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ON THE STRUCTURE OF A CLASS OF DISTRIBUTIONS OBEYING THE PRINCIPLE OF A SINGLE BIG JUMP

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Abstract. In this paper, we present several heavy-tailed distributions belonging to the new class $\mathcal{J}$ of distributions obeying the principle of a single big jump introduced by Beck et al. (2015). We describe the structure of this class from different angles. First, we show that heavy-tailed distributions in the class $\mathcal{J}$ are automatically *strongly heavy-tailed* and thus have tails which are not too irregular. Second, we show that such distributions are not necessarily weakly tail equivalent to a subexponential distribution. We also show that the class of heavy-tailed distributions in $\mathcal{J}$ which are neither long-tailed nor dominatedly-varying-tailed is not only non-empty but even quite rich in the sense that it has a non-empty intersection with several other well-established classes. In addition, the integrated tail distribution of some particular of these distributions shows that the Pakes–Veraverbeke–Embrechts theorem for the class $\mathcal{J}$ does not hold trivially.

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1. INTRODUCTION

In this paper, all distributions have unbounded support contained in $[0, \infty)$. Recall that a distribution F is called *heavy-tailed*, denoted by $F \in \mathcal{K}$, if for all $\alpha > 0$

$$\int_0^\infty e^{\alpha y} dF(y) = \infty;$$

otherwise, F is called *light-tailed*, denoted by $F \in \mathcal{K}^c$. Recently, Beck et al. introduced in [1] the following new distribution class $\mathcal{J}$.

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Let $\{X_i, i \geqslant 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with common distribution F. Define the class $\mathcal{J}$ as the set of distributions F such that, for all $n \geqslant 2$,

$$\lim_{K\to\infty} \liminf \mathbb{P}\big(X_{n,1} > x - K \mid \sum_{i=1}^{n} X_i > x\big) = 1, \tag{1.1}$$

where $X_{n,k}$ means the k-th largest random variable in the sequence $\{X_i, 1 \leqslant i \leqslant n\}, 1 \leqslant k \leqslant n$. Here and in the following all unspecified limits are to be understood for $x \to \infty$. Beck et al. [1] show that each of the following two properties is equivalent to (1.1):

$$\begin{gathered} \lim_{K\to\infty} \liminf \mathbb{P}\big(X_{n,2} \leqslant K \mid \sum_{i=1}^{n} X_i > x\big) = 1, \\ \lim \mathbb{P}\big(X_{n,2} > g(x) \mid \sum_{i=1}^{n} X_i > x\big) = 0 \quad \text{for all } g \nearrow \infty \end{gathered} \tag{1.2}$$

for any (and hence for all) $n \geqslant 2$.

Note that this definition is a way of stating formally that the distribution of X_1 obeys the *principle of a single big jump* which means that conditional on the sum $X_1 + \ldots + X_n$ being unusually large, the probability that a single summand dominates the sum is close to one while the conditional law of the second largest summand remains tight (the letter $\mathcal{J}$ stands for *jump*).

It is natural to ask how the class $\mathcal{J}$ is related to well-established classes like $\mathcal{S}$, $\mathcal{D}$ and $\mathcal{L}$ which are respectively called *subexponential*, *dominatedly varying* and *long tailed* (for their definitions, see [2]). In addition, the class $\mathcal{OS}$ of *generalized subexponential* distributions first introduced in [5] is of interest. By definition, it consists of those distributions for which

$$C^*(F) := \limsup \overline{F^{*2}}(x)\big(\overline{F}(x)\big)^{-1} < \infty, \tag{1.3}$$

where F^{*n} means the n-fold convolution of F with itself for $n \geqslant 2$, and $\overline{F} := 1 - F$ denotes the tail of F. Note that the class $\mathcal{S}$ corresponds to the case $C^*(F) = 2$.

The following relations for these classes are known:

$$\mathcal{S} \subset \mathcal{L} \subset \mathcal{K}, \quad \mathcal{S} \cup \mathcal{D} \subset \mathcal{J} \subset \mathcal{OS}, \quad \mathcal{S} = \mathcal{J} \cap \mathcal{L}, \quad \mathcal{D} \subset \mathcal{K} \tag{1.4}$$

(see [2], respectively [1], for relations not involving $\mathcal{J}$, respectively those involving $\mathcal{J}$). It is not true that all distributions in $\mathcal{J}$ are heavy-tailed (see [1]). The recent paper [8] actually shows that the class of light-tailed distributions in $\mathcal{J}$ is considerably larger than the union of the well-known classes $\mathcal{S}(\gamma)$.

In this paper, the object of study is the class $\mathcal{J} \cap \mathcal{K}$. The list of relations (1.4) says that $\mathcal{J} \cap \mathcal{K}$ contains $\mathcal{S} \cup \mathcal{D}$. We will not only show that this inclusion is proper but that it is even quite large, thus suggesting that the class $\mathcal{J}$ cannot be simply characterized via other known classes. On the other hand, we will show first that $\mathcal{J} \cap \mathcal{K}$ is (strictly) contained in the class of *strongly heavy-tailed* distributions which are characterized by the property that for all $\lambda > 0$ we have

$$\lim \mathrm{e}^{\lambda x}\overline{F}(x) = \infty. \tag{1.5}$$

This class was denoted by $\mathcal{K}^*$ in [1] and by $\mathcal{DK}^c$ in [7]. It is clearly contained in the class of heavy-tailed distributions but excludes some members of $\mathcal{K}$ with irregular tails. The following result will be proved in Section 2.

THEOREM 1.1. *The following inclusions hold*:

$$\mathcal{J} \cap \mathcal{K} \subset \mathcal{OS} \cap \mathcal{K} \subset \mathcal{DK}^c.$$

Note that the first inclusion is clear from (1.4), so we need only to show the second one. In addition, we will show in Example 2.1 that this inclusion is proper. We remark that Theorem 1.1 shows in particular that condition (iii) $F^I \in \mathcal{J} \cap \mathcal{DK}^c$ in Theorem 19 in [1] can be replaced by the equivalent condition $F^I \in \mathcal{J} \cap \mathcal{K}$.

Before we state our second result, we define the class $\mathcal{DK}_1$ which was introduced by Wang et al. [7] as the set of all distributions which satisfy

$$\lim x^{\delta}\overline{F}(x) = \infty \quad \text{for some } \delta > 0. \tag{1.6}$$

Note that $\mathcal{D} \subset \mathcal{DK}_1 \subset \mathcal{DK}^c$. In addition, we will call two distributions F and G *weakly tail equivalent*, denoted by $\overline{F} \approx \overline{G}$, if

$$0 < \liminf \overline{F}(x)\big(\overline{G}(x)\big)^{-1} \leqslant \limsup \overline{F}(x)\big(\overline{G}(x)\big)^{-1} < \infty.$$

THEOREM 1.2. *The class* $(\mathcal{J} \cap \mathcal{K}) \setminus (\mathcal{L} \cup \mathcal{D})$ *is non-empty. Moreover*, *none of its intersections with* $\mathcal{DK}_1$ *and its complement is empty and each of these two subclasses contains both distributions which are weakly tail equivalent to a distribution in* $\mathcal{S}$ *and distributions which are not.*

We will provide four corresponding examples in Sections 3 and 4. Note that it does not matter whether or not we replace $\mathcal{L}$ by $\mathcal{S}$ in Theorem 1.2 since $\mathcal{S} = \mathcal{J} \cap \mathcal{L}$ by (1.4).

Finally, we investigate the class $\mathcal{J}$ with respect to integrated tail distributions. In Theorem 19 of [1], the integrated tail distribution of the claim size in the Sparre Andersen risk model is required to belong to the class $\mathcal{J} \cap \mathcal{K}$, so the question arises whether there exists a distribution F whose integrated tail distribution $F^I \in \mathcal{J} \cap \mathcal{K} \setminus \mathcal{S}$. Otherwise, if $F^I \in \mathcal{S}$, then the corresponding result is the known Pakes–Veraverbeke–Embrechts theorem, see Theorem 16 of [1].

To answer this question, we recall the concepts of an integrated tail distribution and a generalized long-tailed distribution.

For a distribution F, if $0 < \mu := \int_0^\infty \overline{F}(y)dy < \infty$, then the distribution F^I defined by

$$\overline{F^I}(x) = \mathbf{1}(x < 0) + \mu^{-1} \int\limits_x^\infty \overline{F}(y)dy\mathbf{1}(x \geqslant 0), \quad x \in (-\infty, \infty),$$

is called the *integrated tail distribution* of F.

A distribution F is called *generalized long-tailed*, denoted by $F \in \mathcal{OL}$ (see [6]), if for any $t > 0$

$$C(F, t) := \limsup \overline{F}(x - t)\big(\overline{F}(x)\big)^{-1} < \infty.$$

The inclusion $\mathcal{OS} \subset \mathcal{OL}$ is well known.

The following proposition gives a positive answer to the previous question and has important implications concerning Theorem 19 of Beck et al. [1]. At the same time it provides one of the four examples required to prove Theorem 1.2.

PROPOSITION 1.1. *There exists a distribution F such that $F \in \mathcal{DK}^c \setminus \mathcal{OL}$ and $F \notin \mathcal{DK}_1$, thus $F \notin \mathcal{J}$, but $F^I \in (\mathcal{J} \cap \mathcal{DK}^c) \setminus (\mathcal{L} \cup \mathcal{D})$, $F^I \notin \mathcal{DK}_1$, and F^I is not weakly tail equivalent to a distribution in $\mathcal{S}$.*

We prove Proposition 1.1 in Section 4.

2. PROOF OF THEOREM 1.1

We prove that the second inclusion in the theorem holds. Suppose that $F \in \mathcal{OS} \setminus \mathcal{DK}^c$. Then there exists some $\lambda > 0$ and a sequence of positive numbers $\{x_n, n \geqslant 1\}$ such that $x_n > 2x_{n-1}$ and

$$\overline{F}(x_n) \leqslant \exp\{-\lambda x_n\} \tag{2.1}$$

for every $n \in \mathbb{N}$. Since $F \in \mathcal{OS}$, there exist two constants $2 \leqslant C^*(F) < \infty$ and y_0 large enough such that

$$\overline{F^{*2}}(y) \leqslant 2C^*(F)\overline{F}(y)$$

for all $y \geqslant y_0$. Take any $y \geqslant y_0$; then

$$\overline{F}(2^{-1}y) \leqslant \big(\mathbb{P}(S_2 > y)\big)^{2^{-1}} \leqslant \big(2C^*(F)\overline{F}(y)\big)^{2^{-1}}.$$

Iterating, for any positive integer m, we get

$$\begin{aligned} \overline{F}(2^{-m}y) &\leqslant \big(2C^*(F)\big)^{2^{-1}+\ldots+2^{-m}}\big(\overline{F}(y)\big)^{2^{-m}} \\ &\leqslant 2C^*(F)\big(\overline{F}(y)\big)^{2^{-m}} \end{aligned} \tag{2.2}$$

as long as $2^{-m+1}y \geqslant y_0$. Without loss of generality, we assume that $x_1 \geqslant y_0$. For any $x > x_1$, there exists a positive integer $n := n(x) \geqslant 2$ such that $x_{n-1} < x \leqslant x_n$. Further, there exists a positive integer $m := m(x)$ such that

$$\max\{x_{n-1}, 2^{-m}x_n\} < x \leqslant 2^{-m+1}x_n.$$

If $2^{-m}x_n \geqslant x_{n-1} \geqslant y_0$, then, by (2.2) and (2.1), we have

$$\begin{aligned}\overline{F}(x) &\leqslant \overline{F}(2^{-m}x_n) \leqslant 2C^*(F)\big(\overline{F}(x_n)\big)^{2^{-m}}\\ &\leqslant 2C^*(F)\exp\{-\lambda 2^{-m}x_n\} \leqslant 2C^*(F)\exp\{-2^{-1}\lambda x\};\end{aligned}$$

if $2^{-m}x_n < x_{n-1}$, then, by (2.1), we obtain

$$\overline{F}(x) \leqslant \overline{F}(x_{n-1}) \leqslant \exp\{-\lambda x_{n-1}\} \leqslant \exp\{-2^{-1}\lambda 2^{-m+1}x_n\} \leqslant \exp\{-2^{-1}\lambda x\},$$

so F is light-tailed, and therefore the claim follows. ∎

The following example, which was introduced by C. M. Goldie (see Example 4.1 of [4]), shows that the inclusion $\mathcal{OS} \cap \mathcal{K} \subset \mathcal{DK}^c$ is proper.

EXAMPLE 2.1. Take $x_0 = 0,\ x_n = \sum_{k=0}^{n} k^{k-2},\ n \geqslant 2$. Define a distribution F such that

$$\overline{F}(x) = \mathbf{1}(x < 0) + \sum_{n=1}^{\infty} n^{-n}\mathbf{1}\big(x \in [x_{n-1}, x_n)\big)$$

for $x \in (-\infty, \infty)$. Then $F \in \mathcal{DK}^c$, but $F \notin \mathcal{OS} \cap \mathcal{K}$.

Proof. For n large enough, we have

$$\overline{F}(x_n - 1)\big(\overline{F}(x_n)\big)^{-1} = \overline{F}(2^{-1}x_n)\big(\overline{F}(x_n)\big)^{-1} \geqslant n + 1 \to \infty$$

as $n \to \infty$. Thus $F \notin \mathcal{OL} \supset \mathcal{OS} \cap \mathcal{K}$. But when $x \in [x_{n-1}, x_n)$ and $\delta > 1$, from

$$x^{\delta}\overline{F}(x) \geqslant x_{n-1}^{\delta}\overline{F}(x_n) \geqslant (n-1)^{\delta(n-3)}n^{-n} \to \infty$$

as $n \to \infty$, we obtain $F \in \mathcal{DK}_1 \subset \mathcal{DK}^c$. ∎

3. PROOF OF THEOREM 1.2

The first example shows that $(\mathcal{J} \cap \mathcal{K})\backslash(\mathcal{L} \cup \mathcal{D})$ contains distributions which are not in $\mathcal{DK}_1$ and which are weakly tail equivalent to a distribution in $\mathcal{S}$.

EXAMPLE 3.1. Assume that $F_1 \in \mathcal{S}$ is continuous with all (polynomial) moments finite and let $y_0 \geqslant 0$ and $a > 1$ be two constants such that $a\overline{F_1}(y_0) \leqslant 1$. For example, we can take $\overline{F_1}(x) = \mathbf{1}(x < 0) + e^{-x^{2^{-1}}}\mathbf{1}(x \geqslant 0)$ for $x \in (-\infty, \infty)$ and $y_0 = (\ln a)^2$. Define the distribution F by

$$\text{(3.1)} \qquad \overline{F}(x) = \overline{F_1}(x)\mathbf{1}(x < x_1) + \sum_{i=1}^{\infty} \big(\overline{F_1}(x_i)\mathbf{1}(x_i \leqslant x < y_i) + \overline{F_1}(x)\mathbf{1}(y_i \leqslant x < x_{i+1})\big)$$

for $x \in (-\infty, \infty)$, where $\{x_i, i \geqslant 1\}$ and $\{y_i, i \geqslant 1\}$ are two sequences of positive constants satisfying $x_i < y_i < x_{i+1}$ and $\overline{F_1}(x_i) = a\overline{F}_1(y_i)$, $i \geqslant 1$. Then $F \in (\mathcal{J} \cap \mathcal{DK}^c) \setminus (\mathcal{L} \cup \mathcal{D})$ and $F \notin \mathcal{DK}_1$. Obviously, $\overline{F}(x) \approx \overline{F_1}(x)$.

P r o o f. It can be easily seen that $\overline{F_1}(x) \leqslant \overline{F}(x) \leqslant a\overline{F_1}(x)$. Since $F_1 \in \mathcal{S} = \mathcal{L} \cap \mathcal{J}$ and using the fact that $\mathcal{J}$ is closed under weak tail equivalence (see Proposition 8 of [1]), we obtain $F \in \mathcal{J}$. Next it is easy to verify that $F \in \mathcal{K} \setminus \mathcal{DK}_1$ and hence $F \notin \mathcal{D}$, and since $\lim_{n\to\infty} \big(\overline{F}(y_n)\big)^{-1}\overline{F}(y_n - 1) = a$, we get $F \notin \mathcal{L}$. ∎

Next, we provide an example in $(\mathcal{J} \cap \mathcal{K})\setminus(\mathcal{L} \cup \mathcal{D})$ which is in $\mathcal{DK}_1$ and which is weakly tail equivalent to a distribution in $\mathcal{S}$. To this end, we first construct a distribution F_1 belonging to the class $(\mathcal{S} \cap \mathcal{DK}_1) \setminus \mathcal{D}$ and then show that F defined as in the previous example has the required properties.

EXAMPLE 3.2. Let us choose any constants $\alpha \in (0, 1)$, $\beta \in (\alpha, 2\alpha)$ and $x_1 > 2^{\alpha/(\beta-\alpha)}$. For all integers $n \geqslant 1$, let $x_{n+1} = x_n^{\beta\alpha^{-1}}$. Clearly, $x_{n+1} > 2x_n$ and $x_n \to \infty$ as $n \to \infty$. Now, define the distribution F_1 as follows:

$$\text{(3.2)} \quad \overline{F_1}(x) = \mathbf{1}(x < 0) + \big(x_1^{-1}(x_1^{-\alpha} - 1)x + 1\big)\mathbf{1}(0 \leqslant x < x_1) + \sum_{n=1}^{\infty} \big(x_n^{-\alpha} + (x_n^{-\beta} - x_n^{-\alpha})(x_{n+1} - x_n)^{-1}(x - x_n)\big)\mathbf{1}(x_n \leqslant x < x_{n+1})$$

for $x \in (-\infty, \infty)$. Then it follows that F_1 has an infinite mean and belongs to the class $(\mathcal{S} \cap \mathcal{DK}_1) \setminus \mathcal{D}$.

Further, in this example, let us take $a = 2$ and $y_n = 2^{-1}(x_{n+1} + x_n) + 2^{-1}(x_n^{\beta-\alpha} - 1)^{-1}(x_{n+1} - x_n)$ such that $x_n < y_n < x_{n+1}$ and $2\overline{F_1}(y_n) = \overline{F_1}(x_n)$ for all $n \geqslant 1$. Let F be a distribution such that

$$\text{(3.3)} \qquad \overline{F}(x) = \overline{F_1}(x)\mathbf{1}(x < x_1) + \sum_{n=1}^{\infty} \big(\overline{F_1}(x_n)\mathbf{1}(x_n \leqslant x < y_n) + \overline{F_1}(x)\mathbf{1}(y_n \leqslant x < x_{n+1})\big)$$

for $x \in (-\infty, \infty)$. Then $F \in (\mathcal{J} \cap \mathcal{DK}_1) \setminus (\mathcal{L} \cup \mathcal{D})$ and $\overline{F}(x) \approx \overline{F_1}(x)$.

P r o o f. The distribution F_1 clearly has an infinite mean. It belongs to the class $\mathcal{DK}_1 \setminus \mathcal{D}$ due to the following two facts:

$$\big(\overline{F_1}(x_{n+1})\big)^{-1}\overline{F_1}(2^{-1}x_{n+1}) \geqslant 2^{-1}(1 + x_n^{\beta-\alpha}) \to \infty, \qquad n \to \infty,$$

where we used the elementary inequality $(a-c)/(b-c) \leqslant a/b$ for $b \geqslant a > 0$ and $c \geqslant 0$, and for all $x \in [x_n, x_{n+1})$, $n \geqslant 1$,

$$x\overline{F_1}(x) \geqslant x_n \overline{F_1}(x_n) \sim x_n^{1-\alpha} \to \infty, \quad n \to \infty.$$

Next we prove that $F_1 \in \mathcal{S}$. By

$$\overline{F_1^{*2}}(x) = 2\overline{F_1}(x) - \overline{F_1}^2(2^{-1}x) + 2 \int_{2^{-1}x}^{x} \overline{F_1}(x-y) F_1(dy),$$

and

$$\liminf_{x\to\infty} \big(\overline{F_1}(x)\big)^{-1} \overline{F_1^{*2}}(x) = 2,$$

we need only to prove

$$H(x) := \big(\overline{F_1}(x)\big)^{-1} \int_{2^{-1}x}^{x} \overline{F_1}(x-y) F_1(dy) \to 0, \quad x \to \infty.$$

To this end, we estimate $H(x)$ in the two cases: $x_n \leqslant x < 2x_n$ and $2x_n \leqslant x < x_{n+1}$, $n \geqslant 1$. When $x \in [x_n, 2x_n)$, the relation $2^{-1}x \leqslant y \leqslant x$ implies $2^{-1}x_n \leqslant y \leqslant x_n$, $n \geqslant 1$. Thus, by (3.2), we have

$$\begin{aligned}
(3.4)\quad H(x) &\leqslant \big(\overline{F_1}(2x_n)\big)^{-1} (x_n - x_{n-1})^{-1} x_{n-1}^{-\alpha} \Big(\int_0^{x_{n-1}} + \int_{x_{n-1}}^{x_n} \Big) \overline{F_1}(y) dy \\
&\leqslant \big(\overline{F_1}(2x_n)\big)^{-1} (x_n - x_{n-1})^{-1} x_{n-1}^{-\alpha} \big(x_{n-1} + \overline{F_1}(x_{n-1})(x_n - x_{n-1}) \big) \\
&\sim x_n^{-(1-\alpha)(1-\beta^{-1}\alpha)} + x_n^{-\alpha(2\beta^{-1}\alpha - 1)} \to 0, \quad n \to \infty.
\end{aligned}$$

When $x \in [2x_n, x_{n+1})$, the relation $2^{-1}x \leqslant y \leqslant x$ implies $x_n \leqslant y \leqslant x_{n+1}$, $n \geqslant 1$. Thus, by (3.2), we obtain

$$\begin{aligned}
(3.5)\quad H(x) &\leqslant \big(\overline{F_1}(x_{n+1})\big)^{-1} (x_{n+1} - x_n)^{-1} x_n^{-\alpha} \Big(\int_0^{x_n} + \int_{x_n}^{2^{-1}x_{n+1}} \Big) \overline{F_1}(y) dy \\
&\leqslant \big(\overline{F_1}(x_{n+1})\big)^{-1} (x_{n+1} - x_n)^{-1} x_n^{-\alpha} \big(x_n + \overline{F_1}(x_n)(2^{-1}x_{n+1} - x_n) \big) \\
&\sim x_n^{-(\beta-\alpha)(\alpha^{-1}-1)} + 2^{-1} x_n^{-(2\alpha-\beta)} \to 0, \quad n \to \infty.
\end{aligned}$$

By (3.4) and (3.5), we get $F_1 \in \mathcal{S}$.

Now, we prove that $F \in (\mathcal{J} \cap \mathcal{DK}_1) \setminus (\mathcal{L} \cup \mathcal{D})$. By (3.3) we easily get

$$(3.6)\qquad \overline{F_1}(x) \leqslant \overline{F}(x) \leqslant 2\overline{F_1}(x),$$

that is, $\overline{F}(x) \approx \overline{F_1}(x)$. Then, by (3.6) and $F_1 \in \mathcal{S} \subset \mathcal{J}$, we have $F \in \mathcal{J}$. Next, by $F_1 \in \mathcal{DK}_1 \setminus \mathcal{D}$, we immediately get $F \in \mathcal{DK}_1 \setminus \mathcal{D}$. Finally, $F \notin \mathcal{L}$ follows from

$$\big(\overline{F}(y_n)\big)^{-1} \overline{F}(y_n - 1) = 2$$

for all $n \geqslant 1$. ∎

The next example shows that there exist distributions in $(\mathcal{J} \cap \mathcal{DK}_1) \setminus (\mathcal{L} \cup \mathcal{D})$ which are not weakly tail equivalent to a distribution in $\mathcal{S}$.

EXAMPLE 3.3. Let $m \geqslant 1$ be an integer. Let us choose any constants $\alpha \in (2 + 3m^{-1}, \infty)$ and $x_1 > 4^\alpha$. For all integers $n \geqslant 1$, let $x_{n+1} = x_n^{1+\alpha^{-1}}$. Clearly, $x_{n+1} > 4x_n$ and $x_n \to \infty$ as $n \to \infty$. Now, define the distribution F as follows:

$$\begin{aligned} \overline{F}(x) = {} & \mathbf{1}(x<0) + \big(x_1^{-1}(x_1^{-\alpha}-1)x+1\big)\mathbf{1}(0 \leqslant x < x_1) \\ & + \sum_{n=1}^{\infty} \Big(\big(x_n^{-\alpha} + (x_n^{-\alpha-2} - x_n^{-\alpha-1})(x-x_n)\big)\mathbf{1}(x_n \leqslant x < 2x_n) \\ & + x_n^{-\alpha-1}\mathbf{1}(2x_n \leqslant x < x_{n+1})\Big) \end{aligned} \tag{3.7}$$

for $x \in (-\infty, \infty)$. Further, let $m \in \mathbb{N}$ and

$$\overline{G_m}(x) = \big(\overline{F}(x)\big)^m := \overline{F}^m(x), \qquad x \in (-\infty, \infty).$$

Then $G_m \in (\mathcal{J} \cap \mathcal{DK}_1) \setminus (\mathcal{L} \cup \mathcal{D})$ with finite mean, and G_m is not weakly tail equivalent to a distribution in $\mathcal{S}$.

REMARK 3.1. This example shows that there are many such distributions since m, α and x_1 are arbitrary. Further, based on each of the above distributions and using the method of Example 2.1, we can construct new distributions in the class $\mathcal{DK}_1 \setminus (\mathcal{L} \cup \mathcal{D})$ which are not weakly tail equivalent to a distribution in $\mathcal{S}$.

Proof. According to Proposition 12 b) in [1] and Lemma 3.1 below, we need to prove the above conclusion only for $m = 1$, i.e. $G_1 = F$. By (3.7), it is easy to see that when $x \geqslant x_1$, we have

$$x^{-\alpha-1} \leqslant \overline{F}(x) \leqslant 2^\alpha x^{-\alpha}. \tag{3.8}$$

Thus $F \in \mathcal{DK}_1$. Moreover, using (3.7) and (3.8), we see that the distribution F has a finite mean which we denote by μ. In fact, we get

$$\int_0^\infty y^4 \overline{F}(y)dy < \infty. \tag{3.9}$$

Observe that $F \notin \mathcal{L} \cup \mathcal{D}$ by (3.7) and the following facts hold true:

$$\big(\overline{F}(2x_n)\big)^{-1}\,\overline{F}(2x_n - 1) = 2 - x_n^{-1} \to 2, \qquad n \to \infty,$$

and

$$\overline{F}(2x_n)\big(\overline{F}(x_n)\big)^{-1} = x_n^{-1} \to 0, \qquad n \to \infty.$$

Next we prove that $F \in \mathcal{J}$. For $x > 2K$, we obtain

(3.10) $B(x) := P(X_{2,2} \leqslant K | X_1 + X_2 \geqslant x)$

$$= \left(\frac{1}{2}\overline{F}^2\left(\frac{x}{2}\right) + \int\limits_0^{x/2} \overline{F}(x-y)dF(y)\right)^{-1} \int\limits_0^K \overline{F}(x-y)dF(y)$$

$$= 1 - \left(\frac{1}{2}\overline{F}^2\left(\frac{x}{2}\right) + \int\limits_0^{x/2} \overline{F}(x-y)dF(y)\right)^{-1} \left(\int\limits_K^{x/2} \overline{F}(x-y)dF(y) + \frac{1}{2}\overline{F}^2\left(\frac{x}{2}\right)\right)$$

$$\geqslant 1 - \big(\int\limits_0^{x/2} \overline{F}(x-y)dF(y)\big)^{-1}\big(\int\limits_K^{x/2} \overline{F}(x-y)dF(y)\big)$$

$$- \overline{F}^2\left(\frac{x}{2}\right)\left(2\overline{F}(x)F\left(\frac{x}{2}\right)\right)^{-1}$$

$$:= 1 - B_1(x) - B_2(x).$$

By (3.8), we have for $x \geqslant 2x_1$

$$B_2(x) \leqslant \left(F\left(\frac{x}{2}\right)\right)^{-1} 2^{4\alpha-1}x^{1-\alpha} \to 0. \tag{3.11}$$

Next we estimate $B_1(x)$ in each of the five cases: $x_n \leqslant x < x_n + K$, $x_n + K \leqslant x < 2x_n$, $2x_n \leqslant x < 2x_n + K$, $2x_n + K \leqslant x < 4x_n$ and $4x_n \leqslant x < x_{n+1}$.

When $x \in [x_n, x_n + K)$, it follows that $K \leqslant y \leqslant 2^{-1}x$ implies $2x_{n-1} \leqslant x - y \leqslant x_n, n \geqslant 2$. Thus, by (3.7), we obtain

$$\begin{aligned} B_1(x) &\leqslant \big(\int\limits_0^K \overline{F}(x-y)dF(y)\big)^{-1} \int\limits_K^{x/2} \overline{F}(x-y)dF(y) \\ &\leqslant \big(\overline{F}(x_n+K)F(K)\big)^{-1}\,\overline{F}(K)\overline{F}(2x_{n-1}) \\ &\sim \big(F(K)\big)^{-1}\overline{F}(K) \to 0, \quad K \to \infty. \end{aligned} \tag{3.12}$$

When $x \in [x_n + K, 2x_n), n \geqslant 2$, by (3.7), we have,

$$\begin{aligned} B_1(x) &\leqslant \big(\int\limits_0^K \overline{F}(x-y)dF(y)\big)^{-1}\big(\int\limits_K^{x-x_n} + \int\limits_{x-x_n}^{x/2}\big)\overline{F}(x-y)dF(y) \\ &:= B_{11}(x) + B_{12}(x). \end{aligned}$$

Note that $x_n \leqslant x - y \leqslant 2x_n$ for $K \leqslant y \leqslant x - x_n, n \geqslant 2$, so, by (3.7)–(3.9), we have

$$\begin{aligned} B_{11}(x) &\leqslant \big(\overline{F}(x)F(K)\big)^{-1} \int\limits_K^{x-x_n} \big(\overline{F}(x) + x_n^{-\alpha-1}y\big)dF(y) \\ &\leqslant \big(F(K)\big)^{-1} \int\limits_K^{\infty}(1+y)dF(y) \to 0, \quad K \to \infty. \end{aligned} \tag{3.13}$$

Now we deal with $B_{12}(x)$ in the two cases: $x_n + K \leqslant x < 3 \cdot 2^{-1}x_n$ and $3 \cdot 2^{-1}x_n \leqslant x < 2x_n$. When $x \in [x_n + K, 3 \cdot 2^{-1}x_n)$, we get $2x_{n-1} \leqslant x - y \leqslant x_n$ for $x - x_n \leqslant y \leqslant 2^{-1}x,\ n \geqslant 2$, so, by (3.7), we have

$$\begin{aligned} B_{12}(x) &\leqslant \big(\overline{F}(3 \cdot 2^{-1}x_n)F(K)\big)^{-1}\big(\overline{F}(2x_{n-1})\overline{F}(K)\big) \\ &\sim \big(F(K)\big)^{-1}2\overline{F}(K) \to 0, \quad K \to \infty. \end{aligned} \tag{3.14}$$

When $x \in [3 \cdot 2^{-1}x_n, 2x_n)$, we have $2x_{n-1} \leqslant y \leqslant x_n, n \geqslant 2$, and, by (3.7), we get

$$B_{12}(x) = 0. \tag{3.15}$$

When $x \in [2x_n, 2x_n + K)$, if $K \leqslant y \leqslant x_n$, then $x_n \leqslant 2x_n - y \leqslant 2x_n$; if $x_n \leqslant y \leqslant x_n + 2^{-1}K$, then $x_n - 2^{-1}K \leqslant 2x_n - y \leqslant x_n, n \geqslant 2$. Thus, by (3.7)–(3.9), we have

$$\begin{aligned} B_1(x) &\leqslant \big(\overline{F}(x)F(K)\big)^{-1}\Big(\int_K^{x_n} + \int_{x_n}^{x_n+2^{-1}K}\Big)\overline{F}(2x_n - y)dF(y) \\ &\leqslant \big(F(K)\big)^{-1}\Big(\int_K^{x_n}(1+y)dF(y) + \int_{x_n}^{x_n+2^{-1}K} x_n dF(y)\Big) \\ &\leqslant \big(F(K)\big)^{-1}\Big(\int_K^{\infty}(1+y)dF(y) + \int_K^{\infty} y dF(y)\Big) \to 0, \quad K \to \infty. \end{aligned} \tag{3.16}$$

When $x \in [2x_n + K, 4x_n)$, if $K \leqslant y \leqslant x - 2x_n$, then $2x_n \leqslant x - y \leqslant 4x_n$; if $x - 2x_n \leqslant y \leqslant 2^{-1}x$, then $x_n \leqslant x - y \leqslant 2x_n, n \geqslant 2$. Thus, by (3.7)–(3.9), we have

$$\begin{aligned} B_1(x) &\leqslant \Big(\int_0^K \overline{F}(x-y)dF(y)\Big)^{-1}\Big(\int_K^{x-2x_n} + \int_{x-2x_n}^{2^{-1}x}\Big)\overline{F}(x-y)dF(y) \\ &\leqslant \big(\overline{F}(2x_n)F(K)\big)^{-1}\Big(\overline{F}(2x_n)\overline{F}(K) \\ &\quad + \int_{x-2x_n}^{2^{-1}x}\big(x_n^{-\alpha} + (x_n^{-\alpha-2} - x_n^{-\alpha-1})(x - x_n - y)\big)dF(y)\Big) \\ &\leqslant \big(F(K)\big)^{-1}\Big(\overline{F}(K) + \int_K^{\infty}(1+y)dF(y)\Big) \to 0, \quad K \to \infty. \end{aligned} \tag{3.17}$$

When $x \in [4x_n, x_{n+1})$, if $0 \leqslant y \leqslant 2^{-1}x$, then $2x_n \leqslant x - y \leqslant x_{n+1}, n \geqslant 2$, so, by (3.7), we have

$$B_1(x) \leqslant \big(F(K)\big)^{-1}\overline{F}(K) \to 0, \quad K \to \infty. \tag{3.18}$$

By (3.10)–(3.18), we get $F \in \mathscr{J}$.

Next, we prove that F is not weakly tail equivalent to a distribution in $\mathcal{S}$. To see this, we state the following lemma.

LEMMA 3.1. *Assume that the distribution F satisfies*

$$\limsup_{t\to\infty} C(F,t) = \limsup_{t\to\infty} \limsup \overline{F}(x-t)\big(\overline{F}(x)\big)^{-1} = \infty. \tag{3.19}$$

Then F is not weakly equivalent to any long-tailed distribution.

Proof of Lemma 3.1. We assume there exists a distribution $F_1 \in \mathcal{L}$ and $\overline{F_1}(x) \approx \overline{F}(x)$. Then there are two constants $0 < C_1 \leqslant C_2 < \infty$ such that

$$C_1 = \liminf \big(\overline{F}(x)\big)^{-1}\overline{F_1}(x) \leqslant \limsup \big(\overline{F}(x)\big)^{-1}\overline{F_1}(x) = C_2. \tag{3.20}$$

By $F_1 \in \mathcal{L}$ and (3.20), for any $0 < t < \infty$ we have

$$\big(\overline{F}(x)\big)^{-1}\overline{F}(x-t) \leqslant \big(C_1\overline{F_1}(x)\big)^{-1}C_2\overline{F_1}(x-t) \sim C_1^{-1}C_2. \tag{3.21}$$

Obviously, (3.21) contradicts (3.19). Hence the conclusion of the lemma holds. ■

In Example 3.3, we have

$$\big(\overline{F}(2x_n)\big)^{-1}\overline{F}(2x_n - t) = 1 + t - tx_n^{-1} \to 1 + t, \quad n \to \infty,$$

that is, (3.19) holds. Thus, by Lemma 3.1, F is not weakly tail equivalent to a distribution in $\mathcal{S}$. ■

The remaining statement of Theorem 1.2 will be proved in the next section.

4. PROOF OF PROPOSITION 1.1

Next, we will provide an example proving Proposition 1.1 and showing at the same time the remaining claim of Theorem 1.2 that there exists a distribution in $(\mathcal{J} \cap \mathcal{DK}^c) \setminus (\mathcal{L} \cup \mathcal{D})$ which is not in $\mathcal{DK}_1$ and which is not weakly tail equivalent to a distribution in $\mathcal{S}$.

EXAMPLE 4.1. Define the distribution F as follows:

$$\overline{F}(x) = \mathbf{1}(x<0) + 8^{-1}\mathbf{1}(0 \leqslant x < 4) + \sum_{n=2}^{\infty} \Big(\big(2^{-2^{-1}(n+n^2)} - 2^{-2^{-1}(3n+n^2)}\big)\mathbf{1}(2^n \leqslant x < 2^{n+1})\Big) \quad \text{for } x \in (-\infty,\infty). \tag{4.1}$$

Then $F \in \mathcal{DK}^c \setminus \mathcal{OL}$, thus $F \notin \mathcal{J}$, but $F^I \in (\mathcal{J} \cap \mathcal{DK}^c) \setminus (\mathcal{L} \cup \mathcal{D})$, $F^I \notin \mathcal{DK}_1$, and F^I is not weakly tail equivalent to a distribution in $\mathcal{S}$.

P r o o f. First, it is easy to check that

$$\overline{F}(2^n-1)\big(\overline{F}(2^n)\big)^{-1}\sim 2^n\to\infty,\quad n\to\infty.$$

Consequently, $F\notin\mathcal{OL}$. Since $\mathcal{J}\subset\mathcal{OL}$, we get $F\notin\mathcal{J}$. It is easy to verify that $F\in\mathcal{DK}^c\setminus\mathcal{DK}_1$. In addition, since

$$\int_0^\infty \overline{F}(y)dy=2^{-1}+\sum_{n=2}^{\infty}(2^{2^{-1}(n-n^2)}-2^{-2^{-1}(n+n^2)})=1,$$

the distribution F has a finite mean $\mu=1$. Further, we obtain for $x\in(-\infty,\infty)$

$$\overline{F^I}(x)=\mathbf{1}(x<0)+(1-8^{-1}x)\mathbf{1}(0\leqslant x<4)$$
$$+\sum_{n=2}^{\infty}\big(2^{2^{-1}(n-n^2)}+(2^{-2^{-1}(3n+n^2)}-2^{-2^{-1}(n+n^2)})(x-2^n)\big)\mathbf{1}(2^n\leqslant x<2^{n+1}).\tag{4.2}$$

It is easy to see that $F^I\in\mathcal{DK}^c\setminus\mathcal{DK}_1$, and $F^I\notin\mathcal{L}\cup\mathcal{D}$ by (4.2) and the following facts:

$$\overline{F^I}(2^{n+1}-1)\big(\overline{F^I}(2^{n+1})\big)^{-1}=2+2^{-n}\to 2,\quad n\to\infty,$$

and

$$\overline{F^I}(2^{n+1})\big(\overline{F^I}(2^n)\big)^{-1}=2^{-n}\to 0,\quad n\to\infty.$$

Next we prove that $F^I\in\mathcal{J}$. Let $\{X_i,i\geqslant 1\}$ be a sequence of i.i.d. random variables with common distribution F^I. As before, $X_{n,k}$ denotes the k-th largest random variable in the sequence $\{X_i,1\leqslant i\leqslant n\},1\leqslant k\leqslant n$. For $x>2K>0$ we obtain

$$\begin{aligned}
B(x)&:=\mathbb{P}(X_{2,2}\leqslant K|S_2\geqslant x)\\
&=\Big(2^{-1}\overline{F^I}^2(2^{-1}x)+\int_0^{2^{-1}x}\overline{F^I}(x-y)dF^I(y)\Big)^{-1}\int_0^K\overline{F^I}(x-y)dF^I(y)\\
&=1-\Big(-2^{-1}(n+n^2)\overline{F^I}^2(2^{-1}x)+\int_0^{2^{-1}x}\overline{F^I}(x-y)dF^I(y)\Big)^{-1}\\
&\quad\times\Big(\int_K^{2^{-1}x}\overline{F^I}(x-y)dF^I(y)-2^{-1}(n+n^2)\overline{F^I}^2(2^{-1}x)\Big)\\
&\geqslant 1-\Big(\int_0^{2^{-1}x}\overline{F^I}(x-y)dF^I(y)\Big)^{-1}\Big(\int_K^{2^{-1}x}\overline{F^I}(x-y)dF^I(y)\Big)\\
&\quad-\overline{F^I}^2(2^{-1}x)\big(2\overline{F^I}(x)F^I(2^{-1}x)\big)^{-1}\\
&:=1-B_1(x)-B_2(x).
\end{aligned}\tag{4.3}$$

For all $x \in [2^n, 2^{n+1}), n \geqslant 2$, by (4.2), we have

$$B_2(x) \leqslant \big(2\overline{F^I}(2^{n+1})F^I(2^{n-1})\big)^{-1}\overline{F^I}^2(2^{n-1}) \\ = \big(F^I(2^{n-1})\big)^{-1}2^{2^{-1}(7n-6-n^2)} \to 0, \quad n \to \infty. \tag{4.4}$$

Now, we estimate $B_1(x)$ in the two cases: $2^n \leqslant x < 2^n + K$ and $2^n + K \leqslant x < 2^{n+1}$. When $x \in [2^n, 2^n + K), n \geqslant 2$, by (4.2) and $\int_0^\infty ydF^I(y) < \infty$, we have

$$B_1(x) \leqslant \Big(\int_0^K \overline{F^I}(x-y)dF^I(y)\Big)^{-1}\int_K^{x/2}\overline{F^I}(x-y)dF^I(y) \tag{4.5}$$
$$\leqslant \big(\overline{F^I}(x)F^I(K)\big)^{-1}\int_K^{x/2}\big(\overline{F^I}(x-K)+2^{(n-n^2)/2}y\big)dF^I(y)$$
$$\leqslant \big(\overline{F^I}(2^n+K)F^I(K)\big)^{-1}\Big(\overline{F^I}(2^n-K)\overline{F^I}(K)+\int_K^{x/2}2^{(n-n^2)/2}ydF^I(y)\Big)$$
$$= O\Big(\overline{F^I}(K)+\int_K^{+\infty}ydF^I(y)\Big) \to 0, \quad K \to \infty.$$

When $x \in [2^n + K, 2^{n+1}), n \geqslant 2$, by (4.2), we have

$$B_1(x) \leqslant \Big(\int_0^{x-2^n}\overline{F^I}(x-y)dF^I(y)\Big)^{-1}\Big(\int_K^{x-2^n}+\int_{x-2^n}^{x/2}\Big)\overline{F^I}(x-y)dF^I(y) \\ := B_{11}(x)+B_{12}(x). \tag{4.6}$$

By (4.2), (4.6) and $\int_0^\infty ydF^I(y) < \infty$, we have

$$B_{11}(x) = \Big(\int_0^{x-2^n}\overline{F^I}(x-y)dF^I(y)\Big)^{-1}\Big(\int_K^{x-2^n}\overline{F^I}(x-y)dF^I(y)\Big) \tag{4.7}$$
$$\leqslant \big(\overline{F^I}(x)F^I(x-2^n)\big)^{-1}\Big(\int_K^{x-2^n}\big(\overline{F^I}(x)+2^{(-n-n^2)/2}y\big)dF^I(y)\Big)$$
$$\leqslant \big(F^I(K)\big)^{-1}\Big(\int_K^{x-2^n}(1+y)\,dF^I(y)\Big) \to 0, \quad K \to \infty.$$

Now we deal with $B_{12}(x)$ in the two cases: $2^n + K \leqslant x < 3 \cdot 2^{n-1}$ and $3 \cdot 2^{n-1} \leqslant x < 2^{n+1}$. When $x \in [2^n + K, 3 \cdot 2^{n-1}), n \geqslant 2$, by (4.2), (4.6) and

$\int_0^\infty y dF^I(y) < \infty$, we have

$$(4.8)\quad B_{12}(x) \leqslant \left(\overline{F^I}(x)F^I(x-2^n)\right)^{-1}$$
$$\times \Big(\int\limits_{x-2^n}^{x/2} 2^{(3n-2-n^2)/2} + (2^{(2-n-n^2)/2} - 2^{(n-n^2)/2})(x - 2^{n-1} - y) dF^I(y)\Big)$$
$$\leqslant \left(\overline{F^I}(3 \cdot 2^{n-1})F^I(K)\right)^{-1}\Big(\int\limits_K^{x/2} 2^{(n-n^2)/2} dF^I(y) + \int\limits_K^{x/2} 2^{(n-n^2)/2} y dF^I(y)\Big)$$
$$\leqslant \left(F^I(K)\right)^{-1}\Big(2\overline{F^I}(K) + 2\int\limits_K^\infty y dF^I(y)\Big) \to 0, \quad K \to \infty.$$

When $x \in [3 \cdot 2^{n-1}, 2^{n+1}), n \geqslant 2$, by (4.2) and (4.6), we have

$$(4.9)\quad B_{12}(x) \leqslant \left(\overline{F^I}(x)F^I(x-2^n)\right)^{-1}\overline{F^I}(2^{-1}x)\overline{F^I}(x-2^n)$$
$$\leqslant \left(\overline{F^I}(2^{n+1})F^I(2^{n-1})\right)^{-1}\left(\overline{F^I}(2^{n-1})\right)^2 \to 0, \quad n \to \infty.$$

Therefore, by (4.3)–(4.9), we get $F^I \in \mathcal{J}$.

Finally, by Lemma 3.1 and

$$\left(\overline{F^I}(2^{n+1})\right)^{-1}\overline{F^I}(2^{n+1} - t) = 1 + t - 2^{-n}t \sim 1 + t, \quad n \to \infty,$$

we see that F^I is not weakly tail equivalent to a distribution in $\mathcal{S}$. ■

REMARK 4.1. Observe that the example shows that $F^I \in \mathcal{J}$ does not imply that $F \in \mathcal{J}$. Conversely, $F \in \mathcal{J}$ does not imply $F^I \in \mathcal{J}$ either even if we assume that F has a finite first moment (otherwise, F^I is not defined). As an example, we can take the example in Section 3.8 in [3] for which $F \in \mathcal{S}$ and $F^I \notin \mathcal{S}$. Since $\mathcal{S} \subset \mathcal{J}$, we have $F \in \mathcal{J}$. Further, $\mathcal{S} \subset \mathcal{L}$ and $F \in \mathcal{L}$ imply $F^I \in \mathcal{L}$ by Lemma 2.26 in [3]. Since $\mathcal{J} \cap \mathcal{L} = \mathcal{S}$, we obtain $F^I \notin \mathcal{J}$.

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