D}, \quad h = H_D f, \quad \text{and} \quad u = H_D^{\varphi} f.$$

Then, it is obvious that $v \in \mathcal{C}(\overline{D}) \cap \mathcal{C}^2(D)$. Moreover, by an elementary calculus it follows that, for every $x \in D$,

$$\Delta v(x) \leqslant 2\varphi\big(v(x)\big).$$

Therefore, $u \leqslant v$ in D by the comparison principle (Lemma 3.1). Since

$$\lim_{x\to z} \frac{v(x)}{g(x)} = \lim_{t\to 0} \frac{t}{Q(t)} = 0,$$

we get

$$\inf_D \frac{u}{g} \leqslant \inf_D \frac{v}{g} = 0.$$

On the other hand, from the boundary Harnack principle it follows that there exists an open neighborhood W of z such that

$$g \approx h \quad \text{in } W \cap D.$$

This yields that $\inf_D(u/h) = 0$, and consequently $u \not\approx h$. ∎

Since the function $\varphi : t \mapsto t^p$ satisfies (4.7) for $0 < p < 1$, we deduce from the previous theorem that, for small p, $H_D^p \not\approx H_D$, which proves the conjecture given in [3].

In the remainder of this section, we shall proceed to answer the following question: In the case where (4.2) fails, for which function $f \in \mathcal{C}^+(\partial D)$ does the proportionality of $H_D f$ and $H_D^{\varphi} f$ hold?

First, the following proposition is easily obtained.

PROPOSITION 4.1. *Let $f \in \mathcal{C}^+(\partial D)$, $h = H_D f$, and $u = H_D^{\varphi} f$. If*

$$\sup_{x\in D} \frac{1}{h(x)} \int_D G_D(x,y)\varphi\big(h(y)\big)\,dy < 1, \tag{4.8}$$

then $u \approx h$.

P r o o f. It is an immediate consequence of the formula $h = u + G_D\varphi(u)$ and the fact that φ is nondecreasing. ∎

Hence, one direction in solving the question above is to investigate functions f for which condition (4.8) is fulfilled. Let us notice that "< 1" in (4.8) cannot be replaced by "$< \infty$". In fact, as will be shown below, for a smooth domain D we always have

$$\sup_{x\in D} \frac{1}{h(x)} \int_D G_D(x,y)\varphi\big(h(y)\big)\,dy < \infty. \tag{4.9}$$

However, for $\varphi(t) = t^p$ with $0 < p < 1$, Theorem 4.2 proves that there exists a function $f \in \mathcal{C}^+(\partial D)$ such that u and h are not proportional.

LEMMA 4.1. *For every positive harmonic function h in D, there exists a positive constant c such that, for every $x \in D$,*

$$\int_D G_D(x,y)\,dy \leqslant c\,h(x).$$

P r o o f. Let h be a positive harmonic function in D and let ν be the positive Radon measure on ∂D satisfying

$$h = \int_{\partial D} K_D(\cdot,z)\,d\nu(z). \tag{4.10}$$

We claim that there exists $C > 0$ such that, for every $x \in D$ and $z \in \partial D$,

$$h(x) \geqslant C\delta_D(x). \tag{4.11}$$

Indeed, let $x \in D$ and $z \in \partial D$. Then, it is simple to observe that $\delta_D(x)\delta_D(y) \leqslant |x-y|^2$ for every $y \in D$ such that $8|y-z| < \delta_D(x)$. Hence, by (4.6) there exists a constant $c_1 > 0$ such that, for every $y \in D \cap B\big(z, \delta_D(x)/8\big)$,

$$G_D(x,y) \geqslant c_1 \frac{\delta_D(x)\delta_D(y)}{|x-y|^N}.$$

Again by (4.6) there exists $c_2 > 0$ such that

$$G_D(x_0,y) \leqslant c_2 \frac{\delta_D(x_0)\delta_D(y)}{|x_0-y|^N},$$

where x_0 denotes, as was mentioned in Section 2, a reference point. Therefore, for every $y \in D \cap B\big(z, \delta_D(x)/8\big)$,

$$\frac{G_D(x,y)}{G_D(x_0,y)} \geqslant \frac{c_1\,|x_0-y|^N\,\delta_D(x)\delta_D(y)}{c_2\delta_D(x_0)\delta_D(y)\,|x-y|^N}.$$

Hence, letting y tend to z we obtain

$$K_D(x,z) \geqslant c_3 \frac{\delta_D(x)}{|x-z|^N},$$

where c_3 is a positive constant not depending on x and z. This and formula (4.10) yield (4.11). On the other hand, in [17] it is shown that there exists $c_4 > 0$ such that, for every $x, y \in D$,

$$G_D(x,y) \leqslant c_4 \frac{\delta_D(x)}{|x-y|^{N-1}}.$$

Hence, using (4.11) we get

$$\sup_{x\in D} \frac{1}{h(x)} \int_D G_D(x,y)dy = \frac{c_4}{C} \sup_{x\in D} \int_D \frac{dy}{|x-y|^{N-1}} < \infty,$$

which completes the proof. ∎

THEOREM 4.3. *Assume that*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = 0. \tag{4.12}$$

Then for every $f \in \mathcal{C}^+(\partial D)$ there exists a positive constant α_f such that $H_D^\varphi(\alpha f) \approx H_D(\alpha f)$ for every $\alpha \geqslant \alpha_f$.

Proof. Let $f \in \mathcal{C}^+(\partial D)$ be nontrivial and let $h = H_D f$. By the previous lemma, there exists $c > 0$ (depending on h) such that, for every $\alpha > 0$ and every $x \in D$,

$$G_D\varphi(\alpha h)(x) \leqslant \varphi(\alpha\|h\|)G_D 1(x) \leqslant c\varphi(\alpha\|h\|)h(x).$$

Therefore

$$\sup_{x\in D} \frac{G_D\varphi(\alpha h)\,(x)}{\alpha h\,(x)} \leqslant c\frac{\varphi(\alpha\|h\|)}{\alpha}.$$

On the other hand, by (4.12) there exists $A > 0$ such that, for every $t \geqslant A$,

$$\frac{\varphi(t)}{t} < \frac{1}{c\|h\|}.$$

Take $\alpha_f := A/\|h\|$. Then for every $\alpha \geqslant \alpha_f$ we have

$$\sup_{x\in D} \frac{G_D\varphi(\alpha h)\,(x)}{\alpha h\,(x)} < 1,$$

which implies, by Proposition 4.1, that $H_D^\varphi(\alpha f) \approx H_D(\alpha f)$. ∎

5. MORE ABOUT PROBLEM (1.1)

This last section is devoted to investigate the problem (1.1) in the case where φ is nonincreasing. Let us notice that, in this setting, we do not guarantee neither the existence nor the uniqueness of the solution to problem (1.1), and hence the operator H_D^{φ} is no longer defined. As above, we assume that D is a bounded $C^{1,1}$-domain of $\mathbb{R}^N$, $N \geqslant 3$.

THEOREM 5.1. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nonincreasing function, $f \in \mathcal{C}^+(\partial D)$, and let $h = H_D f$ such that*

$$\sup_{x\in D} E_h^x\Big[\int_0^{\tau_D} \frac{1}{h(X_s)} ds\Big] \leqslant \frac{1}{e\,\varphi(0)}. \tag{5.1}$$

Then the problem (1.1) *has a solution $u \in \mathcal{C}^+(\overline{D})$ satisfying $u \approx h$.*

P r o o f. Of course, we assume that $\varphi(0) > 0$ and f is nontrivial. By assumption,

$$c := \sup_{x\in D} \frac{1}{h(x)} \int_D G_D(x,y)\, dy \leqslant \frac{1}{e\,\varphi(0)}.$$

It is easily seen that there exists $b > 0$ such that $e^b\varphi(0)c = b$. Let us observe that the set

$$\Lambda = \{u \in \mathcal{C}(\overline{D}); e^{-b}h \leqslant u \leqslant h\}$$

is closed, bounded, and convex in $\mathcal{C}(\overline{D})$. Consider $T : \Lambda \to \mathcal{C}(\overline{D})$ defined by

$$Tu(x) = E^x\Big[h(X_{\tau_D}) \exp\Big(-\int_0^{\tau_D} \frac{\varphi\big(u(X_s)\big)}{u(X_s)} ds\Big)\Big], \qquad x \in D.$$

Then, it is clear that $Tu \leqslant h$ for every $u \in \Lambda$. Furthermore, for every $x \in D$,

$$\begin{aligned} \frac{Tu(x)}{h(x)} &= E_h^x\Big[\exp\Big(-\int_0^{\tau_D} \frac{\varphi\big(u(X_s)\big)}{u(X_s)} ds\Big)\Big] \\ &\geqslant E_h^x\Big[\exp\Big(-e^b\varphi(0)\int_0^{\tau_D} \frac{1}{h(X_s)} ds\Big)\Big] \\ &\geqslant \exp\Big(-e^b\varphi(0)E_h^x\Big[\int_0^{\tau_D} \frac{1}{h(X_s)} ds\Big]\Big) \\ &\geqslant \exp\big(-e^b\varphi(0)c\big) \\ &= \exp(-b). \end{aligned}$$

This yields $T(\Lambda) \subset \Lambda$. On the other hand, for every $u \in \Lambda$ we have

$$e^{-b}\frac{\varphi(u)}{u}Tu \leqslant \varphi(0).$$

So, in virtue of Lemma 3.2, we deduce that the family

$$\left\{ \int_D G_D(\cdot, y) \frac{\varphi\big(u(y)\big)}{u(y)} Tu(y) dy; u \in \Lambda \right\}$$

is relatively compact in $\mathcal{C}(\overline{D})$. Since

$$Tu(x) + \int_D G_D(x, y) \frac{\varphi\big(u(y)\big)}{u(y)} Tu(y) dy = h(x), \quad x \in D,$$

it follows that $T(\Lambda)$ is relatively compact in $\mathcal{C}(\overline{D})$, and consequently T is continuous. Then, by Schauder's fixed point theorem, there exists $u \in \Lambda$ such that

$$u(x) + \int_D G_D(x, y) \varphi\big(u(y)\big) dy = h(x), \quad x \in D.$$

Hence, u is a solution to the problem (1.1). Moreover, $u \approx h$ since $u \in \Lambda$. ∎

COROLLARY 5.1. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nonincreasing function. For every $f \in \mathcal{C}^+(\partial D)$, there exists $\alpha_f > 0$ such that for every $\alpha \geqslant \alpha_f$ the problem*

$$\begin{cases} \Delta u = 2\varphi(u) & \text{in } D, \\ u = \alpha f & \text{on } \partial D, \end{cases}$$

admits a solution $u \in \mathcal{C}^+(\overline{D})$ satisfying $u \approx H_D f$.

Proof. Let $f \in \mathcal{C}^+(\partial D)$ be nontrivial and let $h = H_D f$. It suffices to consider

$$\alpha_f = e\varphi(0) \sup_{x \in D} \frac{1}{h(x)} \int_D G_D(x, y)\, dy$$

and to apply the previous theorem for αf, $\alpha \geqslant \alpha_f$. ∎

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