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SHARP BOUNDS FOR THE BIAS OF TRIMMED MEANS OF PROGRESSIVELY CENSORED ORDER STATISTICS

BY

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Abstract. We provide sharp upper and lower bounds on the bias of trimmed means of progressively censored type II order statistics from general distributions in various scale units. The results are illustrated with numerical examples. We also discuss this problem for distributions with decreasing density or failure rate, as well as for generalized order statistics.

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1. INTRODUCTION

Progressive censoring type II scheme is a life-testing experiment in which units are removed at various stages of the experiment. More precisely, we consider N identical units with independent lifetimes $X_1, \ldots, X_N$ described by common distribution function (cdf) F, and after the i-th failure a prescribed number R_i of surviving units are randomly removed from the experiment. We observe also a prescribed number $n \leqslant N$ of failures, so that $N = n + R_1 + \ldots + R_n$. The time of the i-th failure is denoted in the literature by $X_{i:n:N}^{\mathbf{R}}$, where $\mathbf{R} = (R_1, \ldots, R_n)$ means the censoring scheme, and $X_{1:n:N}^{\mathbf{R}} \leqslant \ldots \leqslant X_{n:n:N}^{\mathbf{R}}$ are called *progressively* (*type II*) *censored order statistics*. For exhaustive treatment of the subject the reader is referred to the monographs [2] and [3]. Since N is determined by n and $\mathbf{R}$, for brevity we denote the time of the i-th failure by $X_{i:n}^{\mathbf{R}}$. Note that if $R_i = 0$ for $1 \leqslant i \leqslant n$, i.e. there are no withdrawals, then $N = n$, and we get ordinary order statistics $X_{1:n} \leqslant \ldots \leqslant X_{n:n}$ of the sample $(X_1, \ldots, X_n)$.

For any cdf F we denote by F^{-1} its quantile function:

$$F^{-1}(u) = \sup\{x : F(x) \leqslant u\}, \quad 0 \leqslant u \leqslant 1.$$

We study the class of distribution functions F with finite mean

$$\mu = \int_0^1 F^{-1}(u)du$$

and finite absolute central moment σ_p of order $p \in [1,\infty]$, where

$$\sigma_p = \Big(\int_0^1 |F^{-1}(u)-\mu|^p du\Big)^{1/p}, \quad 1 \leqslant p < \infty,$$

and

$$\sigma_\infty = \max\{F^{-1}(1^-)-\mu, \mu - F^{-1}(0)\}.$$

In this paper we address ourselves to the following problem: assume that we are given not the whole sample, not even the order statistics but only progressively censored order statistics and we want to estimate the unknown value of the population mean μ assuming known standard deviation σ_2 (or, more generally, p-th central absolute moment σ_p for some $1 \leqslant p \leqslant \infty$). If we knew order statistics, then we might approximate μ by the sample mean

$$T_{1,n:n} = \frac{1}{n}\sum_{i=1}^{n} X_{i:n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

or by the trimmed mean of the form

$$T_{r,s:n} = \frac{1}{s-r+1}\sum_{i=r}^{s} X_{i:n},$$

where $1 \leqslant r \leqslant s \leqslant n$. However, if we have only progressively censored data, we can approximate μ, for example, by the trimmed mean of progressively censored order statistics

$$T^{\mathbf{R}}_{r,n} = \frac{1}{n-r+1}\sum_{i=r}^{n} X^{\mathbf{R}}_{i:n},$$

where $1 \leqslant r \leqslant n$. Note that without loss of generality we can restrict ourselves to the left-sided trimming procedures only. While for the ordinary order statistics two-sided trimming makes sense, the right censoring for the progressively censored order statistics can be formally replaced by a simple change of sampling scheme. Indeed, by Result 1 of [1], the distribution of $(X^{\mathbf{R}}_{1:n},\ldots,X^{\mathbf{R}}_{s:n})$ for some $1 \leqslant s < n$ does not depend on $R_s,\ldots,R_n$ and is identical with that of $(X^{\mathbf{R}_s}_{1:n:N},\ldots,X^{\mathbf{R}_s}_{s:n:N})$ for the modified censoring scheme $(R_1,\ldots,R_{s-1}, N-s-R_1-\ldots-R_{s-1})$.

Several questions arise immediately here. Namely, what is the error of this approximation and how does it depend on the censoring scheme $\mathbf{R}$? Is it better to withdraw units at earlier or later stages of the experiment? We answer these questions by providing sharp upper and lower bounds on the bias of $T^{\mathbf{R}}_{r,n}$ expressed in σ_p units, i.e. on

$$\frac{ET^{\mathbf{R}}_{r,n}-\mu}{\sigma_p}, \quad 1 \leqslant r \leqslant n.$$

Note that for order statistics we have $E\,(T_{1,n:n}) = \mu$, but for progressive censoring scheme this is not necessarily the case. However, as a special case of our results we show that

$$E\left(T_{1,n}^{\mathbf{R}}\right) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_{i:n}^{\mathbf{R}}\right) \leqslant \mu,$$

which is not a trivial bound.

In the setup of order statistics there is a lot of papers devoted to studying bounds on expectations of various functions of order statistics based on distributions coming from either general or restricted families of distributions. In particular, Danielak and Rychlik [9] studied bounds on expectations of trimmed means of order statistics $T_{r,s:n}$ expressed in σ_p units, and Danielak [8] strengthened these bounds for distributions with decreasing density (DD) or decreasing failure rate (DFR).

In the setup of progressively censored order statistics Balakrishnan et al. [4] considered upper mean-variance bounds on $EX_{r:n}^{\mathbf{R}}$, and their results were extended by Raqab [14] to general p-norm bounds. In his review paper Balakrishnan [1] posed an open problem whether other types of bounds could be generalized to progressive censoring. In many cases the answer is positive since progressive censoring scheme is a special case of generalized order statistics (GOS) which were considered in this context by Bieniek [6], [7] – see also references therein, where the bounds of expectation of GOS or spacings of GOS for DD and DFR classes, as well as p-norm bounds on the differences of GOS were studied. However, to the best of our knowledge, the bounds on expectations of trimmed means of GOS have not been considered yet. We have tried to derive them but this causes some difficulties which made restriction to progressively censored order statistics necessary (see Section 6 for details).

The bounds obtained in this paper are derived by the combination of the Moriguti inequality and the Hölder inequality. The key step is to determine the shape of linear combinations of densities of $U_{i:n}^{\mathbf{R}}$ based on uniform distribution. This is done with the aid of the variation diminishing property of densities for GOS proved in [5]. In Section 2 we describe the essence of our constructions. In Section 3 we derive the forms of desired projections, and in Section 4 we present main results on upper and lower bounds on $T_{r,n}^{\mathbf{R}}$. In Section 5 some numerical example is given, and Section 6 contains some concluding remarks on further research in this direction.

2. AUXILIARY RESULTS

In this section we recall briefly the results on the method used in this paper. For its full explanation as well as numerous applications the reader is referred to the monograph [15]. First we recall a simplified version of Theorem 1 in [13].

LEMMA 2.1. *Let* $g : [0,1] \to \mathbb{R}$ *be any integrable function, and let* $G(x) = \int_0^x g(u)du$, $0 \leqslant x \leqslant 1$, *be its antiderivative. Let* $\overline{G}$ *denote the greatest convex* *minorant of G, and let $\overline{g}$ stand for the right-hand derivative of $\overline{G}$. Then*

$$\int_0^1 f(u)g(u)du \leqslant \int_0^1 f(u)\overline{g}(u)du$$

for all nondecreasing functions f such that both integrals exist. The equality holds iff f is constant on every open interval on which $\overline{G} < G$.

Let $U_{i:n}^{\mathbf{R}}$, $1 \leqslant i \leqslant n$, denote progressively censored order statistics based on uniform $U(0,1)$ distribution, and let $f_{i:n}^{\mathbf{R}}$ denote the density of $U_{i:n}^{\mathbf{R}}$. Using Result 7 of [1] we have

$$E(X_{i:n}^{\mathbf{R}}) = \int_0^1 F^{-1}(u) f_{i:n}^{\mathbf{R}}(u)du,$$

which easily gives

$$E(T_{r,n}^{\mathbf{R}} - \mu) = \int_0^1 \left(F^{-1}(u) - \mu\right) \varphi_{r,n}^{\mathbf{R}}(u)du,$$

where

$$\varphi_{r,n}^{\mathbf{R}}(u) = \frac{1}{n-r+1} \sum_{i=r}^{n} f_{i:n}^{\mathbf{R}}(u).$$

Since $F^{-1} - \mu$ integrates to zero on $[0,1]$, we get

$$E(T_{r,n}^{\mathbf{R}} - \mu) = \int_0^1 \left(F^{-1}(u) - \mu\right) \ell_{r,n}^{\mathbf{R}}(u)du,$$

where $\ell_{r,n}^{\mathbf{R}}(u) = \varphi_{r,n}^{\mathbf{R}}(u) - 1$. Let

$$L_{r,n}^{\mathbf{R}}(x) = \int_0^x \ell_{r,n}^{\mathbf{R}}(u)du, \quad 0 \leqslant x \leqslant 1,$$

denote the antiderivative of $\ell_{r,n}^{\mathbf{R}}$, and let $\overline{L}_{r,n}^{\mathbf{R}}$ stand for the greatest convex minorant of $L_{r,n}^{\mathbf{R}}$. Note that for all $1 \leqslant r \leqslant n$ we have $L_{r,n}^{\mathbf{R}}(0) = L_{r,n}^{\mathbf{R}}(1) = 0$.

Suppose first that $L_{r,n}^{\mathbf{R}}(u^*) < 0$ for some $u^* \in (0,1)$. Then, by Lemma 2.1,

$$\begin{aligned} E(T_{r,n}^{\mathbf{R}} - \mu) &\leqslant \int_0^1 \left(F^{-1}(u) - \mu\right) \overline{\ell}_{r,n}^{\mathbf{R}}(u)du \\ &= \int_0^1 \left(F^{-1}(u) - \mu\right) \left(\overline{\ell}_{r,n}^{\mathbf{R}}(u) - c\right)du, \end{aligned}$$

where $\overline{\ell}^{\mathbf{R}}_{r,n}$ is the non-constant right-hand derivative of $\overline{L}^{\mathbf{R}}_{r,n}$, and $c \in \mathbb{R}$ is any constant. Therefore, applying Hölder's inequality, we have

$$\frac{E(T^{\mathbf{R}}_{r,n} - \mu)}{\sigma_p} \leqslant \|\overline{\ell}^{\mathbf{R}}_{r,n} - c^*\|_q, \tag{2.1}$$

where c^* minimizes the norm $\|\overline{\ell}^{\mathbf{R}}_{r,n} - c\|_q$ over $c \in \mathbb{R}$. The equality is attained if

$$\frac{F^{-1}(u) - \mu}{\sigma_p} = \left(\frac{|\overline{\ell}^{\mathbf{R}}_{r,n}(u) - c^*|}{\|\overline{\ell}^{\mathbf{R}}_{r,n} - c^*\|_q}\right)^{q/p} \operatorname{sgn}\big(\overline{\ell}^{\mathbf{R}}_{r,n}(u) - c^*\big).$$

On the other hand, if $L^{\mathbf{R}}_{r,n}(u) \geqslant 0$ for all $u \in (0,1)$, then $\overline{\ell}^{\mathbf{R}}_{r,n} = 0$ and the above approach gives zero upper bound which need not be sharp. However, proceeding as in the proofs of Theorems 2.3, 2.4 and 2.5 of [10] we obtain sharp bounds:

$$\frac{E(T^{\mathbf{R}}_{r,n} - \mu)}{\sigma_p} \leqslant \begin{cases} 0 & \text{if } 1 < p \leqslant \infty, \\ -\inf\limits_{0<u<1} V^{\mathbf{R}}_{r,n}(u) & \text{if } p = 1, \end{cases} \tag{2.2}$$

where

$$V^{\mathbf{R}}_{r,n}(u) = \frac{L^{\mathbf{R}}_{r,n}(u)}{2u(1-u)}, \quad 0 < u < 1.$$

These bounds are attained (possibly in the limit) by appropriate two-point distributions. Note that the function $V^{\mathbf{R}}_{r,n}$ can be defined at zero and one by continuity as

$$V^{\mathbf{R}}_{r,n}(0) = \frac{1}{2}\big(\varphi^{\mathbf{R}}_{r,n}(0) - 1\big), \quad V^{\mathbf{R}}_{r,n}(1) = \frac{1}{2}\big(1 - \varphi^{\mathbf{R}}_{r,n}(1)\big), \tag{2.3}$$

and the values $\varphi^{\mathbf{R}}_{r,n}(0)$ and $\varphi^{\mathbf{R}}_{r,n}(1)$ are described in (3.3) and (3.4) below. Since these values are finite, the infimum over $(0,1)$ in (2.2) can be replaced by the minimum over $[0,1]$.

To derive lower bounds, Danielak and Rychlik [9] used the symmetry of distributions of ordinary order statistics $Y_{i:n} - \mu = \mu - X_{n+1-i:n}$, where X has cdf F, and Y has cdf $F^-(x) = 1 - F(-x)$. For progressively censored order statistics no symmetry of this kind holds as can be seen, e.g., from the form of marginal density of $U^{\mathbf{R}}_{r:n}$ (see (3.2) below). Therefore, we derive lower bounds performing the whole procedure of the greatest convex minorant. See, e.g., the paper of Goroncy and Rychlik [11] where this approach has been applied to derive the lower sharp bounds for the expectations of record values. It follows that

$$-E(T^{\mathbf{R}}_{r,n} - \mu) \leqslant \big\|\overline{-\ell}^{\mathbf{R}}_{r,n} - c_*\big\|_q \sigma_p, \tag{2.4}$$

where $\overline{-\ell}^{\mathbf{R}}_{r,n}$ denotes the right-hand derivative of $\overline{-L}^{\mathbf{R}}_{r,n}$, which is the greatest convex minorant of $-L^{\mathbf{R}}_{r,n}$, and c_* minimizes the norm $\|\bar{\ell}^{\mathbf{R}}_{r,n} - c\|_q$. However, $\overline{-\ell}^{\mathbf{R}}_{r,n} \neq -\bar{\ell}^{\mathbf{R}}_{r,n}$, so $\overline{-\ell}^{\mathbf{R}}_{r,n}$ has to be determined separately.

Summarizing, we need to determine convexity regions of $L^{\mathbf{R}}_{r,n}$, so in fact we need to study monotonicity properties of $\varphi^{\mathbf{R}}_{r,n}$. Since the function and its derivative are quite complicated functions, this is done in the next section by utilizing distribution theory of generalized order statistics.

3. SHAPES OF PROJECTED FUNCTIONS

In this section we determine the shapes of $\varphi^{\mathbf{R}}_{r,n}$ and those of $\bar{\ell}^{\mathbf{R}}_{r,n}$ and $\overline{-\ell}^{\mathbf{R}}_{r,n}$. By (2.1) and (2.4) this is useful for evaluation of upper and lower bounds.

The joint density of $(U^{\mathbf{R}}_{1:n}, \ldots, U^{\mathbf{R}}_{n:n})$ is given by

$$f_{U^{\mathbf{R}}_{1:n},\ldots,U^{\mathbf{R}}_{n:n}}(u_1, \ldots, u_n) = c_{n-1} \prod_{i=1}^{n} (1-u_i)^{R_i}, \quad 0 \leqslant u_1 \leqslant \ldots \leqslant u_n \leqslant 1,$$

where

$$c_{r-1} = \prod_{j=1}^{r} \gamma_j, \quad 1 \leqslant r \leqslant n,$$

and

$$\gamma_j = N - j + 1 - \sum_{i=1}^{j-1} R_i, \quad 1 \leqslant j \leqslant n \tag{3.1}$$

(see, e.g., [1] or [2]). Therefore, as it is noted in [4], progressively censored order statistics based on uniform $U(0,1)$ distribution are uniform generalized order statistics with parameters $m_i = R_i$, $1 \leqslant i \leqslant n-1$, $k = R_n + 1$. Note that $\gamma_1 > \ldots > \gamma_n \geqslant 1$ (and, in particular, $\gamma_i \neq \gamma_j$ for $i \neq j$), so to derive the shape of $\varphi^{\mathbf{R}}_{r,n}$ we can use distribution theory of generalized order statistics presented in [12].

Let

$$a_{i,r} = \prod_{j=1, j\neq i}^{r} \frac{1}{\gamma_j - \gamma_i}, \quad 1 \leqslant i \leqslant r \leqslant n.$$

Then the density of $U^{\mathbf{R}}_{r:n}$, $1 \leqslant r \leqslant n$, is

$$f^{\mathbf{R}}_{r:n}(u) = c_{r-1} \sum_{i=1}^{r} a_{i,r} (1-u)^{\gamma_i - 1}, \quad u \in (0,1). \tag{3.2}$$

First we study the values of $\varphi^{\mathbf{R}}_{r,n}$ at zero and one. From the above representation of $f^{\mathbf{R}}_{r:n}$ it is easy to derive that

$$\varphi^{\mathbf{R}}_{r,n}(0) = \begin{cases} N/n & \text{if } r = 1, \\ 0 & \text{if } 2 \leqslant r \leqslant n, \end{cases} \tag{3.3}$$

and

$$
\varphi^{\mathbf{R}}_{r,n}(1) = \begin{cases} 0 & \text{if } R_n \geqslant 1, \\ \frac{1}{n-r+1} A_n & \text{if } R_n = 0, \end{cases} \tag{3.4}
$$

with $A_n = c_{n-1} a_{n,n} = \prod_{i=1}^{n-1} \gamma_i/(\gamma_i - 1)$. We shall need the following auxiliary lemma.

LEMMA 3.1. *Fix $n \geqslant 2$ and integers $R_1, \ldots, R_{n-1} \geqslant 0$ and let $R_n = 0$. Then for $\gamma_1, \ldots, \gamma_{n-1}$ defined by* (3.1) *we have*

$$
1 < \prod_{i=1}^{n-1} \frac{\gamma_i}{\gamma_i - 1} \leqslant n,
$$

and the equality on the right-hand side holds iff $R_1 = \ldots = R_{n-1} = 0$.

Proof. The left-hand side inequality is obvious since $\gamma_i > 1$ for $1 \leqslant i \leqslant n-1$, so it suffices to prove the right-hand side. Note that for $1 \leqslant i \leqslant n-1$ we have $\gamma_i \geqslant n - i + 1$, and the equality holds for all $i = 1, \ldots, n-1$ iff $R_1 = \ldots = R_{n-1} = 0$. Therefore,

$$
\frac{\gamma_i}{\gamma_i - 1} \leqslant \frac{n-i+1}{n-i},
$$

and

$$
\prod_{i=1}^{n-1} \frac{\gamma_i}{\gamma_i - 1} \leqslant \prod_{i=1}^{n-1} \frac{n-i+1}{n-i} = n,
$$

which completes the proof of the lemma. ■

The above lemma says that if $R_n = 0$, then the value of $\varphi^{\mathbf{R}}_{1,n}$ at one is less than one, but the values of $\varphi^{\mathbf{R}}_{r,n}(1)$ for $2 \leqslant r < n$ may be greater than one.

Now we study the derivative of $\varphi^{\mathbf{R}}_{r,n}$. If we plug (3.2) to $\varphi^{\mathbf{R}}_{r,n}$, then direct computation of the derivative is easy but analyzing its sign changes is very hard, and therefore some more subtle considerations are necessary. Namely, we have

$$
(f^{\mathbf{R}}_{i:n})'(u) = \frac{1}{1-u} \big(\gamma_i f^{\mathbf{R}}_{i-1:n}(u) - (\gamma_i - 1) f^{\mathbf{R}}_{i:n}(u) \big)
$$

(see, e.g., [5]). Therefore, by using the relations $\gamma_{i+1} - \gamma_i + 1 = -R_i$ and $\gamma_n = R_n + 1$, the derivative of $\varphi^{\mathbf{R}}_{r,n}$ can be written as the following linear combination:

$$
(\varphi^{\mathbf{R}}_{r,n})'(u) = \frac{1}{(n-r+1)(1-u)} \sum_{i=r-1}^{n} a_i f^{\mathbf{R}}_{i:n}(u), \tag{3.5}
$$

where $a_{r-1} = \gamma_r$ and $a_i = -R_i$ for $r \leqslant i \leqslant n$. Here we adopt the convention $\gamma_0 = 0$. To analyze the sign changes of $(\varphi^{\mathbf{R}}_{r,n})'$ we use the variation diminishing property of densities of generalized order statistics proved in [5].

For any function $f : [0,1] \mapsto \mathbb{R}$ let $Z(f)$ denote the number of zeroes of f in $(0,1)$. Moreover, for an arbitrary sequence $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ let

$$H_{\mathbf{a}}(x) = \sum_{j=1}^{n} a_j\, f_{j:n}^{\mathbf{R}}(x), \quad x \in (0,1),$$

and let $S^-(\mathbf{a})$ denote the number of sign changes in the sequence $(a_1, \ldots, a_n)$ after deletion of zeroes. Recall that for progressive censoring scheme we have $\gamma_1 > \ldots > \gamma_n \geqslant 1$.

THEOREM 3.1 (Bieniek [5]). *For any* $\mathbf{a} \neq \mathbf{0}$ *we have*:

(i) $Z(H_{\mathbf{a}}) \leqslant S^-(\mathbf{a})$.

(ii) *The first and the last sign of* $H_{\mathbf{a}}$ *are the same as the signs of the first and the last nonzero element of* $\mathbf{a}$, *respectively.*

Now we can study monotonicity properties of $\varphi_{r,n}^{\mathbf{R}}$, $1 \leqslant r \leqslant n$. By assumption we have $R_i \geqslant 0$, and therefore the coefficients a_i, $r \leqslant i \leqslant n$, in the expansion (3.5) are all non-positive. First we consider some special cases.

The case when $r = n$ corresponds to a single progressively censored order statistic $X_{n:n}^{\mathbf{R}}$ and it was studied in [4] for $p = 2$ and in [14] for arbitrary $p \in [1, \infty]$. So we assume $r < n$.

Now let $r = 1$. If $R_1 = \ldots = R_n = 0$, then $ET_{1,n}^{\mathbf{R}} = ET_{1,n:n} = \mu$. So we assume that $R_i > 0$ for some $1 \leqslant i \leqslant n$. Then, by (3.5) and Theorem 3.1, it is easy to deduce that $(\varphi_{1,n}^{\mathbf{R}})'$ is negative. So $\varphi_{1,n}^{\mathbf{R}}$ is strictly decreasing from $\varphi_{1,n}^{\mathbf{R}}(0) > 1$ by (3.3). By (3.4) and Lemma 3.1, the value $\varphi_{1,n}^{\mathbf{R}}(1)$ is less than one, regardless of the value of R_n. Therefore, $\ell_{1,n}^{\mathbf{R}}$ is positive–negative, and $L_{1,n}^{\mathbf{R}}$ is increasing–decreasing on $(0,1)$. Taking into account that $L_{1,n}^{\mathbf{R}}(0) = L_{1,n}^{\mathbf{R}}(1) = 0$ we infer that $L_{1,n}^{\mathbf{R}}$ is positive on $(0,1)$, and $\overline{\ell}_{1,n}^{\mathbf{R}} = 0$. On the other hand, since $\ell_{1,n}^{\mathbf{R}}$ is decreasing, it follows that $L_{1,n}^{\mathbf{R}}$ is concave. So $-L_{1,n}^{\mathbf{R}}$ is convex and $\overline{-\ell}_{1,n}^{\mathbf{R}} = -\ell_{1,n}^{\mathbf{R}}$.

Now we let $2 \leqslant r < n$. If $R_r = \ldots = R_n = 0$, then $(\varphi_{r,n}^{\mathbf{R}})'$ is positive, and $\varphi_{r,n}^{\mathbf{R}}$ is increasing on $(0,1)$. Moreover, $\varphi_{r,n}^{\mathbf{R}}(0) = 0$ and

$$\varphi_{r,n}^{\mathbf{R}}(1) = \prod_{i=1}^{r-1} \frac{\gamma_i}{\gamma_i - 1} > 1.$$

Therefore, $\ell_{r,n}^{\mathbf{R}}$ is increasing and negative–positive on $(0,1)$. This implies that $L_{r,n}^{\mathbf{R}}$ is negative and convex, so $\overline{\ell}_{r,n}^{\mathbf{R}} = \ell_{r,n}^{\mathbf{R}}$. Moreover, $-L_{r,n}^{\mathbf{R}}$ is positive on $(0,1)$ and $\overline{-\ell}_{r,n}^{\mathbf{R}} = 0$. On the other hand, if $R_i > 0$ for some $r \leqslant i \leqslant n$, then by (3.5) and Theorem 3.1 we infer that there exists a unique $\theta = \theta_{r,n}^{\mathbf{R}} \in (0,1)$ such that $\varphi_{r,n}^{\mathbf{R}}$ is increasing from 0 to $\varphi_{r,n}^{\mathbf{R}}(\theta) > 1$ and then strictly decreasing.

If $R_n > 0$ or $R_n = 0$ and $r < n + 1 - A_n$, then $\varphi_{r,n}^{\mathbf{R}}(1) < 1$, and $\ell_{r,n}^{\mathbf{R}}$ is negative–positive–negative. Therefore, $L_{r,n}^{\mathbf{R}}$ is decreasing–increasing–decreasing and convex on $(0,\theta)$, and concave on $(\theta,1)$. Thus, there exists a unique $\alpha^* = \alpha^{*\mathbf{R}}_{r,n} \in (0,\theta)$ such that the greatest convex minorant of $L^{\mathbf{R}}_{r,n}$ is

$$\overline{L}^{\mathbf{R}}_{r,n}(u) = \begin{cases} L^{\mathbf{R}}_{r,n}(u) & \text{for } 0 \leqslant u \leqslant \alpha^*, \\ \ell^{\mathbf{R}}_{r,n}(\alpha^*)(u-1) & \text{for } \alpha^* \leqslant u \leqslant 1. \end{cases}$$

Indeed, α^* is determined by the equation

$$L^{\mathbf{R}}_{r,n}(\alpha^*) = -(1-\alpha^*)\ell^{\mathbf{R}}_{r,n}(\alpha^*).$$

Therefore, the derivative of $\overline{L}^{\mathbf{R}}_{r,n}$ is

$$\overline{\ell}^{\mathbf{R}}_{r,n}(u) = \begin{cases} \ell^{\mathbf{R}}_{r,n}(u) & \text{for } 0 \leqslant u \leqslant \alpha^*, \\ \ell^{\mathbf{R}}_{r,n}(\alpha^*) & \text{for } \alpha^* \leqslant u \leqslant 1. \end{cases} \tag{3.6}$$

The last formula holds even if $R_r = \ldots = R_n = 0$ if we put $\alpha^* = 1$. Proceeding analogously we infer that if either $R_n > 0$ and $2 \leqslant r < n$, or $R_n = 0$ and $2 \leqslant r < n+1-A_n$, we have

$$\overline{-\ell}^{\mathbf{R}}_{r,n}(u) = \begin{cases} -\ell^{\mathbf{R}}_{r,n}(\alpha_*) & \text{for } 0 \leqslant u \leqslant \alpha_*, \\ -\ell^{\mathbf{R}}_{r,n}(u) & \text{for } \alpha_* \leqslant u \leqslant 1 \end{cases} \tag{3.7}$$

for a unique $\alpha_* = \alpha^{\mathbf{R}}_{*,r,n} \in (\theta,1)$ determined by the equation

$$L^{\mathbf{R}}_{r,n}(\alpha_*) = \alpha_*\ell^{\mathbf{R}}_{r,n}(\alpha_*).$$

The formula (3.7) holds for $r=1$ as well if we put $\alpha_* = 0$.

Finally, if $R_n = 0$ and $r \geqslant n+1-A_n$, then $\varphi^{\mathbf{R}}_{r,n}(1) \geqslant 1$, and $\ell^{\mathbf{R}}_{r,n}$ is negative–positive. Therefore, $L^{\mathbf{R}}_{r,n}$ is decreasing–increasing, so it is negative on $(0,1)$. However, $\overline{\ell}^{\mathbf{R}}_{r,n}$ has the same form as in (3.6). On the other hand, $-L^{\mathbf{R}}_{r,n}$ is positive on $(0,1)$, and again $\overline{-\ell}^{\mathbf{R}}_{r,n} = 0$.

4. MAIN RESULTS

In this section we present the values of upper and lower sharp bounds

$$-\underline{B}^{(p)}_{r,n} \leqslant \frac{E(T^{\mathbf{R}}_{r,n}-\mu)}{\sigma_p} \leqslant \overline{B}^{(p)}_{r,n}$$

for $1 \leqslant p \leqslant \infty$. Obviously, the bounds depend on the censoring scheme $\mathbf{R}$, but since $\mathbf{R}$ is fixed throughout the paper, we supress it from the notation of bounds.

In the next theorems we consider three cases: $1 < p < \infty$, $p = \infty$ and $p = 1$. Since shapes of projected functions are the same as in [9] and [11], the proofs of our theorems follow the ideas there, and therefore we only sketch them briefly. Also the forms of the distributions attaining the bounds are analogous to the ones obtained by [9] and [11] with obvious changes, so we do not specify them here.

Note that in most of the cases the bounds are expressed in terms of the function $\overline{\varphi}^{\mathbf{R}}_{r,n}$ instead of $\overline{\ell}^{\mathbf{R}}_{r,n}$. The function $\overline{\varphi}^{\mathbf{R}}_{r,n}$ is just the right derivative of the greatest convex minorant of the antiderivative of $\varphi^{\mathbf{R}}_{r,n}$ defined by

$$\Phi^{\mathbf{R}}_{r,n}(x) = \int_0^x \varphi^{\mathbf{R}}_{r,n}(u)du, \quad 0 \leqslant x \leqslant 1.$$

But $L^{\mathbf{R}}_{r,n}(u) = \Phi^{\mathbf{R}}_{r,n}(u) - u$, so $\overline{\varphi}^{\mathbf{R}}_{r,n} = \overline{\ell}^{\mathbf{R}}_{r,n} + 1$ and $\overline{-\varphi}^{\mathbf{R}}_{r,n} = \overline{-\ell}^{\mathbf{R}}_{r,n} - 1$.

THEOREM 4.1. *Fix $p \in (1,\infty)$ and let $1/p + 1/q = 1$. For any cdf F with finite σ_p we have*

$$\overline{B}^{(p)}_{r,n} = \begin{cases} 0 & \text{if } r = 1, \\ \|\overline{\varphi}^{\mathbf{R}}_{r,n} - \varphi^{\mathbf{R}}_{r,n}(\eta^*)\|_q & \text{if } 2 \leqslant r \leqslant n, \end{cases}$$

where $\eta^* \in (0,\alpha^*)$ is the unique solution to the equation

$$\int_0^{\eta^*} \left(\varphi^{\mathbf{R}}_{r,n}(\eta^*) - \varphi^{\mathbf{R}}_{r,n}(t)\right)^{q-1} dt = \int_{\eta^*}^{\alpha^*} \left(\varphi^{\mathbf{R}}_{r,n}(t) - \varphi^{\mathbf{R}}_{r,n}(\eta^*)\right)^{q-1} dt + (1-\alpha^*)\left(\varphi^{\mathbf{R}}_{r,n}(\alpha^*) - \varphi^{\mathbf{R}}_{r,n}(\eta^*)\right)^{q-1}.$$

Moreover,

$$\underline{B}^{(p)}_{r,n} = \left\|\overline{-\varphi}^{\mathbf{R}}_{r,n} + \varphi^{\mathbf{R}}_{r,n}(\eta_*)\right\|_q, \quad 1 \leqslant r \leqslant n,$$

where $\eta_* \in (\alpha_*, 1)$ is the unique solution to the equation

$$\alpha_*\left(\varphi^{\mathbf{R}}_{r,n}(\alpha_*) - \varphi^{\mathbf{R}}_{r,n}(\eta_*)\right)^{q-1} + \int_{\alpha_*}^{\eta_*} \left(\varphi^{\mathbf{R}}_{r,n}(t) - \varphi^{\mathbf{R}}_{r,n}(\eta_*)\right)^{q-1} dt = \int_{\eta_*}^1 \left(\varphi^{\mathbf{R}}_{r,n}(\eta_*) - \varphi^{\mathbf{R}}_{r,n}(t)\right)^{q-1} dt.$$

The fact that $\overline{B}^{(p)}_{r,n} = 0$ is implied by (2.2). The form of upper bound for $2 \leqslant r \leqslant n$ follows from (2.1), and the form of lower bound follows from (2.4).

For $p = q = 2$ we easily get $c^* = 1$ and $c_* = -1$, which leads to the following corollary presenting the bounds in standard deviation units.

COROLLARY 4.1. *Fix F with finite variance σ^2. Then $\overline{B}^{(2)}_{1,n} = 0$, and we have for $2 \leqslant r \leqslant n$*

$$\begin{aligned}\overline{B}^{(2)}_{r,n} &= (\|\overline{\varphi}^{\mathbf{R}}_{r,n}\|_2^2 - 1)^{1/2} \\ &= \Big(\int_0^{\alpha_*} \left(\varphi^{\mathbf{R}}_{r,n}(t)\right)^2 dt + (1-\alpha_*)\left(\varphi^{\mathbf{R}}_{r,n}(\alpha_*)\right)^2 - 1\Big)^{1/2},\end{aligned}$$

and for $1 \leqslant r \leqslant n$

$$\begin{aligned}\underline{B}^{(2)}_{r,n} &= \left(\left\|\overline{-\varphi^{\mathbf{R}}_{r,n}}\right\|_2^2 - 1\right)^{1/2} \\ &= \left(\alpha_*\left(\varphi^{\mathbf{R}}_{r,n}(\alpha_*)\right)^2 + \int_{\alpha_*}^{1}\left(\varphi^{\mathbf{R}}_{r,n}(t)\right)^2 dt - 1\right)^{1/2}.\end{aligned}$$

For $p = \infty$ we have the following theorem, whose proof is analogous to the proof of Theorem 4.1.

THEOREM 4.2. *Fix cdf* F *with support on the interval* $[\mu - \sigma_\infty, \mu + \sigma_\infty]$. *Then* $\overline{B}^{(\infty)}_{1,n} = 0$, *and we have for* $2 \leqslant r \leqslant n$

$$\overline{B}^{(\infty)}_{r,n} = \begin{cases} 1 - 2\Phi^{\mathbf{R}}_{r,n}\left(\frac{1}{2}\right) & \text{if } \alpha^* > \frac{1}{2}, \\ \varphi^{\mathbf{R}}_{r,n}(\alpha^*) - 1 & \text{if } \alpha^* \leqslant \frac{1}{2}, \end{cases}$$

and for $1 \leqslant r \leqslant n$

$$\underline{B}^{(\infty)}_{r,n} = \begin{cases} \varphi^{\mathbf{R}}_{r,n}(\alpha_*) - 1 & \text{if } \alpha_* \geqslant \frac{1}{2}, \\ 2\Phi^{\mathbf{R}}_{r,n}\left(\frac{1}{2}\right) - 1 & \text{if } \alpha_* < \frac{1}{2}. \end{cases}$$

For $p = 1$ the result is more complicated due to the fact that $L^{\mathbf{R}}_{1,n}$ and $-L^{\mathbf{R}}_{r,n}$ with $R_n = 0$ and $r \geqslant n + 1 - A_n$ are positive on $(0, 1)$ (see Section 3). Then the Moriguti approach does not yield sharp bounds and we have to apply (2.2).

THEOREM 4.3. *Fix cdf* F *with finite* σ_1. *Then we have*

$$\overline{B}^{(1)}_{r,n} = \begin{cases} -\min\limits_{0 \leqslant u \leqslant 1} V^{\mathbf{R}}_{1,n}(u) & \text{for } r = 1, \\ \frac{1}{2}\varphi^{\mathbf{R}}_{r,n}(\alpha^*) & \text{for } 2 \leqslant r \leqslant n \end{cases} \tag{4.1}$$

and

$$\underline{B}^{(1)}_{r,n} = \begin{cases} \max\limits_{0 \leqslant u \leqslant 1} V^{\mathbf{R}}_{r,n}(u) & \text{if } R_n = 0 \text{ and } n + 1 - A_n \leqslant r \leqslant n, \\ \frac{1}{2}\varphi^{\mathbf{R}}_{r,n}(\alpha_*) & \text{otherwise.} \end{cases}$$

Note that $V^{\mathbf{R}}_{1,n}$ is positive on $[0, 1]$, so the upper bound $\overline{B}^{(1)}_{1,n}$ is negative. Moreover, if $R_n = 0$ and $n + 1 - A_n \leqslant r \leqslant n$, the function $V^{\mathbf{R}}_{r,n}$ is negative, so then the lower bound $-\underline{B}^{(1)}_{r,n}$ is positive. Obviously, it is of interest to know a more specific analytical form of the bounds in these cases. At first glance, one could note that, for instance, the minimum in (4.1) does not exceed the smallest of the values $V^{\mathbf{R}}_{1,n}(0)$, $V^{\mathbf{R}}_{1,n}(1)$ (given by (2.3)) and $V^{\mathbf{R}}_{1,n}\left(\frac{1}{2}\right) = 2\Phi^{\mathbf{R}}_{1,n}\left(\frac{1}{2}\right) - 1$. This gives some rough estimate of $\overline{B}^{(1)}_{1,n}$, and a similar statement can be made about the lower bounds. However, numerical computations exhibit very complicated dependence of the shape of $V^{\mathbf{R}}_{r,n}$ on the censoring scheme $\mathbf{R}$. Therefore, it seems that no more specific general statements can be made about the value of the bounds in these cases, or even about the point at which respective minimum or maximum is achieved.

5. NUMERICAL EXAMPLE

Our results admit immediate numerical implementation. In this section we present and analyze an example of calculations of upper and lower bounds on trimmed means of progressively censored order statistics expressed in standard deviation σ_2 units.

TABLE 1. Comparison of upper and lower bounds for different censoring schemes.
$N = 35$, $n = 7$, $\mathbf{R}_1 = (1, 2, 3, 4, 5, 6, 7)$, $\mathbf{R}_2 = (4, 4, 4, 4, 4, 4, 4)$, $\mathbf{R}_3 = (7, 6, 5, 4, 3, 2, 1)$

	$-\underline{B}^{(2)}_{r,7:7}$			$\overline{B}^{(2)}_{r,7:7}$		
r	$\mathbf{R}_1$	$\mathbf{R}_2$	$\mathbf{R}_3$	$\mathbf{R}_1$	$\mathbf{R}_2$	$\mathbf{R}_3$
1	-1.512	-1.333	-1.106	0	0	0
2	-1.404	-1.195	-0.9006	0.0427	0.045	0.0477
3	-1.319	-1.094	-0.7508	0.1028	0.1114	0.1229
4	-1.238	-1.003	-0.6185	0.156	0.1736	0.2005
5	-1.157	-0.9133	-0.4942	0.2068	0.2361	0.2880
6	-1.067	-0.8174	-0.371	0.2592	0.304	0.3981
7	-0.9546	-0.7012	-0.283	0.3191	0.3858	0.5601

We want to compare three different censoring schemes:

$$\mathbf{R}_1 = (1, 2, 3, 4, 5, 6, 7), \quad \mathbf{R}_2 = (4, 4, 4, 4, 4, 4, 4), \quad \mathbf{R}_3 = (7, 6, 5, 4, 3, 2, 1)$$

for the same values of $N = 35$ and $n = 7$. The results are presented in Table 1. Looking only at upper bounds one might think that the smallest error committed while estimating μ by $T^{\mathbf{R}}_{r,n}$ is obtained if $r = 1$, and that it is better to use $\mathbf{R}_1$ scheme than $\mathbf{R}_2$ or $\mathbf{R}_3$. This would agree with our intuition that we should use as much information as possible (small r) and as long as possible (less withdrawals at early stages of the experiment). But a quick look at the lower bound shows that if we want to minimize lower and upper bounds simultaneously, we should use $\mathbf{R}_3$ scheme and rather large values of r (i.e. close to n). Indeed, it turns out that $E(T^{\mathbf{R}_3}_{5,7:7} - \mu)/\sigma_2$ has the smallest range according to the results presented in Table 1. Similar comparisons are possible for other σ_p units as well as other values of N, n and censoring schemes.

6. DISCUSSION OF FURTHER RESEARCH

One might expect that the results of the paper can easily be extended to other generalized order statistics with arbitrary positive parameters $\gamma_1, \ldots, \gamma_n$. However, then nonnegative integers $R_1, \ldots, R_{n-1}$ are replaced with reals $m_1, \ldots, m_{n-1}$, where $m_i = \gamma_i - \gamma_{i+1} - 1$, and R_n is replaced with $k = \gamma_n$. Therefore, the signs of m_i's can be arbitrary and we cannot infer that the corresponding function $\varphi_{r,n}$ is increasing–decreasing, just based on (3.5) and the variation diminishing property.

The problem is very complicated in view of the dependence of the shape of $\varphi_{r,n}$ on the parameters of generalized order statistics. For instance, numerical computations with Mathematica 9.0 software show that for the parameter vector

$$(\gamma_1, \ldots, \gamma_7) = (13.1, 13, 12.9, 2.3, 2.2, 2.1, 2)$$

the function $\varphi_{3,7}$ is increasing–decreasing–increasing–decreasing. Although there is a general algorithm for calculating the bounds for each fixed generalized L-statistic, a common final formula for bounds valid for various L-statistics is lacking.

Coming back to progressive censoring scheme, one might also suppose that the results of this paper could be strengthened in restricted families of distributions like DD or DFR. This requires analyzing of convexity and concavity regions of $\varphi^{\mathbf{R}}_{r,n}$ or, equivalently, sign changes of the second derivative $(\varphi^{\mathbf{R}}_{r,n})''$. Using the formula

$$\begin{aligned}(f^{\mathbf{R}}_{i:n})''(u) = \frac{1}{(1-u)^2}\{\gamma_i\gamma_{i-1}f^{\mathbf{R}}_{i-2:n}(u) - \gamma_i(\gamma_i+\gamma_{i-1}-3)f^{\mathbf{R}}_{i-1:n}(u) \\ + (\gamma_i-1)(\gamma_i-2)f^{\mathbf{R}}_{i:n}(u)\}\end{aligned}$$

of [5], we easily infer that for $3 \leqslant r \leqslant n-2$

$$(\varphi^{\mathbf{R}}_{r,n})''(u) = \frac{1}{(n-r+1)(1-u)^2}\sum_{i=r-2}^{n} b_i f^{\mathbf{R}}_{i:n}(u),$$

where

$$\begin{aligned}
b_{r-2} &= \gamma_r\gamma_{r-1}, \\
b_{r-1} &= \gamma_r(\gamma_{r+1}-\gamma_r-\gamma_{r-1}+3), \\
b_i &= \gamma_{i+1}(\gamma_{i+2}-\gamma_{i+1}-\gamma_i+3)+(\gamma_i-1)(\gamma_i-2), \quad r \leqslant i \leqslant n-2, \\
b_{n-1} &= -\gamma_n(\gamma_n+\gamma_{n-1}-3)+(\gamma_{n-1}-1)(\gamma_{n-1}-2), \\
b_n &= (\gamma_n-1)(\gamma_n-2).
\end{aligned}$$

This leads to the same problem: since the sings of b_i's can be arbitrary, the arguments based on VDP do not suffice to claim that $(\varphi^{\mathbf{R}}_{r,n})''$ is $+-+$ on $(0,1)$. Numerical computations provide examples where $(\varphi^{\mathbf{R}}_{3,6})''$ is $+-+-$, which may mean that $\varphi^{\mathbf{R}}_{3,6}$ itself is convex–concave–convex–concave increasing, and then concave decreasing. In this case the problem is that there are not known tools for determining projections of functions onto the family of nondecreasing convex functions unless the projected function has a specific simple form.

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