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THE LIMITING BEHAVIOUR OF SUMS AND MAXIMUMS OF IID RANDOM VARIABLES FROM THE VIEWPOINT OF DIFFERENT OBSERVERS

BY

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Abstract. The limiting behaviour of observed and all random variables in the max limit schema was considered by Mladenović and Piterbarg (2006) and Krajka (2011). Here those results are generalised in two directions:
- we allow more than one observer and one superobserver;
- we consider the max limit schema as well as the sum limit schema.

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1. INTRODUCTION

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent identically distributed (iid) random variables. For the distribution function F we will write $\{X_n, n \geqslant 1\} \in D_M(F, a_n, b_n)$ (for short, $D_M(F)$) if

$$\frac{\max\limits_{1 \leqslant i \leqslant n} X_i - b_n}{a_n} \xrightarrow{\mathcal{D}} F \quad \text{as } n \to \infty,$$

and $\{X_n, n \geqslant 1\} \in D_S(F, a_n, b_n)$ (for short, $D_S(F)$) if

$$\frac{\sum\limits_{i=1}^{n} X_i - b_n}{a_n} \xrightarrow{\mathcal{D}} F \quad \text{as } n \to \infty.$$

In Mladenović and Piterbarg [4] the following theorem (in an equivalent form) was proved:

THEOREM 1.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of iid random variables such that $\{X_n\} \in D_M(G)$, and let $\{\varepsilon_n, n \geqslant 1\}$ be a sequence of indicators which is* *independent of* $\{X_n, n \geqslant 1\}$ *and such that for* $p \in [0,1]$

$$\frac{\sum_{i=1}^{n} I[\varepsilon_i = 1]}{n} \xrightarrow{\mathcal{P}} p \quad \Big(\frac{\sum_{i=1}^{n} I[\varepsilon_i = 0]}{n} \xrightarrow{\mathcal{P}} 1-p\Big).$$

Then

$$\lim_{n\to\infty} P[\max_{\substack{\{i:1\leqslant i\leqslant n,\\ \varepsilon_i=1\}}} X_i < a_n x + b_n, \max_{\substack{\{i:1\leqslant i\leqslant n,\\ \varepsilon_i=0\}}} X_i < a_n y + b_n] = G^p(x)G^{1-p}(y),$$

where (*and in the sequel*) *we put* $\max\emptyset = -\infty$.

This theorem may be interpreted as follows: we have the set of two observers, who observe the sequential random variables from the sequence $\{X_n, n \geqslant 1\}$ but so that

1. all random variables are observed;
2. each of them is observed only by one observer.

The random variables $\{\varepsilon_i, i \geqslant 1\}$ indicate the number of observers. The observed random variables are collected in the max schema. In Krajka [3] this theorem was generalised to the case when p is a random variable.

In this paper we generalise Theorem 1.1 in the following directions:

(i) We take more than two observers and allow that some observers collect random variables in the max schema and some observers collect them in the sum schema.

(ii) In consequence, the random variables $\{\varepsilon_i, i \geqslant 1\}$ are not indicators, but must take values from the set $\{1, 2, 3, \ldots, k, k+1, \ldots, k+l\}$, where k is a number of observers in the sum schema and l is a number of observers in the max schema. For clear explanation we consider the sequence $\{Y_n, n \geqslant 1\}$ and collections of sets $\{A_1, A_2, \ldots, A_{k+l}\}$ such that

$$\bigcup_{i=1}^{k+l} A_i = \mathbb{R}, \qquad A_i \cap A_j = \emptyset \ \text{ for } i \neq j,$$

instead of $\{\varepsilon_n, n \geqslant 1\}$, where the indicator $I[Y_i \in A_j]$ shows that the i-th random variable X_i is observed by the j-th observer, assuming that

$$\text{(1.1)} \qquad \begin{gathered} \frac{\sum_{i=1}^{n} I[Y_i \in A_j]}{n} \xrightarrow{\mathcal{P}} p_j, \qquad 1 \leqslant j \leqslant k+l, \\ \sum_{j=1}^{k+l} p_j = 1, \qquad 0 \leqslant p_j \leqslant 1,\ 1 \leqslant j \leqslant k+l. \end{gathered}$$

In particular, we may put $A_i = \{i\},\ Y_i = \varepsilon_i,\ 1 \leqslant i \leqslant k+l-1,\ A_{k+l} = \mathbb{R} \setminus \bigcup_{j=1}^{k+l-1} A_j$. Then $I[Y_i \in A_j] = I[\varepsilon_i = j]$.

(iii) We also consider the convergence with two superobservers. One observes all random variables in the max schema (if $l > 0$) and the second observes all random variables in the sum schema (if $k > 0$).

(iv) Instead of the numbers p_j we consider also the random variables.

Thus in this paper we consider the limiting behaviour of

$$P\Bigg[\frac{\sum_{i=1}^{n} X_i I[Y_i \in A_1] - np_1\mu}{\sqrt{n}\sigma} < x_1, \dots, \frac{\sum_{i=1}^{n} X_i I[Y_i \in A_k] - np_k\mu}{\sqrt{n}\sigma} < x_k,$$
$$\max_{\substack{\{i:1\leqslant i\leqslant n,\\ Y_i\in A_{k+1}\}}} X_i < a_n x_{k+1} + b_n, \dots, \max_{\substack{\{i:1\leqslant i\leqslant n,\\ Y_i\in A_{k+l}\}}} X_i < a_n x_{k+l} + b_n\Bigg]$$

for $\{X_n, n \geqslant 1\} \in D_S(\Phi)$ and $\{X_n, n \geqslant 1\} \in D_M(G)$, where Φ is the standard normal distribution function, and G is the distribution function – one of the three possible limits in the max schema:

(i) $G_1(x) = \exp\{-e^{-x}\}, \quad x \in \mathbb{R}$,

(ii) $G_{2,\gamma}(x) = \begin{cases} \exp(-x^{-\gamma}), & x > 0, \\ 0, & x \leqslant 0, \end{cases}$

(iii) $G_{3,\gamma}(x) = \begin{cases} 1, & x \geqslant 0, \\ \exp\big(-(-x)^{\gamma}\big), & x < 0, \end{cases}$

for some $\gamma > 0$. Throughout the paper we put $0^0 = 1$. We denote by A^c the complement of the set A. Furthermore, we note that $\sum_{i=b}^{a} = -\sum_{i=a}^{b}$ for $a < b$.

2. MAIN RESULTS

Let $\{X_n, n \geqslant 1\}$ be a sequence of iid random variables with the distribution function F. We denote by $A_i, 1 \leqslant i \leqslant k + l$, Borel measurable pairwise disjoint sets ($A_i \in \mathcal{B}(\mathbb{R})$), where k and l are nonnegative integer constants indicating the number of observers in the sum schema and max schema, respectively, and such that $k + l > 0$. Let $\{Y_i, i \geqslant 1\}$ be a sequence of random variables independent of $\{X_i, i \geqslant 1\}$ and such that $I[Y_i \in A_j]$ indicates that the random variable X_i is observed by the j-th observer. Let $\lambda(A_i) = \lambda(A_i, \omega)$ be a nonnegative random measure, $\lambda : \mathcal{B}(\mathbb{R}) \times \Omega \to [0, 1]$, defined for all $A_i, 1 \leqslant i \leqslant k + l$, and any $\omega \in \Omega$ (in fact, we may consider a finite family of random variables, indexed by sets). Let us define, for an arbitrary nonnegative random variable η,

$$S_{i,j} = \sum_{m=1}^{j} I[Y_m \in A_i], \qquad 1 \leqslant i \leqslant k + l, j \geqslant 1,$$
$$\Phi(x, \eta) = \Phi\left(\frac{x}{\sqrt{\eta}}\right) I[\eta > 0] + I[x > 0, \eta = 0],$$

where Φ is a standard normal distribution function.

THEOREM 2.1. *If $k > 0$ we assume that $EX_1 = \mu$, $EX_1^2 - (EX_1)^2 = \sigma^2$ and $\{X_n, n \geqslant 1\} \in D_S(\Phi, n\mu, \sigma\sqrt{n})$; on the other hand, if $l > 0$ then we assume $\{X_n, n \geqslant 1\} \in D_M(G, a_n, b_n)$ for some distribution function G $(G_1, G_{2,\gamma}$, or $G_{3,\gamma})$, some sequence of positive numbers $\{a_n, n \geqslant 1\}$, and a sequence of arbitrary numbers $\{b_n, n \geqslant 1\}$. Assume that, for some pairwise disjoint sets $A_j \in \mathcal{B}(\mathbb{R})$,*

$$\frac{\sum_{i=1}^{n} I[Y_i \in A_j]}{n} \xrightarrow{\mathcal{P}} \lambda(A_j) \quad \text{as } n \to \infty, \quad j = 1, 2, \ldots, k+l. \tag{2.1}$$

We have

(2.2)

$$P\Big[\bigcap_{i=1}^{k}\Big[\frac{\sum_{j=1}^{n}(X_j - \mu) I[Y_j \in A_i]}{\sqrt{n}\sigma} < x_i\Big] \cap \bigcap_{i=k+1}^{k+l}\big[\max_{\substack{\{j:1\leqslant j\leqslant n,\\ Y_j \in A_i\}}} X_j < a_n x_i + b_n\big]\Big] \to E\prod_{i=1}^{k}\Phi\big(x_i, \lambda(A_i)\big)\prod_{i=k+1}^{k+l} G^{\lambda(A_i)}(x_i) \quad \text{as } n \to \infty$$

for $x_i \in \mathbb{R}$ such that $x_i \neq 0$ for those i for which $P[\lambda(A_i) = 0] > 0, 1 \leqslant i \leqslant k$, and such that $G(x_i) \neq 0$ for those i for which $P[\lambda(A_i) = 0] > 0, k+1 \leqslant i \leqslant k+l$.

In the case of nonrandom strong limits we have:

COROLLARY 2.1. *Under the assumptions of Theorem* 2.1, *if*

$$\sum_{i=1}^{k+l}\left|\frac{\sum_{j=1}^{n} I[Y_j \in A_i]}{n} - p_i\right| \xrightarrow{\mathcal{P}} 0 \quad \text{as } n \to \infty, \tag{2.3}$$

instead of (2.1), *for a set of reals $\{0 < p_i \leqslant 1, 1 \leqslant i \leqslant k+l\}$, then*

$$P\Big[\bigcap_{i=1}^{k}\Big[\frac{\sum_{j=1}^{n}(X_j - \mu) I[Y_j \in A_i]}{\sqrt{n}\sigma} < x_i\Big] \cap \bigcap_{i=k+1}^{k+l}\big[\max_{\substack{\{j:1\leqslant j\leqslant n,\\ Y_j \in A_i\}}} X_j < a_n x_i + b_n\big]\Big] \to \prod_{i=1}^{k}\Phi(x_i, p_i)\prod_{i=k+1}^{k+l} G^{p_i}(x_i) \quad \text{as } n \to \infty.$$

The next results generalise Theorem 2.1 to the case when superobservers arise.

COROLLARY 2.2. *Under the assumptions of Theorem* 2.1, *if*

$$E\frac{1}{\sqrt{\prod_{i=1}^{k}\lambda(A_i)}} < \infty,$$

then

$$P\Big[\bigcap_{i=1}^{k}\Big[\frac{\sum_{j=1}^{n}(X_j-\mu)I[Y_j\in A_i]}{\sqrt{n}\sigma} < x_i\Big], \frac{\sum_{j=1}^{n}(X_j-\mu)I\big[Y_j\in\bigcup_{i=1}^{k}A_i\big]}{\sqrt{n}\sigma} < y \tag{2.4}$$
$$\cap\bigcap_{i=k+1}^{k+l}[\max_{\substack{\{j:1\leqslant j\leqslant n,\\ Y_j\in A_i\}}} X_j < a_n x_i + b_n], \max_{\substack{\{j:1\leqslant j\leqslant n\\ Y_j\in\bigcup_{i=k+1}^{k+l}A_i\}}} X_j < a_n z + b_n\Big]$$
$$\to E\big[\int_{-\infty}^{x_1}\ldots\int_{-\infty}^{x_k} I[u_1+\ldots+u_k<y]d\prod_{i=1}^{k}\Phi\big(u_i,\lambda(A_i)\big)$$
$$\times\prod_{\{i:x_i<z,i>k\}} G^{\lambda(A_i)}(x_i)\cdot G^{\lambda(B)}(z)\big],\quad \textit{where } B=\bigcup_{\{i:x_i\geqslant z,i>k\}}A_i,$$

as $n\to\infty$, *and* x_i, y, z *are arbitrary reals.*

COROLLARY 2.3. *Under the assumptions of Theorem* 2.1, *if*

$$E\frac{1}{\sqrt{\prod_{i=1}^{k}\lambda(A_i)}} < \infty$$

then

$$P\Big[\bigcap_{i=1}^{k}\Big[\frac{\sum_{j=1}^{n}(X_j-\mu)I[Y_j\in A_i]}{\sqrt{n}\sigma} < x_i\Big], \frac{\sum_{j=1}^{n}(X_j-\mu)I\big[Y_j\in\bigcup_{i=1}^{k}A_i\big]}{\sqrt{n}\sigma} < y$$
$$\cap\bigcap_{i=k+1}^{k+l}[\max_{\substack{\{j:1\leqslant j\leqslant n,\\ Y_j\in A_i\}}} X_j < a_n x_i + b_n], \max_{\substack{\{j:1\leqslant j\leqslant n,\\ Y_j\in\mathbb{R}\setminus\bigcup_{i=1}^{k}A_i\}}} X_j < a_n z + b_n\Big]$$
$$\to E\big[\int_{-\infty}^{x_1}\ldots\int_{-\infty}^{x_k} I[u_1+\ldots+u_k<y]d\Phi\big(u_1,\lambda(A_1)\big)\ldots d\Phi\big(u_k,\lambda(A_k)\big)$$
$$\times\prod_{\{i:x_i<z,i>k\}} G^{\lambda(A_i)}(x_i)\cdot G^{\lambda(B\cup(\mathbb{R}\setminus\bigcup_{i=1}^{k+l}A_i))}(z)\big],\quad \textit{where } B=\bigcup_{\{i:x_i\geqslant z,i>k\}}A_i,$$

as $n\to\infty$, *and* x_i, y, z *are arbitrary reals.*

COROLLARY 2.4. *Under the assumptions of Theorem* 2.1, *if*

$$E\frac{1}{\sqrt{\prod_{i=1}^{k}\lambda(A_i)\cdot\lambda\big(\mathbb{R}\setminus\bigcup_{i=k+1}^{k+l}A_i\big)}}<\infty,$$

then

$$P\Bigg[\bigcap_{i=1}^{k}\Bigg[\frac{\sum_{j=1}^{n}(X_j-\mu)I[Y_j\in A_i]}{\sqrt{n}\sigma}<x_i\Bigg],\frac{\sum_{j=1}^{n}(X_j-\mu)I\big[Y_j\in\mathbb{R}\setminus\bigcup_{i=k+1}^{k+l}A_i\big]}{\sqrt{n}\sigma}<y$$

$$\cap\bigcap_{i=k+1}^{k+l}\big[\max_{\substack{\{j:1\leqslant j\leqslant n,\\ Y_j\in A_i\}}}X_j<a_nx_i+b_n\big],\ \max_{\substack{\{j:1\leqslant j\leqslant n,\\ Y_j\in\bigcup_{i=k+1}^{k+l}A_i\}}}X_j<a_nz+b_n\Bigg]$$

$$\to E\Bigg[\int_{-\infty}^{x_1}\ldots\int_{-\infty}^{x_k}\Phi\Bigg(\frac{y-u_1-\ldots-u_k}{\sqrt{\lambda\big(\mathbb{R}\setminus\bigcup_{i=k+1}^{k+l}A_i\big)}}\Bigg)d\Phi\big(u_1,\lambda(A_1)\big)\ldots d\Phi\big(u_k,\lambda(A_k)\big)$$

$$\times\prod_{\{i:x_i<z,i>k\}}G^{\lambda(A_i)}(x_i)\cdot G^{\lambda(B)}(z)\Bigg],\quad \textit{where } B=\bigcup_{\{i:x_i\geqslant z,i>k\}}A_i,$$

as $n\to\infty$, *and* x_i, y, z *are arbitrary reals.*

As a consequence of Corollary 2.4 we get the main result (but for an iid sequence only) of the paper [3].

COROLLARY 2.5. *Let us suppose that the following conditions are satisfied*:

(i) $F\in D_M(G,a_n,b_n)$ *for some real constants* $a_n>0$, b_n, *and every real* x;

(ii) $\{X_n,n\geqslant 1\}$ *is an iid random sequence*;

(iii) $\mathbf{Y}=\{Y_n,n\geqslant 1\}$ *is a sequence of indicators which is independent of* $\{X_n,n\geqslant 1\}$ *and such that*

$$\frac{\sum_{i=1}^{n}Y_i}{n}\xrightarrow{\mathcal{P}}\lambda\quad \textit{as } n\to\infty$$

for some random variable λ.

Then for all reals x *and* y (*if* $x<y$ *and* $P[\lambda=0]>0$ *then* x *must be such that* $G(x)>0$)

$$P\big[\max_{\substack{\{i:1\leqslant i\leqslant n,\\ Y_i=1\}}}X_i\leqslant a_nx+b_n,\ \max_{1\leqslant i\leqslant n}X_i\leqslant a_ny+b_n\big]$$

$$\to\begin{cases}E[G^{\lambda}(x)G^{1-\lambda}(y)] & \textit{if } x<y,\\ G(y) & \textit{if } x\geqslant y,\end{cases}\qquad \textit{as } n\to\infty.$$

3. PROOFS

LEMMA 3.1. *Let* $\{X_n, n \geqslant 1\}$ *and* $\{Y_n, n \geqslant 1\}$ *be two sequences of random variables. Let*

$$P[X_n < x, Y_n < y] \to E\Phi(x, \lambda_1)\Phi(y, \lambda_2) \quad \textit{as } n \to \infty,$$

where $\{\Phi(x, y), y \geqslant 0\}$ *are families of Gaussian distribution functions as defined previously,* $0 \leqslant \lambda_i \leqslant 1$ *a.s.,* $i = 1, 2$, *are arbitrary random variables and* $x, y \in \mathbb{R}$. *If* $E(1/\sqrt{\lambda_1\lambda_2}) < \infty$, *then*

$$P[X_n < x, Y_n < y, X_n + Y_n < z] \to E\int_{-\infty}^{x}\int_{-\infty}^{y} I[z_1 + z_2 < z] d_{z_1}\Phi(z_1, \lambda_1) d_{z_2}\Phi(z_2, \lambda_2)$$

as $n \to \infty$.

P r o o f o f L e m m a 3.1. It is easy to check that $E\Phi(u, \lambda_1)\Phi(v, \lambda_2)$ is a two-dimensional distribution function. Thus there exists a random vector (X, Y) such that

$$(X_n, Y_n) \xrightarrow{\mathcal{D}} (X, Y) \quad \text{as } n \to \infty.$$

Putting the set $A = \{u, v : u < z_1, v < z_2, u + v < z_3\}$ for arbitrary reals z_1, z_2, z_3 in Theorem 29.1 (iv) of [2] we get

$$(X_n, Y_n, X_n + Y_n) \xrightarrow{\mathcal{D}} (X, Y, X + Y) \quad \text{as } n \to \infty.$$

For arbitrary u, v and $h_1, h_2 > 0$, by (3.4), p. 114, in [5], we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\big(\Phi(u + h_1, t_1) - \Phi(u, t_1)\big)\big(\Phi(v + h_2, t_2) - \Phi(v, t_2)\big)}{h_1 h_2} P[\lambda_1 < dt_1, \lambda_2 < dt_2] \leqslant C^2 E\frac{1}{\sqrt{\lambda_1\lambda_2}} < \infty$$

as

$$\frac{|\Phi(u + h, t) - \Phi(u, t)|}{h} \leqslant \frac{C}{\sqrt{t}}.$$

Thus from the Lebesgue dominated convergence theorem it follows that

$$\frac{\partial^2}{\partial u \partial v}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(u, t_1)\Phi(v, t_2) P[\lambda_1 < dt_1, \lambda_2 < dt_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2}{\partial u \partial v}\Phi(u, t_1)\Phi(v, t_2) P[\lambda_1 < dt_1, \lambda_2 < dt_2],$$

and in consequence

$$\iint_{\substack{u < z_1,\, v < z_2,\\ u + v < z_3}} d_{uv} E\Phi(u, \lambda_1)\Phi(v, \lambda_2) = E \iint_{\substack{u < z_1,\, v < z_2,\\ u + v < z_3}} d_{uv}\Phi(u, \lambda_1)\Phi(v, \lambda_2). \quad \blacksquare$$

LEMMA 3.2. *Let* $\{X_n, n \geqslant 1\}$ *and* $\{Z_n, n \geqslant 1\}$ *be sequences of random variables*, $\{A_n, n \geqslant 1\}$ *be a sequence of events such that* $\lim_{n\to\infty} P[A_n] < \varepsilon$ *and* $Z_n \xrightarrow{\mathcal{D}} H$ *as* $n \to \infty$. *Then the following convergences are true*:

$$\lim_{n\to\infty} P[Z_n < z, A_n] < \varepsilon$$

and

$$\lim_{n\to\infty} |P[Z_n < z, A_n^c] - H(z)| < \varepsilon$$

for every point z of continuity of H, *even when* A_n *and* Z_n *are dependent.*

Proof of Lemma 3.2. The first inequality is a consequence of the relation $P[Z_n < z, A_n] \leqslant P[A_n]$, whereas the second follows from

$$|P[Z_n < z, A_n^c] - H(z)| \leqslant |P[Z_n < z] - H(z)| + P[A_n]. \quad \blacksquare$$

Proof of Theorem 2.1. Let $\{X_{i,j}, 1 \leqslant i \leqslant k+l, j \geqslant 1\}$ be an array of iid random variables with the distribution function F, independent of $\{Y_j, j \geqslant 1\}$. Because the iid sequence of random variables is exchangeable, we get

$$P\Bigg[\bigcap_{i=1}^{k}\Bigg[\frac{\sum_{j=1}^{n}(X_j-\mu)I[Y_j\in A_i]}{\sqrt{n}\sigma} < x_i\Bigg] \cap \bigcap_{i=k+1}^{k+l}\Big[\max_{\substack{\{j:1\leqslant j\leqslant n,\\ Y_j\in A_i\}}} X_j < a_n x_i + b_n\Big]\Bigg]$$
$$= P\Bigg[\bigcap_{i=1}^{k}\Bigg[\frac{\sum_{j=1}^{S_{i,n}}(X_{i,j}-\mu)}{\sqrt{n}\sigma} < x_i\Bigg], \bigcap_{i=k+1}^{k+l}\Big[\max_{1\leqslant j\leqslant S_{i,n}} X_{i,j} < a_n x_i + b_n\Big]\Bigg] = P[B_n],$$

say, and denote the right-hand side of (2.2) by B. To see that $P[B_n] \to B$, firstly, we evaluate the difference between $P[B_n]$ and this same $P[B_n]$, but with $S_{i,n}$ replaced by $\lfloor n\lambda(A_i)\rfloor, 1 \leqslant i \leqslant k+l$, using (2.1). Secondly, we evaluate the speed of convergence (2.2) for $k = 1$ and $l = 0$. Thirdly, we bound the convergence (2.2) for $k = 0$, $l = 1$, and finally we prove (2.2).

Let us put, for some nonnegative sequences of reals $\{\gamma_n, \eta_n, n \geqslant 1\}$,

$$D_n = \bigcup_{i=1}^{k+l}\Bigg[\Bigg|\frac{S_{i,n} - \lfloor n\lambda(A_i)\rfloor}{n}\Bigg| > 2\gamma_n\Bigg],$$
$$E_n = \bigcup_{i=1}^{k}\Bigg[\sup_{\lfloor n\lambda(A_i)\rfloor - 2\gamma_n n \leqslant t \leqslant \lfloor n\lambda(A_i)\rfloor + 2\gamma_n n}\Bigg|\sum_{j=S_{i,n}}^{t}\frac{X_{i,j}-\mu}{\sqrt{n}\sigma}\Bigg| > \eta_n\Bigg].$$

We have

$$(3.1)\quad P[B_n] \leqslant$$

$$\leqslant P\Big[\bigcap_{i=1}^{k} \frac{\sum_{j=1}^{S_{i,n}}(X_{i,j}-\mu)}{\sigma\sqrt{n}} < x_i, \bigcap_{i=k+1}^{k+l}(\max_{1\leqslant j\leqslant S_{i,n}} X_{i,j} < a_n x_i + b_n), E_n^c, D_n^c\Big]$$

$$+ P[D_n^c, E_n] + P[D_n]$$

$$\leqslant P\Big[\bigcap_{i=1}^{k} \frac{\sum_{j=1}^{\lfloor n\lambda(A_i)\rfloor}(X_{i,j}-\mu)}{\sigma\sqrt{n}} < x_i + \eta_n, \bigcap_{i=k+1}^{k+l}(\max_{\substack{1\leqslant j\\ \leqslant \lfloor n\lambda(A_i)\rfloor - 2n\gamma_n}} X_{i,j} < a_n x_i + b_n)\Big]$$

$$+ P[D_n^c, E_n] + P[D_n],$$

and similarly

$$(3.2)\quad P[B_n] \geqslant$$

$$\geqslant P\Big[\bigcap_{i=1}^{k} \frac{\sum_{j=1}^{\lfloor n\lambda(A_i)\rfloor}(X_{i,j}-\mu)}{\sigma\sqrt{n}} < x_i - \eta_n, \bigcap_{i=k+1}^{k+l}(\max_{\substack{1\leqslant j\\ \leqslant \lfloor n\lambda(A_i)\rfloor + 2n\gamma_n}} X_{i,j} < a_n x_i + b_n)\Big]$$

$$- P[D_n^c, E_n] - P[D_n].$$

In order to consider these two evaluations together we introduce the value ϑ equal to 1 in the case of upper bound of $P[B_n]$ and -1 in the case of lower bound. Furthermore, let $x = \min_{J(\lambda)} |x_i|$, where $J(\lambda) = \{1 \leqslant i \leqslant k : P[\lambda(A_i) = 0] > 0\}$ (by our assumptions $x > 0$). Then because for $x_i > 0,\ u \in [0,1]$, and some sequence $\{\delta_n, n \geqslant 1\},\ 0 \leqslant \delta_n \leqslant 1$,

$$P\Big[\frac{\sum_{j=1}^{\lfloor nu\rfloor}(X_{i,j}-\mu)}{\sigma\sqrt{n}} < x_i + \vartheta\eta_n, u \leqslant \delta_n\Big]$$

$$= P[u \leqslant \delta_n] - P\Big[\frac{\sum_{j=1}^{\lfloor nu\rfloor}(X_{i,j}-\mu)}{\sigma\sqrt{n}} \geqslant x_i + \vartheta\eta_n, u \leqslant \delta_n\Big],$$

and from Kolmogorov's maximal inequality it follows that, for any $i \in J(\lambda)$,

$$P\left[\left|\frac{\sum_{j=1}^{\lfloor nu\rfloor}(X_{i,j}-\mu)}{\sigma\sqrt{n}}\right| > |x_i+\vartheta\eta_n|, u \leqslant \delta_n\right]$$
$$\leqslant P\left[\sup_{1\leqslant k\leqslant \lfloor n\delta_n\rfloor}\left|\frac{\sum_{j=1}^{k}(X_{i,j}-\mu)}{\sigma\sqrt{n}}\right| > |x_i+\vartheta\eta_n|\right] \leqslant \frac{\delta_n}{(x+\vartheta\eta_n)^2},$$

we get, for arbitrary $1 \leqslant i \leqslant k$,

$$\text{(3.3)}\quad \int_0^1 \left|P\left[\frac{\sum_{j=1}^{\lfloor nu\rfloor}(X_{i,j}-\mu)}{\sigma\sqrt{n}} < x_i+\vartheta\eta_n\right] - \Phi(x_i,u)\right| F_{\lambda(A_i)}(du)$$
$$\leqslant \int_0^{\delta_n} |I[x_i>0]-\Phi(x_i,u)|F_{\lambda(A_i)}(du) + \frac{\delta_n}{(x+\vartheta\eta_n)^2}$$
$$+ \int_{\delta_n}^1 \left|P\left[\frac{\sum_{j=1}^{\lfloor nu\rfloor}(X_{i,j}-\mu)}{\sigma\sqrt{n}} < x_i+\vartheta\eta_n\right] - \Phi\left(\frac{x_i+\vartheta\eta_n}{\sqrt{u}}\right)\right| F_{\lambda(A_i)}(du)$$
$$+ \int_{\delta_n}^1 \left|\Phi\left(\frac{x_i+\vartheta\eta_n}{\sqrt{u}}\right) - \Phi\left(\frac{x_i}{\sqrt{u}}\right)\right| F_{\lambda(A_i)}(du) + \int_{\delta_n}^1 \Phi\left(\frac{x_i}{\sqrt{u}}\right) F_{\lambda(A_i)}(du).$$

The third integral can be evaluated by (3.4), p. 114, in [5], as

$$\text{(3.4)}\qquad \int_{\delta_n}^1 \left|\Phi\left(\frac{x_i+\vartheta\eta_n}{\sqrt{u}}\right) - \Phi\left(\frac{x_i}{\sqrt{u}}\right)\right| F_{\lambda(A_i)}(du) \leqslant \frac{\eta_n}{\sqrt{2\pi\delta_n}},$$

while for the second one, by Theorem 8, p. 118, in [5], we get

$$\text{(3.5)}\quad \int_{\delta_n}^1 \left|P\left[\frac{\sum_{j=1}^{\lfloor nu\rfloor}(X_{i,j}-\mu)}{\sigma\sqrt{n}} < x_i+\vartheta\eta_n\right] - \Phi\left(\frac{x_i+\vartheta\eta_n}{\sqrt{u}}\right)\right| F_{\lambda(A_i)}(du)$$
$$\leqslant \int_{\delta_n}^1 \frac{\lfloor nu\rfloor}{(\sigma\sqrt{\lfloor nu\rfloor})^2} E(X_{i,1}-\mu)^2 I[|X_{i,1}-\mu| > \varepsilon_n\sigma\sqrt{\lfloor nu\rfloor}]$$
$$+ \frac{\lfloor nu\rfloor}{(\sigma\sqrt{\lfloor nu\rfloor})^3} E(X_{i,1}-\mu)^3 I[|X_{i,1}-\mu| \leqslant \varepsilon_n\sigma\sqrt{\lfloor nu\rfloor}]dF_{\lambda(A_i)}(u_i)$$
$$\leqslant \frac{C}{\sigma^2} E(X_1-\mu)^2 I[|X_1-\mu| > \varepsilon_n\sigma\sqrt{\lfloor n\delta_n\rfloor}] + C\varepsilon_n$$

for some sequence of reals $\{\varepsilon_n, n \geqslant 1\}$. We also have for arbitrary $k < i \leqslant k + l$

(3.6)
$$\begin{aligned}
&\int_0^1 |P[\max_{1\leqslant j\leqslant \lfloor nu\rfloor - 2n\vartheta\gamma_n} X_{i,j} < a_n x_i + b_n] - G^u(x_i)| dF_{\lambda(A_i)}(u_i) \\
&\leqslant \int_0^1 |F^{\lfloor nu\rfloor - 2n\vartheta\gamma_n}(a_n x_i + b_n) - F^{nu}(a_n x_i + b_n)| dF_{\lambda(A_i)}(u_i) \\
&\quad + \int_0^1 |F^{nu}(a_n x_i + b_n) - G^u(x_i)| dF_{\lambda(A_i)}(u_i) \\
&\leqslant |F^{-2n\vartheta\gamma_n}(a_n x_i + b_n) - 1| EG^{\lambda(A_i)}(x_i) \\
&\quad + 2\int_0^1 |F^{nu}(a_n x_i + b_n) - G^u(x_i)| dF_{\lambda(A_i)}(u_i).
\end{aligned}$$

On the other hand, we have

(3.7)
$$\begin{aligned}
P[D_n] &\leqslant \\
&\leqslant \sum_{i=1}^{k+l}\left(P\left[\left|\frac{S_{i,n}}{n} - \lambda(A_i)\right| > \gamma_n\right] + P\left[\left|\frac{n\lambda(A_i) - \lfloor n\lambda(A_i)\rfloor}{n}\right| > \gamma_n\right]\right) \\
&\leqslant \sum_{i=1}^{k+l}\left(P\left[\left|\frac{S_{i,n}}{n} - \lambda(A_i)\right| > \gamma_n\right] + I\left[\frac{1}{n} > \gamma_n\right]\right),
\end{aligned}$$

and from Kolmogorov's maximal inequality it follows that

$$P[D_n, E_n^c] \leqslant \frac{4k\gamma_n}{\eta_n^2}. \tag{3.8}$$

Because $\{\lambda(A_i), 1 \leqslant i \leqslant k + l\}$ and $\{X_{i,j}, i, j \geqslant 1\}$ are independent, taking into account (3.1)–(3.8) and the inequality

$$|\prod_{i=1}^n a_i - \prod_{i=1}^n b_i| \leqslant \sum_{i=1}^n |a_i - b_i| \tag{3.9}$$

valid for $0 \leqslant a_i, b_i \leqslant 1$, we get for some absolute constants C

$$
\begin{aligned}
|P[B_n] - B| \leqslant & \sum_{i=1}^{k} \int_0^1 \left| P\left[\frac{\sum_{j=1}^{\lfloor nu \rfloor} (X_{i,j} - \mu)}{\sigma\sqrt{n}} < x_i + \eta_n \right] - \Phi(x_i, u) \right| dF_{\lambda(A_i)}(du) \\
& + \sum_{i=k+1}^{k+l} \int_0^1 |P[\max_{\substack{1 \leqslant j \\ \leqslant \lfloor nu \rfloor + n\gamma_n}} X_{i,j} < a_n x_i + b_n] - G^u(x_i)| dF_{\lambda(A_i)}(u) \\
\leqslant & \, C \Big\{ \frac{\delta_n}{(x - \eta_n)^2} + \frac{\eta_n}{\sqrt{2\pi\delta_n}} \\
& \quad + \frac{1}{\sigma^2} E(X_1 - \mu)^2 I[|X_1 - \mu| > \varepsilon_n \sigma \sqrt{\lfloor n\delta_n \rfloor}] + \varepsilon_n \Big\} \\
& + \sum_{i=1}^{k} \left(\int_0^{\delta_n} |I[x_i > 0] - \Phi(x_i, u)| F_{\lambda(A_i)}(du) + \frac{4\gamma_n}{\eta_n^2} \right) \\
& + \sum_{i=1}^{k} P\left[\left| \frac{S_{i,n}}{n} - \lambda(A_i) \right| > \gamma_n \right] + (k+l) I\left[\frac{1}{n} > \gamma_n \right] \\
& + \sum_{i=k+1}^{k+l} |F^{-\vartheta n \gamma_n}(a_n x_i + b_n) - 1| E G^{\lambda(A_i)}(x_i) \\
& + 2 \sum_{i=k+1}^{k+l} \int_0^1 |F^{nu}(a_n x_i + b_n) - G^u(x_i)| dF_{\lambda(A_i)}(u_i).
\end{aligned}
$$

Because, by (2.1),

$$
\frac{S_{i,j}}{n} \xrightarrow{\mathcal{P}} \lambda(A_i), \qquad 1 \leqslant i \leqslant k + l,
$$

there exists a sequence $\gamma_n' \to 0$, as $n \to \infty$, such that

$$
P\left[\left| \frac{S_{i,n}}{n} - \lambda(A_i) \right| > \gamma_n' \right] \to 0.
$$

Thus, taking

$$
\gamma_n = \max\left\{ \gamma_n', \frac{2}{n} \right\}
$$

we have

$$
\sum_{i=1}^{k} P\left[\left| \frac{S_{i,n}}{n} - \lambda(A_i) \right| > \gamma_n \right] \to 0 \qquad \text{as } n \to \infty
$$

and

$$
I\left[\frac{1}{n} > \gamma_n \right] = 0.
$$

Now, putting $\delta_n = \eta_n = \sqrt[3]{\gamma_n}, \varepsilon_n = 1/\sqrt[6]{n}, n \geqslant 1$, for sufficiently large n (such that $\eta_n < x/2$), because $\varepsilon_n\sqrt{\lfloor n\delta_n \rfloor} \geqslant \sqrt[6]{n}$, we have

$$\begin{aligned} \frac{\delta_n}{(x-\eta_n)^2} \leqslant 4\frac{\delta_n}{x} &\to 0, \\ \frac{\eta_n}{\sqrt{2\pi\delta_n}} = \frac{\sqrt[6]{\gamma_n}}{\sqrt{2\pi}} &\to 0, \\ \frac{4\gamma_n}{\eta_n^2} = 4\sqrt[3]{\gamma_n} &\to 0, \\ \varepsilon_n &\to 0, \\ E(X_1-\mu)^2 I[|X_1-\mu| > \varepsilon_n\sigma\sqrt{\lfloor n\delta_n \rfloor}] &\to 0, \\ \sum_{i=1}^{k} \Big(\int_0^{\delta_n} |I[x_i > 0] - \Phi(x_i, u)| F_{\lambda(A_i)}(du) \Big) &\to 0 \end{aligned}$$

as $n \to \infty$. Furthermore, if $G(x_i) > 0$, then $|F^{-2n\vartheta\gamma_n}(a_n x_i + b_n) - 1| \to 0$ as $n \to \infty$, otherwise (then $P[\lambda(A_i) = 0] = 0$) $EG^{\lambda(A_i)}(x_i) = 0$. By Lebesgue's dominated convergence theorem, as the function $G^u(x_i)$ is monotonic, the last integral also converges to zero. ■

P r o o f o f C o r o l l a r y 2.1. It is enough to take $\lambda(A_i, \omega) \equiv p_i$ for $1 \leqslant i \leqslant k + l$ in the previous theorem. ■

P r o o f o f C o r o l l a r y 2.2. From Lemma 3.1 we have

$$\begin{aligned} &P\Bigg[\bigcap_{i=1}^{k} \Bigg[\frac{\sum_{j=1}^{n}(X_j-\mu)I[Y_j \in A_i]}{\sqrt{n}\sigma} < x_i\Bigg], \frac{\sum_{j=1}^{n}(X_j-\mu)I\big[Y_j \in \bigcup_{i=1}^{k} A_i\big]}{\sqrt{n}\sigma} < y\Bigg] \\ &= \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_k} I[u_1 + \ldots + u_k < y] dF\bigg(\frac{u_1}{\sqrt{\lambda(A_1)}}\bigg) \ldots dF\bigg(\frac{u_k}{\sqrt{\lambda(A_k)}}\bigg). \end{aligned}$$

As for any sets D and E

$$\begin{aligned} &P[\max_{k\in D} X_k < a, \max_{k\in(D\cup E)} X_k < b] \\ &\qquad = \begin{cases} P[\max_{k\in(D\cup E)} X_k < b] & \text{for } b < a, \\ P[\max_{k\in D} X_k < a, \max_{k\in E} X_k < b] & \text{otherwise,} \end{cases} \end{aligned}$$

we have

$$\lim_{n\to\infty} P\Big[\bigcap_{i=k+1}^{k+l}\big[\max_{\substack{\{j:1\leqslant j\leqslant n,\\ Y_j\in A_i\}}}\{X_j\}<x_i\big],\ \max_{\substack{\{j:1\leqslant j\leqslant n,\\ Y_j\in\bigcup_{i=k+1}^{k+l}A_i\}}}\{X_j\}<z\Big]$$

$$= E[G^{\lambda(A_i)}(x_i)\cdot G^{\lambda(B)}(z)],\quad \text{where } B=\bigcup_{\{i:x_i\geqslant z,i>k\}} A_i.$$

Since both of these intersections of random variables are independent, we get the assertion. ∎

Proofs of Corollaries 2.3 and 2.4 are similar. Corollary 2.5 follows directly from Corollary 2.3.

4. EXAMPLES AND APPLICATIONS

EXAMPLE 4.1. Let $\{Y_n, n\geqslant 1\}$ be a sequence of pairwise independent random variables such that $Y_n \xrightarrow{\mathcal{D}} Y$ for some random variable Y. Let us assume that $\{A_1, A_2, \ldots, A_{k+l}\}$ is a collection of the sets such that

$$\bigcup_{i=1}^{k+l} A_i=\mathbb{R},\quad A_i\cap A_j=\emptyset \text{ for } i\neq j,\qquad P[Y\in\partial A_i]=0,\quad 1\leqslant i\leqslant k+l.$$

Let $\{X_n, n\geqslant 1\}$ be a sequence of iid random variables independent of $\{Y_n, n\geqslant 1\}$ such that $\{X_n, n\in\mathbb{N}\}\in D_S(\Phi, n\mu, \sigma\sqrt{n})$ and $\{X_n, n\in\mathbb{N}\}\in D_M(G, a_n, b_n)$ for some distribution function G. Then Corollary 2.1 holds with $p_i=P[Y\in A_i]$, $1\leqslant i\leqslant k+l$. Particularly, for two independent random variables U_1 and U_2, uniformly distributed on the unit interval, let us put

$$Y_j=\mathrm{Re}(e^{2\pi(U_1+jU_2)i})=\cos\big(2\pi(U_1+jU_2)\big),\qquad j\geqslant 1,$$
$$A_1=\Big[-1,-\frac{\sqrt{3}}{2}\Big], A_2=\Big[-\frac{\sqrt{3}}{2},-\frac{1}{2}\Big], A_3=\Big[-\frac{1}{2},0\Big],$$
$$A_4=\Big[0,\frac{1}{2}\Big], A_5=\Big[\frac{1}{2},\frac{\sqrt{3}}{2}\Big], A_6=\Big[\frac{\sqrt{3}}{2},1\Big].$$

Then we have $Y_n\xrightarrow{\mathcal{D}} Y_1=Y$ and $P[Y\in\partial A_i]=0$, $P[Y\in A_i]=\frac{1}{6}$. Let us assume that $\{X_n, n\geqslant 1\}$ is a sequence of iid random variables with

$$P[X_1\leqslant x]=\begin{cases}0, & x<0,\\ 1-e^{-x}, & x\geqslant 0.\end{cases}$$

Then $\{X_n, n \geqslant 1\} \in D_S(\Phi, n, \sqrt{n})$ and $\{X_n, n \in \mathbb{N}\} \in D_M(G_1, 1, \ln n)$. Consequently,

$$P\Bigg[\frac{\sum\limits_{j=1}^{n}(X_j - 1)I[Y_j \in A_i]}{\sqrt{n}} < x_i, 1 \leqslant i \leqslant 3, \max_{\substack{\{j:1\leqslant j\leqslant n,\\ Y_j\in A_i\}}} X_j \leqslant x_i + \ln n, 4 \leqslant i \leqslant 6\Bigg]$$

$$\to \prod_{i=1}^{3} \Phi(\sqrt{6}x_i) \prod_{i=4}^{6} \exp\left\{ -\frac{1}{6}e^{-x_i}\right\} \quad \text{as } n \to \infty$$

for arbitrary $x_i \in \mathbb{R}$, $i = 1, \ldots, 6$.

EXAMPLE 4.2. For an arbitrary random variable λ such that $0 \leqslant \lambda \leqslant 1$ a.s. we define the sequence

$$\varepsilon_n(\lambda) = \begin{cases} 0 & \text{for } \lambda \in \bigcup\limits_{r=1}^{n-1} \left(\dfrac{r-1}{n-1}, \dfrac{r}{n}\right] \cup \{0\}, \\ 1 & \text{for } \lambda \in \bigcup\limits_{r=1}^{n-1} \left(\dfrac{r}{n}, \dfrac{r}{n-1}\right]. \end{cases}$$

It is easy to check that $\sum_{k=1}^{n} \varepsilon_k(\lambda)/n \xrightarrow{\mathcal{P}} \lambda$. For two different random variables $\lambda_1,\ \lambda_2$ we put

$$Y_k = \begin{cases} \varepsilon_l(\lambda_1) & \text{if } k = 2l, \\ 2\varepsilon_l(\lambda_2) & \text{if } k = 2l+1, \end{cases}$$

and $A_1 = \{1\},\ A_2 = \{2\}$. Let $\{X_n, n \geqslant 1\}$ be a sequence of iid random variables with uniform law on $(0, 1)$. Then $\{X_n, n \geqslant 1\} \in D_S(\Phi, n/2, \sqrt{3n}/6)$ and $\{X_n, n \geqslant 1\} \in D_M(G_{3,1}, 1/n, 1)$. Consequently,

$$P\Bigg[\frac{\sum\limits_{j=1}^{n}(X_j - 1/2)I[Y_j \in A_1]}{\sqrt{3n}/6} < x, \max_{\substack{\{j:1\leqslant j\leqslant n,\\ Y_j\in A_2\}}} X_j < \frac{y}{n} + 1\Bigg]$$

$$\xrightarrow{\mathcal{D}} E\Phi\left(\frac{x}{\sqrt{\lambda_1/2}}\right) G_{3,1}^{\lambda_2/2}(y)$$

for $x \neq 0$ if $P[\lambda_1 = 0] > 0$.

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