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BSDES WITH JUMPS AND FINITE OR INFINITE TIME HORIZON

BY

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Abstract. This paper is devoted to solving a real-valued backward stochastic differential equation with jumps where the time horizon may be finite or infinite. Under a linear growth generator, we prove the existence of a minimal solution. Using a comparison theorem we show the existence and uniqueness of solution to such equations when the generator is uniformly continuous and satisfies a weakly monotonic condition.

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1. INTRODUCTION

After the pioneer work of Pardoux and Peng [12] on linear backward stochastic differential equation (BSDE in short) with Lipschitz generator, the interest in such stochastic equations has increased thanks to many domains of applications including stochastic representation of solutions of partial differential equations (PDEs in short). For example, Pardoux and Peng [11] and Peng [13] proved that BSDEs provide a probabilistic formula for solutions of quasilinear parabolic PDEs.

BSDEs with Poisson process (BSDEP in short) were first discussed by Tang and Li [16] and Wu [18]. Studying such equations, Barles et al. [2] generalized the result in [11], and obtained a probabilistic interpretation of a solution of a parabolic integral-partial differential equation (PIDE). This was done by means of a real-valued BSDEP with Lipschitzian generator. Since then many efforts have been made in relaxing the Lipschitz assumption of the generator of the BSDEs (see [1] and [7]–[9] among others) and the BSDEP (see [10], [14], [15], [21]). Pardoux [10] solved a multidimensional BSDEP and showed an existence result under monotonicity in the second variable of the drift and the Lipschitz condition in the other ones. Royer [14] focused on weakening the Lipschitz condition required on the last variable of the generator and improved upon the results given in [2]. The key point is a strict comparison theorem and a representation of solution of the one-dimensional BSDEP in terms of non-linear expectation. But all these results are established with a fixed time horizon T. A natural question is: under which condition on the coefficients does the stochastic equation still have a solution given a square-integrable terminal value ξ? In fact, this problem has been investigated by Peng [13], Darling and Pardoux [4], and other researchers when the terminal value ξ is null or satisfies the integrability condition $\mathbf{E}(e^{\lambda T}\xi^2) < \infty$ for some $\lambda > 0$ and random terminal time T. Chen and Wang [3] were the first to establish the existence and uniqueness of solution to BSDE with infinite time horizon when the generator satisfies a Lipschitz type condition. Recently Fan et al. [6] weakened assumptions required in [3] and proved an existence and uniqueness result under mild conditions of the generator with finite or infinite time horizon.

The aim of this paper is to extend the result established in [6] to the case of BSDEP. Our motivation comes from the recent work of Yao [19]. The author proves an existence and uniqueness result of BSDEP with infinite time interval and some monotonicity condition stronger than those in [6]. In this work we show that the results obtained in [6] can be extended to BSDEP. The paper is organized as follows. We first prove the existence of a minimal solution in Section 2 and a comparison theorem in Section 3. Using these statements we deal with the solvability of finite or infinite BSDEP in Section 4.

2. BSDE WITH POISSON JUMPS

2.1. Definitions and preliminary results. Let Ω be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of Ω, and $\mathbf{P}$ a probability measure defined on $\mathcal{F}$. The triplet $(\Omega, \mathcal{F}, \mathbf{P})$ defines a probability space, which is assumed to be complete. We are given two mutually independent processes:

- a d-dimensional Brownian motion $(W_t)_{t\geqslant 0}$,
- a random Poisson measure μ on $E \times \mathbf{R}^+$ with compensator $\nu(dt, de) = \lambda(de)dt$, where the space $E = \mathbf{R} - \{0\}$ is equipped with its Borel field $\mathcal{E}$ such that $\{\widetilde{\mu}([0,t] \times A) = (\mu - \nu)[0,t] \times A\}$ is a martingale for any $A \in \mathcal{E}$ satisfying $\lambda(A) < \infty$. λ is a σ-finite measure on $\mathcal{E}$ and satisfies

$$\int_E (1 \wedge |e|^2)\lambda(de) < \infty.$$

We consider the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ given by $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu$, where for any process $\{\eta_t\}_{t\geqslant 0}$ we have $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s,\ s \leqslant r \leqslant t\} \vee \mathcal{N}$, and $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$, $\mathcal{N}$ denotes the class of $\mathbf{P}$-null sets of $\mathcal{F}$. For $Q \in \mathbf{N}^*$, $|\cdot|$ stands for the Euclidean norm in $\mathbf{R}^Q$. We consider the following sets (where $\mathbf{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbf{P}$), and a non-random horizon time T, $0 < T \leqslant +\infty$:

- $S^2(\mathbf{R}^Q)$, the space of $\mathcal{F}_t$-adapted càdlàg processes

$$\Psi : [0,T] \times \Omega \to \mathbf{R}^Q, \quad \|\Psi\|^2_{S^2(\mathbf{R}^Q)} = \mathbf{E}(\sup_{0\leqslant t\leqslant T} |\Psi_t|^2) < \infty.$$

• $H^2(\mathbf{R}^Q)$, the space of $\mathcal{F}_t$-progressively measurable processes

$$\Psi : [0,T] \times \Omega \to \mathbf{R}^Q, \qquad \|\Psi\|^2_{H^2(\mathbf{R}^Q)} = \mathbf{E} \int_0^T |\Psi_t|^2 \, dt < \infty.$$

• $L^2(\widetilde{\mu}, \mathbf{R}^Q)$, the space of mappings $U \colon \Omega \times [0,T] \times E \to \mathbf{R}^Q$ which are $\mathcal{P} \otimes \mathcal{E}$-measurable and such that

$$\|U\|^2_{L^2(\mathbf{R}^Q)} = \mathbf{E} \int_0^T \|U_t\|^2_{L^2(E,\mathcal{E},\lambda,\mathbf{R})} dt < \infty,$$

where $\mathcal{P}$ denotes the σ-algebra of $\mathcal{F}_t$-predictable sets of $\Omega \times [0,T]$, and

$$\|U_t\|^2_{L^2(E,\mathcal{E},\lambda,\mathbf{R})} = \int_E |U_t(e)|^2 \, \lambda(de).$$

We may often write $|\cdot|$ instead of $\|\cdot\|_{L^2(E,\mathcal{E},\lambda,\mathbf{R})}$ for the sake of simplicity.

Notice that the space $\mathcal{B}^2(\mathbf{R}^Q) = S^2(\mathbf{R}^Q) \times H^2(\mathbf{R}^Q) \times L^2(\widetilde{\mu}, \mathbf{R}^Q)$ endowed with the norm

$$\|(Y,Z,U)\|^2_{\mathcal{B}^2(\mathbf{R}^Q)} = \|Y\|^2_{S^2(\mathbf{R}^Q)} + \|Z\|^2_{H^2(\mathbf{R}^Q)} + \|U\|^2_{L^2(\mathbf{R}^Q)}$$

is a Banach space.

Finally, let $\mathbf{S}$ be the set of all non-decreasing continuous functions $\varphi(\cdot) \colon \mathbf{R}^+ \to \mathbf{R}^+$ satisfying $\varphi(0) = 0$ and $\varphi(s) > 0$ for $s > 0$, and put

$$\mathcal{W} = \mathbf{R} \times \mathbf{R}^d \times L^2(E, \mathcal{E}, \lambda, \mathbf{R}).$$

Let $f : \Omega \times [0,T] \times \mathcal{W} \to \mathbf{R}$ be jointly measurable. Given an $\mathcal{F}_T$-measurable $\mathbf{R}$-valued random variable ξ, we are interested in the BSDEP with parameters (ξ, f, T):

$$Y_t = \xi + \int_t^T f(r, \Theta_r) \, dr - \int_t^T Z_r dW_r - \int_t^T \int_E U_r(e) \widetilde{\mu}(dr, de), \qquad 0 \leqslant t \leqslant T, \tag{2.1}$$

where Θ_r stands for the triple (Y_r, Z_r, U_r).

For instance, let us precise the notion of solution to (2.1).

DEFINITION 2.1. A triplet of processes $(Y_t, Z_t, U_t)_{0 \leqslant t \leqslant T}$ is called a *solution to equation* (2.1) if $(Y_t, Z_t, U_t)_{0 \leqslant t \leqslant T} \in \mathcal{B}^2(\mathbf{R})$ and satisfies (2.1).

First we state some results in the case of Lipschitz type conditions of the generator. Suppose that for all $0 < T \leqslant \infty$ the following assumption (A) holds:

(A1) For all $(y,z,u) \in \mathcal{W}$, $f(\cdot, y, z, u)$ is a progressively measurable process and satisfies $\mathbf{E}\big[\big(\int_0^T |f(r,0,0,0)| dr\big)^2\big] < \infty$.

(A2) There exist two non-random functions $\gamma(\cdot)$, $\rho(\cdot) : [0,T] \to \mathbf{R}^+$ such that, for $0 \leqslant t \leqslant T$ and $(y,y') \in \mathbf{R}^2$, $(z,z') \in (\mathbf{R}^d)^2$, and $u \in L^2(E, \mathcal{E}, \lambda, \mathbf{R})$,

$$|f(t,y,z,u) - f(t,y',z',u)| \leqslant \gamma(t)|y-y'| + \rho(t)|z-z'|.$$

(A3) There exist $-1 < c \leqslant 0$ and $C > 0$, a deterministic function $\sigma(\cdot)\colon [0,T] \to \mathbf{R}^+$ and $\beta : \Omega \times [0,T] \times E \to \mathbf{R}$, $\mathcal{P} \otimes \mathcal{E}$-measurable and satisfying $c(1 \wedge |e|) \leqslant \beta_t(e) \leqslant C(1 \wedge |e|)$ such that, for all $y \in \mathbf{R},\ z \in \mathbf{R}^d$ and $u, u' \in \big(L^2(E, \mathcal{E}, \lambda, \mathbf{R})\big)^2$,

$$f(t,y,z,u) - f(t,y,z,u') \leqslant \sigma(t) \int_E \big(u(e) - u'(e)\big)\, \beta_t(e)\lambda(de). \tag{2.2}$$

(A4) The integrability condition holds: $\int_0^T \big(\gamma(s) + \rho^2(s) + \sigma^2(s)\big) ds < \infty$.

REMARK 2.1. Let us mention that (A3) implies that f is $\sigma(t)$-Lipschitz in u since we have

$$\begin{aligned}
|f(t,y,z,u) - f(t,y,z,u')| &\leqslant \widetilde{c}\,\sigma(t) \int_E |u(e) - u'(e)|(1 \wedge |e|)\lambda(de) \\
&\leqslant \widetilde{c}\,\sigma(t)\Big(\int_E |u(e) - u'(e)|^2 \lambda(de)\Big)^{1/2} \\
&:= \widetilde{c}\,\sigma(t)\|u - u'\|_{L^2(E,\mathcal{E},\lambda,\mathbf{R})},
\end{aligned}$$

where $\widetilde{c}$ is a universal positive constant.

We have the following result which is a consequence of Lemma 2.2 in [21].

LEMMA 2.1. *Let* $0 < T \leqslant \infty$ *and* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P})$. *If* (A) *holds, then equation* (2.1) *with parameters* (ξ, f, T) *has a unique solution* $(Y_t, Z_t, U_t)_{0 \leqslant t \leqslant T}$.

The proof of our main result needs a comparison theorem for infinite time horizon. Given two parameters (ξ^i, f^i, T), $i = 1, 2$, we consider the BSDEPs

$$Y_t^i = \xi^i + \int_t^T f^i(r, \Theta_r^i)dr - \int_t^T Z_r^i dW_r - \int_t^T \int_E U_r^i(e)\widetilde{\mu}(dr, de), \qquad 0 \leqslant t \leqslant T, \tag{2.3}$$

where, for $i = 1, 2$, $\Theta_\cdot^i$ stands for the triple $(Y_\cdot^i, Z_\cdot^i, U_\cdot^i)$.

Assume in addition that

(A5) $\xi^1 \leqslant \xi^2$ and, for all (ω, t, y, z, u), $f^1(\omega, t, y, z, u) \leqslant f^2(\omega, t, y, z, u)$.

We have the following theorem, which can be regarded as a corollary to Theorem 3.2 in Section 3.

THEOREM 2.1. *Suppose that* f^1 *and* f^2 *satisfy* (A1)–(A5) *and* $0 < T \leqslant +\infty$. *If* $(Y^i, Z^i, U^i),\ i = 1, 2$, *are solutions to* (2.3), *then we have*

$$\forall\, 0 \leqslant t \leqslant T,\ Y_t^1 \leqslant Y_t^2\ \mathbf{P}\text{-}a.s.$$

REMARK 2.2. Theorem 2.1 establishes a comparison theorem in the case of Lipschitz coefficients for either $T < \infty$ or $T = +\infty$. Basically it improves the well-known result for the finite time horizon.

Let us now deal with our problem.

2.2. Existence of a minimal solution. In this section, we will prove the existence of a minimal solution for BSDEPs when their generators are continuous and have a linear growth (see Theorem 2.2 below). First let us give the following

DEFINITION 2.2. A solution $(Y_t, Z_t, U_t)_{0\leqslant t\leqslant T}$ of equation (2.1) is called *minimal* if for any other solution $(\widetilde{Y}_t, \widetilde{Z}_t, \widetilde{U}_t)_{0\leqslant t\leqslant T}$ to (2.1) we have, for each $0 \leqslant t \leqslant T$, $Y_t \leqslant \widetilde{Y}_t$.

Now we introduce the list of conditions weaker than those required in [2], [14], [19], [21]. We assume that $0 < T \leqslant +\infty$ and the generator f satisfies the following assumptions (H1):

(H1.1) There exist three functions $\gamma(\cdot)$, $\rho(\cdot)$, $\sigma(\cdot) : [0,T] \to \mathbf{R}^+$ satisfying assumption (A4).

(H1.2) There exists an $\mathcal{F}_t$-progressively measurable nonnegative process $(f_t)_{0\leqslant t\leqslant T}$ such that $\mathbf{E}\big[\big(\int_0^T f_t dt\big)^2\big] < \infty$ and, for $(t, y, z, u) \in [0,T] \times \mathbf{R} \times \mathbf{R}^d \times L^2(E, \mathcal{E}, \lambda, \mathbf{R})$,

$$|f(\omega, t, y, z, u)| \leqslant f_t(\omega) + \gamma(t)|y| + \rho(t)|z| + \sigma(t)|u|.$$

(H1.3) $f(\omega, t, \cdot, \cdot, \cdot) : \mathbf{R} \times \mathbf{R}^d \times L^2(E, \mathcal{E}, \lambda, \mathbf{R}) \to \mathbf{R}$ is continuous.

As in [8], we are led to consider the sequence $f_n : \Omega \times [0,T] \times \mathcal{W} \to \mathbf{R}$ associated with f, which for $(\omega, t, y, z, u) \in \Omega \times [0,T] \times \mathcal{W}$ is given by

$$f_n(\omega, t, y, z, u) = \inf_{(y',z',u')\in\mathcal{W}} [f(\omega, t, y', z', u') + n(|y-y'| + |z-z'| + |u-u'|)].$$

Using similar computations to those in the proof of Lemma 1 in [8], one can obtain the following proposition. We omit its proof.

PROPOSITION 2.1. *Assume that f satisfies* (H1). *Then the sequence of functions f_n is well defined for each $n \geqslant 1$ and satisfies the following conditions $d\mathbf{P} \times dt$-a.s.*:

(i) *Linear growth*: *for all $n \geqslant 1$ and for all y, z, u,*

$$|f_n(\omega, t, y, z, u)| \leqslant f_t(\omega) + \gamma(t)|y| + \rho(t)|z| + \sigma(t)|u|.$$

(ii) *Monotonicity in n*: *for all y, z, u, $f_n(\omega, t, y, z, u)$ increases in n.*

(iii) *Convergence*: *If $(y_n, z_n, u_n) \to (y, z, u)$ in $\mathbf{R} \times \mathbf{R}^d \times L^2(E, \mathcal{E}, \lambda, \mathbf{R})$ as $n \to \infty$, then for each $(\omega, t) \in \Omega \times [0,T]$ we have*

$$f_n(\omega, t, y_n, z_n, u_n) \to f(\omega, t, y, z, u) \quad \text{as } n \to \infty. \tag{2.4}$$

(iv) *The Lipschitz condition*: *for all $n \geqslant 1$ and for all y, y', z, z', u, u', we have*

$$|f_n(\omega, t, y, z, u) - f_n(\omega, t, y', z', u')| \leqslant n\gamma(t)|y-y'| + n\rho(t)|z-z'| + n\sigma(t)|u-u'|.$$

Thus, by Lemma 2.1, the following BSDEP with parameters (ξ, f_n, T):
(2.5)
$$Y_t^n = \xi + \int_t^T f_n\,(r, \Theta_r^n)\,dr - \int_t^T Z_r^n dW_r - \int_t^T \int_E U_r^n(e)\widetilde{\mu}(dr, de), \qquad 0 \leqslant t \leqslant T,$$
has a unique solution $(\Theta_t^n)_{0\leqslant t\leqslant T} = (Y_t^n, Z_t^n, U_t^n)_{0\leqslant t\leqslant T}$.

The main result in this section is the following

THEOREM 2.2. *Let* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P})$ *and* $0 < T \leqslant \infty$. *Under the assumption* (H1), *the BSDEP* (2.1) *has a minimal solution* $(Y_t, Z_t, U_t)_{0\leqslant t\leqslant T}$.

Proof. We follow the proof of Theorem 1 in [6]. Consider $F : \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d \times L^2(E, \mathcal{E}, \lambda, \mathbf{R}) \to \mathbf{R}$ given by
$$\forall(\omega, t, y, z, u),\ F(\omega, t, y, z, u) = f_t(\omega) + \gamma(t)|y| + \rho(t)|z| + \sigma(t)|u|.$$
It follows from Lemma 2.1 that the BSDEP with parameters (ξ, F, T) admits a unique solution $(\widetilde{Y}_t, \widetilde{Z}_t, \widetilde{U}_t)_{0\leqslant t\leqslant T}$. Applying Theorem 2.1 and Proposition 2.1, we deduce that
$$\forall(\omega, t) \in \Omega \times [0,T],\ Y_t^1(\omega) \leqslant Y_t^n(\omega) \leqslant Y_t^{n+1}(\omega) \leqslant \widetilde{Y}_t(\omega).$$
Hence there exists an $\mathcal{F}_t$-progressively measurable process $(Y_t)_{0\leqslant t\leqslant T}$ such that $\lim_{n\to+\infty} Y_t^n(\omega) = Y_t(\omega)$. Putting $G = \sup_n \sup_{0\leqslant s\leqslant T} |Y_s^n(\omega)|$ and arguing as in [6], Theorem 1, we have
$$\mathbf{E}\big(\sup_{0\leqslant s\leqslant T} |Y_s(\omega)|^2\big) \leqslant \mathbf{E}(G^2) < \infty.$$
Itô's formula applied to equation (2.5) yields, for $0 \leqslant t \leqslant T$,
$$\begin{aligned}\mathbf{E}|Y_t^n|^2 + \mathbf{E}\int_t^T |Z_r^n|^2 dr + \mathbf{E}\int_t^T\int_E |U_r^n(e)|^2\lambda(de)dr &\leqslant \mathbf{E}|\xi|^2 + 2\mathbf{E}\int_t^T Y_r^n f_n(r, \Theta_r^n)dr\\ &\leqslant \mathbf{E}|\xi|^2 + 2\mathbf{E}\int_t^T |Y_r^n|\big(f_r + \gamma(r)|Y_r^n| + \rho(r)|Z_r^n| + \sigma(r)|U_r^n|\big)dr.\end{aligned}$$
Using the inequality $2ab \leqslant a^2\varepsilon + b^2/\varepsilon$ for every $a \geqslant 0, b \geqslant 0$, and $\varepsilon > 0$, we deduce that
$$\begin{aligned}&\mathbf{E}\int_0^T |Z_r^n|^2 dr + \mathbf{E}\int_0^T\int_E |U_r^n(e)|^2\lambda(de)dr\\ &\quad\leqslant \mathbf{E}|\xi|^2 + (1 + \delta + \delta')\mathbf{E}(G^2) + \mathbf{E}\big[\big(\int_0^T f_r dr\big)^2\big] + 2\mathbf{E}(G^2)\cdot\int_0^T \gamma(r)dr\\ &\quad\quad + \frac{1}{\delta}\mathbf{E}\big[\big(\int_0^T \rho(r)|Z_r^n|dr\big)^2\big] + \frac{1}{\delta'}\mathbf{E}\big[\big(\int_0^T\int_E \sigma(r)|U_r^n(e)|\lambda(de)dr\big)^2\big],\end{aligned}$$

where $\delta = 2\int_0^T \rho^2(s)ds$ and $\delta' = 2\int_0^T \sigma^2(s)ds$. Applying Hölder's inequality in the last two integrals, we obtain

$$\mathbf{E}\big[\int_0^T |Z_r^n|^2 dr + \int_0^T\int_E |U_r^n(e)|^2 \lambda(de)dr\big] \leqslant \frac{1}{2}\mathbf{E}\big[\int_0^T |Z_r^n|^2 dr + \int_0^T\int_E |U_r^n(e)|^2\lambda(de)dr\big] + M,$$

where

$$M = \mathbf{E}|\xi|^2 + (1+\delta+\delta')\mathbf{E}(G^2) + \mathbf{E}\big[\big(\int_0^T f_r dr\big)^2\big] + 2\mathbf{E}(G^2)\cdot\int_0^T \gamma(r)dr > 0$$

and depends only on the parameters f, ξ, and T. Consequently, we have

$$\sup_{n\in\mathbf{N}} \mathbf{E}\int_0^T |Z_r^n|^2 dr \leqslant 2M \quad \text{and} \quad \sup_{n\in\mathbf{N}} \mathbf{E}\int_0^T\int_E |U_r^n(e)|^2\lambda(de)dr \leqslant 2M.$$

Let us define, for $\delta \in \{Y, Z, U\}$ and integers $n, m \geqslant 1$, $\delta^{n,m} = \delta^n - \delta^m$. Applying again Itô's formula, we deduce from (2.5) the relation

$$\mathbf{E}|Y_t^{n,m}|^2 + \mathbf{E}\int_t^T |Z_r^{n,m}|^2 dr + \mathbf{E}\int_t^T\int_E |U_r^{n,m}(e)|^2\lambda(de)dr \leqslant 2\mathbf{E}\int_t^T Y_r^{n,m}\big(f_n(r,\Theta_r^n) - f_m(r,\Theta_r^m)\big)dr, \quad 0 \leqslant t \leqslant T.$$

Using once again Hölder's inequality and the assumption (H1) we obtain

$$\begin{aligned}
&\mathbf{E}|Y_0^{n,m}|^2 + \mathbf{E}\int_0^T |Z_r^{n,m}|^2 dr + \mathbf{E}\int_0^T\int_E |U_r^{n,m}(e)|^2\lambda(de)dr \\
&\leqslant 4\mathbf{E}\int_0^T |Y_r^{n,m}| f_r dr + 4\big(\mathbf{E}(G^2)\big)^{1/2}\cdot\Big(\mathbf{E}\big[\big(\int_0^T |Y_r^{n,m}|\gamma(r)dr\big)^2\big]\Big)^{1/2} \\
&\quad + 2\sqrt{8M}\big(\mathbf{E}\int_0^T |Y_r^{n,m}|^2\rho^2(r)dr\big)^{1/2} + 2\sqrt{8M}\big(\mathbf{E}\int_0^T |Y_r^{n,m}|^2\sigma^2(r)dr\big)^{1/2}.
\end{aligned}$$

In particular, Lebesgue's dominated convergence theorem implies that $\{Z^n\}$ (respectively, $\{U^n\}$) is a Cauchy sequence in $H^2(\mathbf{R}^d)$ (respectively, $L^2(\widetilde{\mu}, \mathbf{R})$). Hence there exists $(Z, U) \in H^2(\mathbf{R}^d) \times L^2(\widetilde{\mu}, \mathbf{R})$ such that

$$\|Z^n - Z\|^2_{H^2(\mathbf{R}^d)} \to 0 \quad \text{and} \quad \|U^n - U\|^2_{L^2(\mathbf{R})} \to 0 \quad \text{as } n \to \infty,$$

which implies along a subsequence, if necessary,

$$Z^n \xrightarrow{H^2(\mathbf{R}^d)} Z \quad \text{and} \quad U^n \xrightarrow{L^2(\widetilde{\mu},\mathbf{R})} U \quad \text{as } n \to \infty.$$

Further, by virtue of (2.4), we get $f_n(s, Y_s^n, Z_s^n, U_s^n) \to f(s, Y_s, Z_s, U_s)$ as $n \to \infty$, $0 \leqslant s \leqslant T$, and arguing as in [6], Theorem 1, we obtain

$$\lim_{n\to\infty} \mathbf{E}\Big(\int_0^T |f_n(r,\Theta_r^n) - f(r,\Theta_r)|dr\Big)^2 = 0 \text{ and } \lim_{n\to\infty} \mathbf{E}\big(\sup_{0\leqslant t\leqslant T} |Y_t^n - Y_t|^2\big) = 0.$$

This is enough to deduce that $Y \in S^2(\mathbf{R})$. Letting $n \to +\infty$ in (2.5), we prove that $(Y_s, Z_s, U_s)_{0\leqslant s\leqslant T}$ is a solution to (2.1).

Let $(Y', Z', U') \in \mathcal{B}^2(\mathbf{R})$ be a solution of equation (2.1). By Theorem 2.1, for all $n \geqslant 1$, we have $Y^n \leqslant Y'$. Letting $n \to \infty$, we get $Y \leqslant Y'$. This implies that Y is the minimal solution to (2.1). ■

3. COMPARISON THEOREM

We intend to prove a comparison theorem under mild conditions on the drift of the BSDEP. This result is useful for the proof of the existence and uniqueness of solution.

Let us introduce, for $0 < T \leqslant +\infty$, the following assumptions (H2) on the generator f:

(H2.1) f is weakly monotonic in y, i.e., there exist $\gamma(\cdot)\colon [0,T] \to \mathbf{R}^+$ satisfying $\int_0^T \gamma(t)dt < \infty$ and a concave function $\varrho \in \mathbf{S}$ such that $\int_{0^+} \big(\varrho(r)\big)^{-1} dr = +\infty$, and for any $(y, y') \in \mathbf{R}^2,\ z \in \mathbf{R}^d,\ u \in L^2(E, \mathcal{E}, \lambda, \mathbf{R})$,

$$(3.1) \qquad (y - y')\big(f(t,y,z,u) - f(t,y',z,u)\big) \leqslant |y - y'|\gamma(t)\varrho(|y - y'|),$$

and we assume that $\varrho(x) \leqslant k(x+1)$, where k denotes a linear growth constant of ϱ.

(H2.2) f is uniformly continuous in z and there exist $\rho(\cdot)\colon [0,T] \to \mathbf{R}^+$ satisfying $\int_0^T \rho^2(t)dt < \infty$ and $\phi \in \mathbf{S}$ such that

$$|f(t,y,z,u) - f(t,y,z',u)| \leqslant \rho(t)\phi(|z - z'|),$$

and we assume that $\phi(x) \leqslant ax + b,\ a > 0, b \geqslant 0$. Futhermore, we assume that $\int_0^T \rho(t)dt < \infty$ when $b \neq 0$.

(H2.3) There exist $-1 < c \leqslant 0$ and $C > 0$, a deterministic function $\sigma(\cdot)\colon [0,T] \to \mathbf{R}^+$ satisfying $\int_0^T \sigma^2(s)ds < \infty$, and $\beta : \Omega \times [0,T] \times E \to \mathbf{R}$, $\mathcal{P} \otimes \mathcal{E}$-measurable satisfying $c(1 \wedge |e|) \leqslant \beta_t \leqslant C(1 \wedge |e|)$ such that, for all $y \in \mathbf{R}$, $z \in \mathbf{R}^d$ and $u, u' \in \big(L^2(E, \mathcal{E}, \lambda, \mathbf{R})\big)^2$,

$$(3.2) \qquad f(t,y,z,u) - f(t,y,z,u') \leqslant \sigma(t) \int_E \big(u(e) - u'(e)\big)\beta_t(e)\lambda(de).$$

Given two parameters (ξ^1, f^1) and (ξ^2, f^2), we are interested in the following two one-dimensional BSDEPs with $0 \leqslant t \leqslant T$:

$$Y_t^1 = \xi^1 + \int_t^T f^1\big(r, \Theta_r^1\big)\, dr - \int_t^T Z_r^1 dW_r - \int_t^T \int_E U_r^1(e)\widetilde{\mu}(dr, de), \tag{3.3}$$

$$Y_t^2 = \xi^2 + \int_t^T f^2\big(r, \Theta_r^2\big)\, dr - \int_t^T Z_r^2 dW_r - \int_t^T \int_E U_r^2(e)\widetilde{\mu}(dr, de), \tag{3.4}$$

and we assume in addition that

(H2.4) For all (t, y, z, u) we have $f^1(t, y, z, u) \leqslant f^2(t, y, z, u)$ and $\xi^1 \leqslant \xi^2$.

We state the following result (see [6], Lemma 3), which will be useful in the sequel.

LEMMA 3.1. *Let $\Psi(\cdot) : \mathbf{R}^+ \to \mathbf{R}^+$ be a nondecreasing function with linear growth, which means that there exists $K > 0$ such that, for all $x \in \mathbf{R}^+$, $\Psi(x) \leqslant K(x+1)$. Then for each $n \geqslant 2K$ we have*

$$\Psi(x) \leqslant nx + \Psi\left(\frac{2K}{n}\right), \qquad x \geqslant 0.$$

Before proving the main statement of this section, let us recall the Girsanov theorem for discontinuous processes. If $\mathcal{M}^2$ denotes the set of square-integrable martingales, we can define, using the martingale representation (see [16], Lemma 2.3), a mapping

$$\begin{aligned} &\Phi : \mathcal{M}^2 \to H^2(\mathbf{R}^d) \times L^2(\widetilde{\mu}, \mathbf{R}), \\ &\qquad M \mapsto (\theta, \upsilon) \quad \text{such that } M_t = \int_0^t \theta_s dW_s + \int_0^t \int_E \upsilon_r(e)\widetilde{\mu}(de, dr). \end{aligned}$$

Let $\mathcal{M} = \big\{M = (M_t)_{t\geqslant 0} \in \mathcal{M}^2 \big| \|\theta_s\| \leqslant C, \upsilon_s(x) > -1,\ |\upsilon_s(x)| \leqslant C(1 \wedge |x|)$ a.s. with $\Phi(M) = (\theta, \upsilon)\big\}$. For $M \in \mathcal{M}$, the Doléans-Dade exponential of M is defined by

$$\mathcal{E}(M)_T = \exp\left(M_T - \frac{1}{2}\langle M^c\rangle_T\right) \prod_{0<s\leqslant T} (1 + \Delta M_s)\exp(-\Delta M_s).$$

We have

THEOREM 3.1 (Girsanov's theorem). *Let $(\overline{Z}, \overline{U}) \in H^2(\mathbf{R}^d) \times L^2(\widetilde{\mu}, \mathbf{R})$ and $K_t = \int_0^t \overline{Z}_s dW_s + \int_0^t \int_E \overline{U}_r(e)\widetilde{\mu}(de, dr)$. If $M \in \mathcal{M}$, then the process $\widetilde{K} = K - \langle K, M\rangle$ is a martingale under the probability measure $\widetilde{\mathbf{P}}$ such that $d\widetilde{\mathbf{P}}/d\mathbf{P} = \mathcal{E}(M)_T$.*

Here is the main result of this section.

THEOREM 3.2. *Let* $0 < T \leqslant +\infty$. *Assume we are given* f^1, f^2 *and* $(\xi^1, \xi^2) \in \left(L^2(\Omega, \mathcal{F}_T, \mathbf{P})\right)^2$ *such that the conditions* (H2) *hold true. If* $(Y_t^1, Z_t^1, U_t^1)_{0 \leqslant t \leqslant T}$ *and* $(Y_t^2, Z_t^2, U_t^2)_{0 \leqslant t \leqslant T}$ *are solutions of equations* (3.3) *and* (3.4), *respectively, then, for all* $0 \leqslant t \leqslant T$, *we have*

$$Y_t^1 \leqslant Y_t^2 \ \ \mathbf{P}\text{-}a.s.$$

Proof. We assume $d = 1$. Putting

$$\widehat{\Theta}_t = (\widehat{Y}_t, \widehat{Z}_t, \widehat{U}_t) = (Y_t^1 - Y_t^2, Z_t^1 - Z_t^2, U_t^1 - U_t^2), \quad \widehat{\xi} = \xi^1 - \xi^2, \tag{3.5}$$

we can see that $(\widehat{\Theta}_t)_{0 \leqslant t \leqslant T}$ satisfies the following BSDEP for $0 \leqslant t \leqslant T$:

$$\widehat{Y}_t = \widehat{\xi} + \int_t^T [f^1(r, \Theta_r^1) - f^2(r, \Theta_r^2)] dr - \int_t^T \widehat{Z}_r dW_r - \int_t^T \int_E \widehat{U}_r(e) \widetilde{\mu}(dr, de). \tag{3.6}$$

Tanaka–Meyer's formula yields

$$\begin{aligned} \widehat{Y}_t^+ \leqslant \widehat{\xi}^+ &+ \int_t^T \mathbf{1}_{\{\widehat{Y}_r > 0\}} [f^1(r, \Theta_r^1) - f^2(r, \Theta_r^2)] dr - \int_t^T \mathbf{1}_{\{\widehat{Y}_r > 0\}} \widehat{Z}_r dW_r \\ &- \int_t^T \int_E \mathbf{1}_{\{\widehat{Y}_r > 0\}} \widehat{U}_r(e) \widetilde{\mu}(de, dr), \quad 0 \leqslant t \leqslant T, \ x^+ = \max(x, 0). \end{aligned} \tag{3.7}$$

Further we have

$$f^1(r, \Theta_r^1) - f^2(r, \Theta_r^2) = [f^1(r, \Theta_r^1) - f^1(r, \Theta_r^2)] + [f^1(r, \Theta_r^2) - f^2(r, \Theta_r^2)]$$

and the assumption (H2.4) implies that the right-hand side is less than

$$\begin{aligned} [f^1(r, \Theta_r^1) - f^1(r, Y_r^2, Z_r^1, U_r^1)] + [f^1(r, Y_r^2, Z_r^1, U_r^1) - f^1(r, Y_r^2, Z_r^2, U_r^1)] \\ + [f^1(r, Y_r^2, Z_r^2, U_r^1) - f^1(r, \Theta_r^2)]. \end{aligned}$$

Hence applying (H2.1) and (H2.3) we deduce that

$$\begin{aligned} \mathbf{1}_{\{\widehat{Y}_r > 0\}} [f^1(r, \Theta_r^1) - f^2(r, \Theta_r^2)] \leqslant \gamma(r) \varrho(\widehat{Y}_r^+) + \mathbf{1}_{\{\widehat{Y}_r > 0\}} \rho(r) \phi(|\widehat{Z}_r|) \\ + \int_E \mathbf{1}_{\{\widehat{Y}_r > 0\}} \widehat{U}_r(e) \sigma(r) \beta_r(e) \lambda(de). \end{aligned}$$

By Lemma 3.1 with $\Psi(\cdot) = \phi(\cdot)$ and $K = c = a + b$, we have

$$\mathbf{1}_{\{\widehat{Y}_r > 0\}} \rho(r) \phi(|\widehat{Z}_r|) \leqslant \mathbf{1}_{\{\widehat{Y}_r > 0\}} n \rho(r) |\widehat{Z}_r| + \mathbf{1}_{\{\widehat{Y}_r > 0\}} \rho(r) \phi\left(\frac{2c}{n}\right), \quad n \geqslant 2c.$$

Putting pieces together, we infer from (3.7) that

$$\widehat{Y}_t^+ \leqslant a_n + \int_t^T \gamma(r)\varrho(\widehat{Y}_r^+)dr + \widetilde{K}_t, \tag{3.8}$$

where

$$\widetilde{K}_t = \int_t^T \left[\mathbf{1}_{\{\widehat{Y}_r>0\}}\widehat{Z}_r\left(\frac{n\rho(r)\widehat{Z}_r}{|\widehat{Z}_r|}\mathbf{1}_{\{\widehat{Z}_r\neq 0\}}\right) + \int_E \mathbf{1}_{\{\widehat{Y}_r>0\}}\widehat{U}_r(e)\sigma(r)\beta_r(e)\lambda(de)\right]dr \\ - \int_t^T \mathbf{1}_{\{\widehat{Y}_r>0\}}\widehat{Z}_r dW_r - \int_t^T\int_E \mathbf{1}_{\{\widehat{Y}_r>0\}}\widehat{U}_r(e)\widetilde{\mu}(de,dr)$$

and

$$a_n = \mathbf{1}_{b\neq 0}\phi\left(\frac{2c}{n}\right)\cdot\int_0^T \rho(r)dr \to 0 \quad \text{as } n\to\infty,$$

b being given in (H2.2). Define

$$M_t = \int_0^t \left(\frac{n\rho(r)\widehat{Z}_r}{|\widehat{Z}_r|}\mathbf{1}_{\{\widehat{Z}_r\neq 0\}}\right)dW_r + \int_0^t\int_E \sigma(r)\beta_r(e)\widetilde{\mu}(de,dr), \quad 0\leqslant t\leqslant T,$$
$$K_t = \int_0^t \mathbf{1}_{\{\widehat{Y}_r>0\}}\widehat{Z}_r dW_r + \int_0^t\int_E \mathbf{1}_{\{\widehat{Y}_r>0\}}\widehat{U}_r(e)\widetilde{\mu}(de,dr), \quad 0\leqslant t\leqslant T.$$

By Theorem 3.1, it follows that $\widetilde{K}_t$ is a martingale under the probability measure $\widetilde{\mathbf{P}} = \mathcal{E}(M)_T\cdot\mathbf{P}$. Hence, taking $\widetilde{\mathbf{E}}\,(\cdot|\mathcal{F}_t)$, the conditional expectation given $\mathcal{F}_t$ under the probability measure $\widetilde{\mathbf{P}}$, and taking into account that ϱ is concave, we deduce that

$$\widetilde{\mathbf{E}}(\widehat{Y}_s^+|\mathcal{F}_t) \leqslant a_n + \int_s^T \gamma(r)\varrho(\widetilde{\mathbf{E}}[\widehat{Y}_r^+|\mathcal{F}_t])dr, \quad t\leqslant s\leqslant T.$$

Thus Lemma 5 in [6] implies that $\widehat{Y}_t^+ = 0$, which is true if and only if $Y_t^1 \leqslant Y_t^2$. ∎

The following corollary is immediate.

COROLLARY 3.1. *Let* $0 < T \leqslant +\infty$. *If* $\xi \in L^2(\Omega,\mathcal{F}_T,\mathbf{P})$ *and* f *satisfies* (H2), *then the BSDEP* (2.1) *with parameters* (ξ, f, T) *has at most one solution.*

4. EXISTENCE AND UNIQUENESS OF SOLUTION

Using the results established in the previous section, we can now investigate the solvability of our equation under weaker conditions on the generator.

Assume that $f : \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d \times L^2(E,\mathcal{E},\lambda,\mathbf{R}) \to \mathbf{R}$ is uniformly continuous with respect to its variables and satisfies (H3):

$$|f(t,y,z,u) - f(t,y',z',u)| \leqslant \gamma(t)\varrho(|y-y'|) + \rho(t)\phi(|z-z'|),$$
$$f(t,y,z,u) - f(t,y,z,u') \leqslant \sigma(t) \int_E \big(u(e) - u'(e)\big)\,\beta_t(e)\lambda(de),$$

where $\gamma, \rho, \sigma, \phi$, and β are as in (H2).

We claim

THEOREM 4.1. *Let* $0 < T \leqslant +\infty$ *and* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P})$. *If* f *satisfies* (H3) *and* (A1), *then equation* (2.1) *admits a unique solution.*

Proof. The uniqueness follows from Corollary 3.1 since (H3) implies (H2). Moreover, from (H3) one can infer that

$$\begin{aligned}|f(\omega,t,y,z,u)| &\leqslant \gamma(t)\varrho(|y|) + \rho(t)\phi(|z|) + \widetilde{c}\,\sigma(t)\Big(\int_E |u(e)|^2\lambda(de)\Big)^{1/2}\\ &\quad + |f(\omega,t,0,0,0)|\\ &\leqslant f_t + k\gamma(t)|y| + a\rho(t)|z| + \widetilde{c}\,\sigma(t)|u|,\end{aligned}$$

where $f_t = k\gamma(t) + b\rho(t) + |f(\omega,t,0,0,0)|$. Hence Theorem 2.2 ensures the existence of a minimal solution. This completes the proof. ∎

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