# CROSS-VARIATION OF YOUNG INTEGRAL WITH RESPECT TO LONG-MEMORY FRACTIONAL BROWNIAN MOTIONS

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*Abstract.* We study the asymptotic behaviour of the cross-variation of two-dimensional processes having the form of a Young integral with respect to a fractional Brownian motion of index $H > \frac{1}{2}$. When $H$ is smaller than or equal to $\frac{3}{4}$, we show asymptotic mixed normality. When $H$ is stricly greater than $\frac{3}{4}$, we obtain a limit that is expressed in terms of the difference of two independent Rosenblatt processes.

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## 1. INTRODUCTION

**1.1. Foreword and main results.** In the near past, there have been many applications of stochastic differential equations (SDE) driven by fractional Brownian motion in different areas of mathematical modelling. To name but a few, we mention the use of such equations as a model for meteorological phenomena [1], [12], protein dynamics [7], [6], or noise in electrical networks [8].

Here, we consider more generally a two-dimensional process $\{X_t\}_{t \in [0,T]} = \{(X_t^{(1)}, X_t^{(2)})\}_{t \in [0,T]}$ of the form

$$(1.1) \qquad X_t^{(i)} = x_i + \int_0^t \sigma_s^{i,1} dB_s^{(1)} + \int_0^t \sigma_s^{i,2} dB_s^{(2)}, \quad t \in [0,T], \, i = 1, 2,$$

where $B = (B^{(1)}, B^{(2)})$ is a two-dimensional fractional Brownian motion of Hurst index $H > \frac{1}{2}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$, whereas $x = (x_1, x_2) \in \mathbb{R}^2$, and $\sigma$ is a $2 \times 2$ matrix-valued process. The case where $X$ solves a fractional SDE corresponds to $\sigma_t = \sigma(X_t)$, with $\sigma : \mathbb{R}^2 \to \mathcal{M}_2(\mathbb{R})$ deterministic. Since we are assuming that $H > \frac{1}{2}$, by imposing appropriate conditions on $\sigma$ (see Section 2 for details), we may and will assume throughout the paper that

$\int_0^t \sigma_s^{i,j} dB_s^{(j)}$ is understood in the Young [16] sense (see, again, Section 2 for the details).

In this paper, we are concerned with the asymptotic behaviour of the cross-variation associated with $X$ on $[0, T]$, which is a sequence of stochastic processes defined as

$$(1.2) \qquad J_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} \Delta X_{k/n}^{(1)} \Delta X_{k/n}^{(2)}, \quad n \geqslant 1,\, t \in [0, T].$$

Here, and the same anywhere else, we use the notation $\Delta X_{k/n}^{(i)}$ to indicate the increment $X_{k/n}^{(i)} - X_{(k-1)/n}^{(i)}$. We shall show the following two theorems which might be of interest for solving problems arising from statistics, e.g., the problem of testing the hypothesis $(H_0)$: "$\sigma^{1,2} = \sigma^{2,1} = 0$" in (1.1).

THEOREM 1.1. *For any $t \in [0, T]$,*

$$(1.3) \qquad n^{2H-1} J_n(t) \xrightarrow{P} \int_0^t (\sigma_s^{1,1} \sigma_s^{1,2} + \sigma_s^{2,1} \sigma_s^{2,2}) ds \quad \text{as } n \to \infty.$$

THEOREM 1.2. *Assume $\sigma^{1,2} = \sigma^{2,1} = 0$ and let*

$$(1.4) \qquad a_n := \begin{cases} n^{2H-1/2} & \text{if } \frac{1}{2} < H < \frac{3}{4}, \\ n/\sqrt{\log n} & \text{if } H = \frac{3}{4}, \\ n & \text{if } \frac{3}{4} < H < 1. \end{cases}$$

*Then, as $n \to \infty$,*

$$(1.5) \qquad a_n J_n \xrightarrow{\mathcal{L}} \int_0^\cdot \sigma_s^{1,1} \sigma_s^{2,2} dZ_s \quad \text{in the Skorokhod space } D[0, T].$$

*In (1.5), the definition of $Z$ is according to the value of $H$. More precisely, $Z = C_H/2 \cdot W$ when $H \in \left(\frac{1}{2}, \frac{3}{4}\right]$, with $C_H$ given by (3.1) and (3.2) below, and $W$ a Brownian motion independent of $\mathcal{F}$; and $Z = \frac{1}{2}(R^{(1)} - R^{(2)})$ when $H \in \left(\frac{3}{4}, 1\right)$, with $R^{(k)}$ the Rosenblatt process constructed from the fractional Brownian motion*

$$\beta^{(k)} = \frac{1}{\sqrt{2}} \big(B^{(1)} + (-1)^{k+1} B^{(2)}\big), \quad k = 1, 2$$

(see Definition 3.1 for details).

**1.2. Link to the existing literature.** Our results are close in spirit to those contained in [4] (which has been a strong source of inspiration to us), where central limit theorems for power variations of integral fractional processes are investigated.

As will be seen, our analysis of $J_n$, that requires similar but different efforts compared to [4] (as we are here dealing with a *two-dimensional* fractional Brownian motion on one hand and we also consider[1] the case where $H > \frac{3}{4}$ on the other hand), is actually greatly simplified by the use of a recent, nice result obtained in [3] about the asymptotic behaviour of weighted random sums.

**1.3. Outline of the paper.** The rest of the paper is as follows. Section 2 contains a thorough description of the framework in which our study takes place (in particular, we recall the definition of the Young integral and we provide its main properties). Section 3 gathers several preliminary results that will be essential for proving our main results. Finally, proofs of Theorems 1.1 and 1.2 are given in Section 4.

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**2. OUR FRAMEWORK**

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In this section, we describe the framework used throughout the paper and we fix a parameter $\alpha \in (0, 1)$.

We let $C^\alpha$ denote the set of Hölder continuous functions of index $\alpha \in (0, 1)$, that is, the set of the functions $f : [0, T] \to \mathbb{R}$ satisfying

$$(2.1) \qquad |f|_\alpha := \sup_{0 \leqslant s < t \leqslant T} \frac{|f(t) - f(s)|}{(t - s)^\alpha} < \infty.$$

Also, we set $\|f\|_\alpha := |f|_\alpha + |f|_\infty$, with $|f|_\infty = \sup_{0 \leqslant t \leqslant T} |f(t)|$.

For a fixed $f \in C^\alpha$, we consider the operator $T_f : \mathcal{C}^1 \to \mathcal{C}^1$ defined as

$$T_f(g)(t) = \int\limits_0^t f(u)g'(u)du, \quad t \in [0, T].$$

Let $\gamma \in (0, 1)$ be such that $\alpha + \gamma > 1$. Then $T_f$ extends, in a unique way, to an operator $T_f : C^\gamma \to C^\gamma$ which further satisfies

$$\|T_f(g)\|_\gamma \leqslant (1 + C_{\alpha,\gamma})(1 + T^\gamma)\|f\|_\alpha\|g\|_\gamma,$$

with $C_{\alpha,\gamma} = \frac{1}{2}\sum_{n=1}^\infty 2^{-n(\alpha+\gamma-1)} < \infty$. See, e.g., [9], Theorem 3.1, for a proof.

DEFINITION 2.1. Let $\alpha, \gamma \in (0, 1)$ be such that $\alpha + \gamma > 1$. Let $f \in C^\alpha$ and $g \in C^\gamma$. The *Young integral* $\int_0^\cdot f(u)dg(u)$ is then defined as being $T_f(g)$.

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[1] The authors of [4] did not consider the case where $H > \frac{3}{4}$ since, quoting them, "the problem is more involved because non-central limit theorems are required".

The Young integral satisfies (see, e.g., [9], inequality (3.3)), for any $a, b \in [0, T]$ with $a < b$, the condition

$$(2.2) \qquad \left| \int_a^b \big( f(u) - f(a) \big) dg(u) \right| \leqslant C_{\alpha, \gamma} |f|_\alpha |g|_\gamma (b - a)^{\alpha + \gamma}.$$

As we said in Introduction, we let $B = (B^{(1)}, B^{(2)})$ be a two-dimensional fractional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. We assume further that $\mathcal{F}$ is the $\sigma$-field generated by $B$. We also suppose that the Hurst parameter $H$ of $B$ is the same for the two components and that it is strictly greater than $\frac{1}{2}$.

Let $\alpha \in (0, 1)$ and let $\sigma^{i,j} : \Omega \times [0, T] \to \mathbb{R}$, $i, j = 1, 2$, be four given stochastic processes that are measurable with respect to $\mathcal{F}$. We will assume throughout the text that the following two additional assumptions on $\alpha$ and $\sigma^{i,j}$ are satisfied:

(A) $\alpha \in \left( \frac{1}{4} + \frac{H}{2}, H \right)$.

(B) For each pair $(i, j) \in \{1, 2\}^2$, the random variable $\|\sigma^{i,j}\|_\alpha$ has moments of all orders.

Observe that $\alpha + H > 1$ due to both (A) and $H > \frac{1}{2}$, so that the integrals in (1.1) are well defined in the Young sense. Also, recall the following variant of the Garcia–Rodemich–Rumsey lemma [5]: for any $q > 1$, there exists a constant $c_{\alpha,q} > 0$ (depending only on $\alpha$ and $q$) such that

$$(2.3) \qquad |B^{(i)}|_\alpha^q \leqslant c_{\alpha,q} \iint\limits_{[0,T]^2} \frac{|B_u^{(i)} - B_v^{(i)}|^q}{|u - v|^{2 + q\alpha}} du \, dv.$$

Using (2.3), we infer that $|B^{(i)}|_\alpha$ has moments of all orders.

## 3. PRELIMINARIES

**3.1. Breuer–Major theorem.** The next statement is a direct consequence of the celebrated Breuer–Major [2] theorem (see [9], Section 7.2, for a modern proof). We write 'fdd' to indicate the convergence of all the finite-dimensional distributions.

THEOREM 3.1 (Breuer–Major [2]). *Let $\beta$ be a (one-dimensional) fractional Brownian motion of index $H \in \left( 0, \frac{3}{4} \right]$. Then, as $n \to \infty$ and with $W$ a standard Brownian motion,*
   (i) *if $H < \frac{3}{4}$, then*

$$\left\{ \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} [(\beta_k - \beta_{k-1})^2 - 1] \right\}_{t \in [0,T]}$$

$$\xrightarrow{\text{fdd}} \frac{1}{2} \sum_{k \in \mathbb{Z}} (|k + 1|^{2H} + |k - 1|^{2H} - 2|k|^{2H})^2 \, \{W_t\}_{t \in [0,T]};$$

(ii) *if $H = \frac{3}{4}$, then*

$$\left\{ \frac{1}{\sqrt{n \log n}} \sum_{k=1}^{\lfloor nt \rfloor} [(\beta_k - \beta_{k-1})^2 - 1] \right\}_{t \in [0,T]} \xrightarrow{\text{fdd}} \frac{3}{4} (\log 2) \{W_t\}_{t \in [0,T]}.$$

By a scaling argument (to pass from $k$ to $k/n$) and by using the seminal result of Peccati and Tudor [11] (to allow an extra $F$), we can immediately obtain from Theorem 3.1 the following corollary.

COROLLARY 3.1. *Let $\beta = (\beta^{(1)}, \beta^{(2)})$ be a two-dimensional fractional Brownian motion of index $H \in \left(0, \frac{3}{4}\right]$. Then, as $n \to \infty$ and with $W$ a (one-dimensional) standard Brownian motion independent of $\beta$, we have, for any random vector $F = (F_1, \ldots, F_d)$ measurable with respect to $\beta$,*
  (i) *if $H < \frac{3}{4}$, then*

$$\left\{ F, n^{2H-1/2} \sum_{k=1}^{\lfloor nt \rfloor} [(\beta_{k/n}^{(1)} - \beta_{(k-1)/n}^{(1)})^2 - (\beta_{k/n}^{(2)} - \beta_{(k-1)/n}^{(2)})^2] \right\}_{t \in [0,T]}$$

$$\xrightarrow{\text{fdd}} \{F, C_H W_t\}_{t \in [0,T]},$$

*where*

$$(3.1) \qquad C_H = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} (|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H})^2;$$

  (ii) *if $H = \frac{3}{4}$, then*

$$\left\{ F, \frac{n}{\sqrt{\log n}} \sum_{k=1}^{\lfloor nt \rfloor} [(\beta_{k/n}^{(1)} - \beta_{(k-1)/n}^{(1)})^2 - (\beta_{k/n}^{(2)} - \beta_{(k-1)/n}^{(2)})^2] \right\}_{t \in [0,T]}$$

$$\xrightarrow{\text{fdd}} \{F, C_{3/4} W_t\}_{t \in [0,T]},$$

*where*

$$(3.2) \qquad C_{3/4} = \frac{3\sqrt{2}}{4} \log 2.$$

**3.2. Taqqu's theorem and the Rosenblatt process.** Taqqu's theorem [13] describes the fluctuations of the quadratic variation of the fractional Brownian motion when the Hurst index $H$ is strictly greater than $\frac{3}{4}$, that is, for the range of values which are not covered by the Breuer–Major Theorem 3.1. We state here a version that fits into our framework. With respect to the original statement, it is worth noting that in Theorem 3.2 (whose proof may be found in [10]) the convergence is in $L^2(\Omega)$ (and not only in law). This latter fact will reveal to be crucial in our proof of Theorem 1.2, as it will allow us to apply the main result of [3] recalled in Section 3.4.

THEOREM 3.2 (Taqqu [13]). *Let $\beta$ be a (one-dimensional) fractional Brownian motion of index $H \in \left(\frac{3}{4}, 1\right)$. Then, for any $t \in [0, T]$, the sequence*

$$(3.3) \qquad n^{1-2H} \sum_{k=1}^{\lfloor nt \rfloor} [n^{2H}(\beta_{k/n} - \beta_{(k-1)/n})^2 - 1]$$

*converges in $L^2(\Omega)$ as $n \to \infty$.*

DEFINITION 3.1. Let the assumption of Theorem 3.2 be satisfied, and denote by $R_t$ the limit of (3.3). The process $R = \{R_t\}_{t \in [0,T]}$ is called the *Rosenblatt process* constructed from $\beta$.

For the main properties of the Rosenblatt process $R$, we refer the reader to Taqqu [14] or Tudor [15]. See also [9], Section 7.3. An immediate corollary of Theorem 3.2 is as follows.

COROLLARY 3.2. *Let $\beta = (\beta^{(1)}, \beta^{(2)})$ be a two-dimensional fractional Brownian motion of index $H \in \left(\frac{3}{4}, 1\right)$. Then, for any $t \in [0, T]$,*

$$n \sum_{k=1}^{\lfloor nt \rfloor} [(\beta_{k/n}^{(1)} - \beta_{(k-1)/n}^{(1)})^2 - (\beta_{k/n}^{(2)} - \beta_{(k-1)/n}^{(2)})^2] \xrightarrow{L^2(\Omega)} R_t^{(1)} - R_t^{(2)}$$

*as $n \to \infty$, where $R^{(i)}$ is the Rosenblatt process constructed from the fractional Brownian motion $\beta^{(i)}$, $i = 1, 2$.*

**3.3. Two simple auxiliary lemmas.** To complete the proofs of Theorems 1.1 and 1.2 we will, among other things, need the following two simple lemmas.

LEMMA 3.1. *Let $B$ and $\sigma$ be as in Section 2. Then there exists a constant $C = C(\alpha, H, T, \sigma) > 0$ such that, for any $i, j = 1, 2$, any $n \geqslant 1$ and any $k \in \{1, \ldots, \lfloor nT \rfloor\}$,*

$$(3.4) \qquad \Big\| \int_{(k-1)/n}^{k/n} (\sigma_s^{i,j} - \sigma_{k/n}^{i,j}) dB_s^j \Big\|_{L^2(\Omega)} \leqslant C n^{-2\alpha},$$

$$(3.5) \qquad \Big\| \int_{(k-1)/n}^{k/n} \sigma_s^{i,j} dB_s^j \Big\|_{L^2(\Omega)} \leqslant C n^{-H}.$$

P r o o f. Without loss of generality, we may and will assume that $i = j = 1$. Using (2.2) with $\beta = \alpha$, we have, almost surely,

$$\Big| \int_{(k-1)/n}^{k/n} (\sigma_s^{1,1} - \sigma_{k/n}^{1,1}) dB_s^1 \Big| \leqslant C_{\alpha,\alpha} |\sigma^{1,1}|_\alpha |B^1|_\alpha n^{-2\alpha}.$$

Using the Cauchy–Schwarz inequality, we obtain

$$E\big[\big(\int\limits_{(k-1)/n}^{k/n}(\sigma_s^{1,1}-\sigma_{k/n}^{1,1})dB_s^1\big)^2\big]$$

$$\leqslant C_{\alpha,\alpha}^2\sqrt{E\left[\|\sigma^{1,1}\|_\alpha^4\right]}\sqrt{[E|B^1|_\alpha^4]}\,n^{-4\alpha}=Cn^{-4\alpha},$$

thus yielding (3.4). On the other hand, we have

$$\big\|\int\limits_{(k-1)/n}^{k/n}\sigma_s^{i,j}dB_s^j\big\|_{L^2(\Omega)}\leqslant\big\|\int\limits_{(k-1)/n}^{k/n}(\sigma_s^{i,j}-\sigma_{k/n}^{i,j})dB_s^j\big\|_{L^2(\Omega)}+\|\sigma_{k/n}^{i,j}\Delta B_{k/n}^j\|_{L^2(\Omega)}$$

$$\leqslant Cn^{-2\alpha}+Cn^{-2H}\quad\text{(by (3.4) and because of (B))},$$

$$\leqslant Cn^{-H}\quad\text{(using (A))},$$

which is the desired claim (3.5). ∎

LEMMA 3.2. *Let* $g,h:[0,T]\to\mathbb{R}$ *be two continuous functions, let* $\gamma\in\mathbb{R}$, *and let us write* $\Delta h_{k/n}$ *to denote the increment* $h(k/n)-h\big((k-1)/n\big)$. *If*

$$(3.6)\qquad\forall t\in[0,T]\cap\mathbb{Q}:\quad\lim_{n\to\infty}n^\gamma\sum_{k=1}^{\lfloor nT\rfloor}\mathbf{1}_{[0,t]}(k/n)(\Delta h_{k/n})^2=t,$$

*then, for all* $t\in[0,T]$,

$$\lim_{n\to\infty}n^\gamma\sum_{k=1}^{\lfloor nT\rfloor}g(k/n)\mathbf{1}_{[0,t]}(k/n)(\Delta h_{k/n})^2=\int\limits_0^t g(s)ds.$$

P r o o f. Since $t\mapsto n^\gamma\sum_{k=1}^n\mathbf{1}_{[0,t]}(k/n)(\Delta h_{k/n})^2$ is non-decreasing, it is straightforward to infer from (3.6) that, for *all* $t\in[0,T]$,

$$\lim_{n\to\infty}n^\gamma\sum_{k=1}^{\lfloor nT\rfloor}\mathbf{1}_{[0,t]}(k/n)(\Delta h_{k/n})^2=t.$$

Otherwise stated, the cumulative distribution function (cdf) of the compactly supported measure

$$\nu_n(dx)=n^\gamma\sum_{k=1}^{\lfloor nT\rfloor}(\Delta h_{k/n})^2\delta_{k/n}(dx),$$

where $\delta_a$ stands for the Dirac mass at $a$, converges pointwise to the cdf of the Lebesgue measure on $[0,T]$. Since $g$ is continuous, it is then a routine exercise to deduce that our desired claim holds true. ∎

**3.4. Asymptotic behaviour of weighted random sums, following Corcuera, Nualart and Podolskij [3].** The following result is a central ingredient in the proof of both Theorems 1.1 and 1.2.

PROPOSITION 3.1. *Let $u = \{u_t\}_{t \in [0,T]}$ be a Hölder continuous process with index $\alpha > \frac{1}{2}$, set*

$$K_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} u_{k/n} \Delta B_{k/n}^{(1)} \Delta B_{k/n}^{(2)}, \quad t \in [0,T],$$

*and let $a_n$ be given by (1.4). Then, as $n \to \infty$,*

$$(3.7) \qquad a_n K_n \overset{\mathcal{L}}{\to} \int_0^{\cdot} u_s dZ_s \quad \text{in the Skorokhod space } D[0,T].$$

*Here, $Z$ is as in the statement of Theorem* 1.2.

The proof of our Proposition 3.1 heavily relies on a nice result taken from Corcuera, Nualart and Podolskij [3]. Actually, we will need a slight extension of the result of [3], which we state here for convenience (and also because we do not share the same notation). The only difference between Theorem 3.3 as stated below and its original version appearing in [3] is that $Z$ need not be a Brownian motion. A careful inspection of the proof given in [3] indeed reveals that the Brownian feature of $Z$ plays actually no role; the only property of $Z$ which is used is that the sum of its Hölder exponent and that of $u$ is strictly greater than one, see (H1).

THEOREM 3.3 (Corcuera, Nualart and Podolskij [3]). *The underlying probability space is $(\Omega, \mathcal{F}, P)$. Let $u = \{u_t\}_{t \in [0,T]}$ be a Hölder continuous process with index $\alpha \in (0,1)$, and let $\xi = \{\xi_{k,n}\}_{n \in \mathbb{N}, 1 \leqslant k \leqslant \lfloor nT \rfloor}$ be a family of random variables. Let us set*

$$g_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} \xi_{k,n}, \quad t \in [0,T].$$

*Assume the following two hypotheses on the double sequence $\xi$:*

(H1) $\{g_n(t)\}_{t \in [0,T]} \overset{\text{fdd}}{\to} \{Z(t)\}_{t \in [0,T]}$ $\mathcal{F}$-*stably, where $Z$ is Hölder continuous with index $\beta$ such that $\alpha + \beta > 1$.*

(H2) *There is a constant $C > 0$ such that, for any $1 \leqslant i < j \leqslant [nT]$,*

$$E\Big[\Big(\sum_{k=i+1}^{j} \xi_{k,n}\Big)^4\Big] \leqslant C \left(\frac{j-i}{n}\right)^2.$$

*Then*

$$\sum_{k=1}^{\lfloor n \cdot \rfloor} u_{k/n} \xi_{k,n} \overset{\mathcal{L}}{\to} \int_0^{\cdot} u_s dZ_s \quad \text{in the Skorokhod space } D[0,T],$$

*where $\int_0^{\cdot} u_s dZ_s$ is understood as a Young integral.*

Equipped with Theorem 3.3, we are now ready to prove Proposition 3.1.

P r o o f  o f  P r o p o s i t i o n  3.1. Set $\xi_{k,n} = a_n \Delta B^{(1)}_{k/n} \Delta B^{(2)}_{k/n}$ and $g_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} \xi_{k,n}$, $t \in [0, T]$. We shall check the two assumptions, (H1) and (H2), of Theorem 3.3.

S t e p  1:  C h e c k i n g  (H1). We make use of the rotation trick. More precisely, let

$$\beta^{(1)} = \frac{1}{\sqrt{2}}(B^{(1)} + B^{(2)}) \quad \text{and} \quad \beta^{(2)} = \frac{1}{\sqrt{2}}(B^{(1)} - B^{(2)}),$$

so that

$$\xi_{k,n} = \frac{a_n}{2}\big((\Delta\beta^{(1)}_{k/n})^2 - (\Delta\beta^{(2)}_{k/n})^2\big).$$

It is easy to check that $\beta^{(1)}$ and $\beta^{(2)}$ are two independent fractional Brownian motions of index $H$. As a result, assumption (H1) is satisfied due to Corollary 3.1 (resp. Corollary 3.2) when $H \leqslant \frac{3}{4}$ (resp. $H > \frac{3}{4}$).

S t e p  2:  C h e c k i n g  (H2). Since all the $L^p(\Omega)$-norms are equivalent inside a given Wiener chaos (here: the second Wiener chaos), it suffices to check the existence of a constant $C > 0$ such that, for any $1 \leqslant i < j \leqslant [nT]$,

$$(3.8) \qquad E\big[\big(\sum_{k=i+1}^{j} \xi_{k,n}\big)^2\big] \leqslant C\,\frac{j-i}{n}.$$

Using the independence of $B^{(1)}$ and $B^{(2)}$, we obtain

$$E\big[\big(\sum_{k=i+1}^{j} \xi_{k,n}\big)^2\big] = a_n^2\, n^{-4H} \sum_{k,k'=i+1}^{j} \rho(k-k')^2,$$

with $\rho(r) = \frac{1}{2}(|r+1|^{2H} + |r-1|^{2H} - 2|r|^{2H})$. As a consequence, for any $i < j$ in $\{1, \ldots, [nT]\}$,

$$E\big[\big(\sum_{k=i+1}^{j} \xi_{k,n}\big)^2\big] \leqslant a_n^2\, n^{-4H}(j-i) \sum_{r=-[nT]}^{[nT]} \rho(r)^2.$$

It is straightforward to show that $a_n^2\, n^{1-4H} \sum_{r=-[nT]}^{[nT]} \rho(r)^2 = O(1)$ as $n \to \infty$. Thus, (3.8) is satisfied, and so is (H2).

To conclude the proof of Proposition 3.1, it remains to apply Theorem 3.3 with $\xi_{k,n} = a_n \Delta B^{(1)}_{k/n} \Delta B^{(2)}_{k/n}$. ∎

## 4. PROOF OF OUR MAIN RESULTS

**4.1. Proof of Theorem 1.1.** We divide it into several steps.

S t e p  1. Recall $J_n$ from (1.2). One can write

$$J_n(t) =$$

$$= \sum_{k=1}^{\lfloor nt \rfloor} \Big( \int_{(k-1)/n}^{k/n} \sigma_s^{1,1} dB_s^1 + \int_{(k-1)/n}^{k/n} \sigma_s^{1,2} dB_s^2 \Big) \Big( \int_{(k-1)/n}^{k/n} \sigma_s^{2,1} dB_s^1 + \int_{(k-1)/n}^{k/n} \sigma_s^{2,2} dB_s^2 \Big)$$

$$=: A_n(t) + R_{1,n}(t) + R_{2,n}(t),$$

with

$$(4.1) \quad A_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} (\sigma_{k/n}^{1,1} \Delta B_{k/n}^1 + \sigma_{k/n}^{1,2} \Delta B_{k/n}^2)(\sigma_{k/n}^{2,1} \Delta B_{k/n}^1 + \sigma_{k/n}^{2,2} \Delta B_{k/n}^2),$$

$$(4.2) \quad R_{1,n}(t) = \sum_{k=1}^{\lfloor nt \rfloor} \Big( \int_{(k-1)/n}^{k/n} \sigma_s^{1,1} dB_s^1 + \int_{(k-1)/n}^{k/n} \sigma_s^{1,2} dB_s^2 \Big)$$

$$\times \Big( \int_{(k-1)/n}^{k/n} (\sigma_s^{2,1} - \sigma_{k/n}^{2,1}) dB_s^1 + \int_{(k-1)/n}^{k/n} (\sigma_s^{2,2} - \sigma_{k/n}^{2,2}) dB_s^2 \Big),$$

$$(4.3) \quad R_{2,n}(t) = \sum_{k=1}^{\lfloor nt \rfloor} (\sigma_{k/n}^{2,1} \Delta B_{k/n}^1 + \sigma_{k/n}^{2,2} \Delta B_{k/n}^2)$$

$$\times \Big( \int_{(k-1)/n}^{k/n} (\sigma_s^{1,1} - \sigma_{k/n}^{1,1}) dB_s^1 + \int_{(k-1)/n}^{k/n} (\sigma_s^{1,2} - \sigma_{k/n}^{1,2}) dB_s^2 \Big).$$

S t e p  2. Let us prove the convergence of $n^{2H-1} R_{i,n}(t)$, $i = 1, 2$, $t \in [0, T]$, in $L^1(\Omega)$ towards zero. By the Cauchy–Schwarz inequality and Lemma 3.1, we get

$$\|R_{1,n}(t)\|_{L^1(\Omega)} \leqslant \sum_{k=1}^{\lfloor nt \rfloor} \Big\| \int_{(k-1)/n}^{k/n} \sigma_s^{1,1} dB_s^1 + \int_{(k-1)/n}^{k/n} \sigma_s^{1,2} dB_s^2 \Big\|_{L^2(\Omega)}$$

$$\times \Big\| \int_{(k-1)/n}^{k/n} (\sigma_s^{2,1} - \sigma_{k/n}^{2,1}) dB_s^1 + \int_{(k-1)/n}^{k/n} (\sigma_s^{2,2} - \sigma_{k/n}^{2,2}) dB_s^2 \Big\|_{L^2(\Omega)}$$

$$\leqslant Cn^{-(H+2\alpha-1)}.$$

By our assumption (A), we infer that $n^{2H-1} \|R_{1,n}(t)\|_{L^1(\Omega)} \to 0$ as $n \to \infty$. We can similarly prove that $n^{2H-1} \|R_{2,n}(t)\|_{L^1(\Omega)} \to 0$.

S t e p  3. Let us now consider $A_n$. We have

$$A_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} (\sigma_{k/n}^{1,1} \Delta B_{k/n}^1 + \sigma_{k/n}^{1,2} \Delta B_{k/n}^2)(\sigma_{k/n}^{2,1} \Delta B_{k/n}^1 + \sigma_{k/n}^{2,2} \Delta B_{k/n}^2)$$

$$=: A_{1,n}(t) + A_{2,n}(t) + S_n(t),$$

with

$$(4.4) \qquad A_{i,n}(t) = \sum_{k=1}^{\lfloor nt \rfloor} \sigma_{k/n}^{1,i} \sigma_{k/n}^{2,i} (\Delta B_{k/n}^i)^2, \quad i = 1, 2,$$

$$(4.5) \qquad S_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} (\sigma_{k/n}^{1,1} \sigma_{k/n}^{2,2} + \sigma_{k/n}^{1,2} \sigma_{k/n}^{2,1}) \Delta B_{k/n}^1 \Delta B_{k/n}^2.$$

Using Proposition 3.1 and whatever the value of $H$ compared to $\frac{3}{4}$, we can immediately check that $n^{2H-1} S_n(t)$ converges in law to zero, thus in probability. On the other hand, fix $i \in \{1, 2\}$ and recall the well-known fact that, for any $t \in [0, T]$,

$$\lim_{n \to \infty} n^{2H-1} \sum_{k=1}^{\lfloor nT \rfloor} \mathbf{1}_{[0,t]}(k/n)(\Delta B_{k/n}^i)^2 = t \quad \text{almost surely.}$$

We then infer that, with probability one, assumption (3.6) holds true with $h = B^i$ and $\gamma = 2H - 1$. Lemma 3.2 applies and yields

$$n^{2H-1} A_{i,n}(t) \to \int_0^t \sigma_s^{1,i} \sigma_s^{2,i} ds \quad \text{almost surely.}$$

S t e p 4. Plugging together the conclusions of Steps 1–3 completes the proof of Theorem 1.1. ∎

**4.2. Proof of Theorem 1.2.** Recall from the previous section that $J_n = A_{1,n} + A_{2,n} + S_n + R_{1,n} + R_{2,n}$, with $A_{i,n}$, $S_n$, $R_{1,n}$ and $R_{2,n}$ given by (4.4), (4.5), (4.2) and (4.3) respectively. Using the estimates of Step 2 in the previous section, we easily see that, under (A), $a_n R_{i,n}(t)$ tends to zero in $L^1(\Omega)$ as $n \to \infty$, $i = 1, 2$, $t \in [0, T]$. Moreover, the quantities $A_{1,n}$ and $A_{2,n}$ given by (4.4) equal zero when $\sigma^{1,2} = \sigma^{2,1} = 0$. As a result, the asymptotic behaviour of $a_n J_n$ is the same as that of $a_n S_n$, and the desired conclusion follows directly from Proposition 3.1. ∎

**REFERENCES**

[1] F. E. Benth, *On arbitrage-free pricing of weather derivatives based on fractional Brownian motion*, Appl. Math. Finance 10 (2003), pp. 303–324.

[2] P. Breuer and P. Major, *Central limit theorems for non-linear functionals of Gaussian fields*, J. Multivariate Anal. 13 (1983), pp. 425–441.

[3] J. M. Corcuera, D. Nualart, and M. Podolskij, *Asymptotics of weighted random sums*, Commun. Appl. Ind. Math. 6 (1) (2014).

[4] J. M. Corcuera, D. Nualart, and J. H. C. Woerner, *Power variation of some integral long memory process*, Bernoulli 12 (4) (2006), pp. 713–735.

[5]  A. M. Garcia, E. Rodemich, and H. Rumsey, *A real variable lemma and the continuity of paths of some Gaussian processes*, Indiana Univ. Math. J. 20 (1978), pp. 565–578.

[6]  S. C. Kou, *Stochastic modeling in nanoscale biophysics: subdiffusion within proteins*, Ann. Appl. Stat. 2 (2008), pp. 501–535.

[7]  S. C. Kou and X. S. Xie, *Generalized Langevin equation with fractional Gaussian noise: subdiffusion within a single protein molecule*, Phys. Rev. Lett. 93 (2004).

[8]  D. Meintrup, G. Denk, and S. Scheffler, *Transient noise simulation: modeling and simulation of $1/f$ noise*, in: *Modeling, Simulation, and Optimization of Integrated Circuits*, K. Antreich et al. (Eds.), Internat. Ser. Numer. Math. 146 (2001), pp. 251–267.

[9]  I. Nourdin, *Selected Aspects of Fractional Brownian Motion*, Bocconi Springer Ser., Vol. 4, Springer Milan, 2012.

[10]  I. Nourdin, D. Nualart, and C. A. Tudor, *Central and non-central limit theorems for weighted power variations of fractional Brownian motion*, Ann. Inst. H. Poincaré Probab. Statist. 46 (4) (2009), pp. 1055–1079.

[11]  G. Peccati and C. A. Tudor, *Gaussian limits for vector-valued multiple stochastic integrals*, in: *Séminaire de Probabilités XXXVIII*, Lecture Notes in Math., Vol. 1857, Springer, 2005, pp. 247–262.

[12]  J. Syroka, D. Brody, and M. Zervos, *Dynamical pricing of weather derivatives*, Quant. Finance 2 (2002), pp. 189–198.

[13]  M. S. Taqqu, *Weak convergence to fractional Brownian motion and to the Rosenblatt process*, Z. Wahrsch. Verw. Gebiete 31 (1975), pp. 287–302.

[14]  M. S. Taqqu, *The Rosenblatt process*, in: *Selected Works of Murray Rosenblatt*, R. A. Davis, K.-S. Lii, and D. N. Politis (Eds.), Sel. Works Probab. Stat., 2011, pp. 29–45.

[15]  C. A. Tudor, *Analysis of the Rosenblatt process*, ESAIM Probab. Stat. 12 (2008), pp. 230–257.

[16]  L. C. Young, *An inequality of the Hölder type connected with Stieltjes integration*, Acta Math. 67 (1936), pp. 251–282.

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