

ON EXTREMAL INDEX OF MAX-STABLE STATIONARY PROCESSES

BY

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*To Tomasz Rolski,
thankful for all the support and ideas you shared with us!*

Abstract. In this contribution we discuss the relation between Pickands-type constants defined for certain Brown–Resnick stationary process $W(t)$, $t \in \mathbb{R}$, as

$$\mathcal{H}_W^\delta = \lim_{T \rightarrow \infty} T^{-1} \mathbb{E} \left\{ \sup_{t \in \delta \mathbb{Z} \cap [0, T]} e^{W(t)} \right\}, \quad \delta \geq 0$$

(set $0\mathbb{Z} = \mathbb{R}$ if $\delta = 0$) and the extremal index of the associated max-stable stationary process ξ_W . We derive several new formulas and obtain lower bounds for \mathcal{H}_W^δ if W is a Gaussian or a Lévy process. As a by-product we show an interesting relation between Pickands constants and lower tail probabilities for fractional Brownian motions.

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1. INTRODUCTION

The motivation for this contribution comes from the importance and the intriguing properties of the classical Pickands constants \mathcal{H}_W^δ which are defined for any $\delta \geq 0$ by (interpret $0\mathbb{Z}$ as \mathbb{R})

$$(1.1) \quad \mathcal{H}_W^\delta = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left\{ \sup_{t \in \delta \mathbb{Z} \cap [0, T]} e^{W(t)} \right\},$$

where

$$W(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha, \quad t \in \mathbb{R},$$

with B_α a standard fractional Brownian motion with Hurst index $\alpha \in (0, 2]$, that is a centered Gaussian process with stationary increments and variance function $\text{Var}(B_\alpha(t)) = |t|^\alpha, t \in \mathbb{R}$.

It is well known (but not trivial to prove) that \mathcal{H}_W^δ is finite and positive for any $\delta \geq 0$. The only values known for \mathcal{H}_W^δ are for $\delta = 0$ and $\alpha \in \{1, 2\}$, see, e.g., [41], [42]. Surprisingly, Pickands and related constants appear in numerous unrelated asymptotic problems, see, e.g., the recent papers [17], [25], [26], [15].

The contribution [19] derived a new formula for Pickands constants, which in fact indicates a direct connection between those constants and max-stable stationary processes, see [12]. The definition of \mathcal{H}_W^δ in (1.1) is extended in [12] for some general process W provided that it defines a max-stable and stationary process. More precisely, assume throughout in the sequel that

$$(1.2) \quad W(t) = B(t) - \ln \mathbb{E}\{e^{B(t)}\}, \quad t \in \mathbb{R},$$

where $B(t), t \in \mathbb{R}$, is a random process on the space D of càdlàg functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with

$$(1.3) \quad B(0) = 0, \quad \mathbb{E}\{e^{B(t)}\} < \infty, \quad t \in \mathbb{R}.$$

Hence $X(t) = e^{W(t)}$ satisfies $X(0) = 1$ almost surely, and $\mathbb{E}\{X(t)\} = 1, t \in \mathbb{R}$. If $\Pi = \sum_{i=1}^{\infty} \varepsilon_{P_i}$ is a Poisson point process (PPP) with intensity $x^{-2}dx$ on $(0, \infty)$, and $X_i = e^{W_i}, i \geq 1$, are independent copies of the random process $X = e^W$ being independent of Π , then the random process ξ_W defined by

$$(1.4) \quad \xi_W(t) = \max_{i \geq 1} P_i X_i(t) = \max_{i \geq 1} P_i e^{W_i(t)}, \quad t \in \mathbb{R},$$

has unit Fréchet marginals and is max-stable. Here ε_x denotes the unit Dirac measure at $x \in \mathbb{R}$. Adopting the definition from [30], we shall refer to W as the *Brown–Resnick stationary process* whenever the associated max-stable process ξ_W is stationary. Note that stationarity of ξ_W means that $\{\xi_W(t), t \in \mathbb{R}\}$ and $\{\xi_W(t+h), t \in \mathbb{R}\}$ have the same distribution for any $h \in \mathbb{R}$.

In the sequel, for the case $\delta = 0$ we shall assume that

$$\mathbb{E}\{\sup_{t \in K} e^{W(t)}\} < \infty$$

for any compact $K \subset \mathbb{R}$. A direct consequence of stationarity of ξ_W and the fact that for any $t_1, \dots, t_n \in \mathbb{R}$ and $x_1, \dots, x_n > 0$ (see, e.g., [18], [39])

$$(1.5) \quad \mathbb{P}\{\xi_W(t_i) \leq x_i, \forall i \leq n\} = \exp\left(-\mathbb{E}\left\{\max_{1 \leq i \leq n} (e^{W(t_i)}/x_i)\right\}\right)$$

is that, for any $b \geq 0, \delta \geq 0, T > 0$, we have

$$H_W^\delta([0, T]) := \mathbb{E}\left\{\sup_{t \in \delta\mathbb{Z} \cap [0, T]} e^{W(t)}\right\} = \mathbb{E}\left\{\sup_{t \in \delta\mathbb{Z} \cap [b, b+T]} e^{W(t)}\right\}.$$

Consequently, \mathcal{H}_W^δ defined in (1.1) exists and is given by (see [12])

$$(1.6) \quad \mathcal{H}_W^\delta = \inf_{T>0} \frac{1}{T} H_W^\delta([0, T]) \in [0, \infty).$$

Note that if $\delta > 0$, then (1.6) implies that

$$\mathcal{H}_W^\delta \leq \frac{H_W^\delta([0, \delta - \varepsilon])}{\delta - \varepsilon} = \frac{1}{\delta - \varepsilon}$$

for any $\varepsilon \in (0, \delta)$, hence letting ε tend to zero yields $\mathcal{H}_W^\delta \in [0, 1/\delta]$.

Interestingly, \mathcal{H}_W^δ is related to the extremal index of the stationary process

$$\xi_W^\delta(t) = \xi_W(\delta t), \quad t \in \mathbb{Z}, \delta > 0,$$

where we set $\xi_W^\delta(t) = \xi_W(t)$ if $\delta = 0$. Indeed, by (1.5),

$$(1.7) \quad \begin{aligned} \lim_{T \rightarrow \infty} \mathbb{P}\{ \max_{i \in \delta\mathbb{Z} \cap [0, T]} \xi_W(t) \leq Tx \} \\ &= \exp\left(- \lim_{T \rightarrow \infty} \mathbb{E}\{ \max_{i \in \delta\mathbb{Z} \cap [0, T]} (e^{W(i)}/T) \} x^{-1}\right) \\ &= (e^{-1/x})^{\mathcal{H}_W^\delta}, \quad x > 0. \end{aligned}$$

Thus the Fréchet limit result in (1.7), which is already shown in [50] (see also Proposition 3.1 in [10] and [18]), states that the extremal index of the stationary process $\xi_W^\delta(t), t \in \mathbb{Z}$, is given for any $\delta > 0$ by

$$(1.8) \quad \theta_W^\delta = \delta \mathcal{H}_W^\delta \in [0, 1].$$

Clearly, the constant \mathcal{H}_W^δ is positive if and only if the extremal index θ_W^δ of the stationary process ξ_W^δ is positive.

Numerous papers in the literature have discussed the calculation and estimation of extremal index of stationary processes, see, e.g., the recent articles [46], [10], [38], [35], [33], [21] and the references therein. The primary goal of this contribution is to study Pickands-type constants \mathcal{H}_W^δ by exploring the properties of the extremal index θ_W^δ . In particular, we are interested in establishing tractable conditions that guarantee the positivity of \mathcal{H}_W^δ .

By our assumptions it is clear that ξ_W^δ is stationary and *jointly regularly varying*, hence in view of Theorem 2.1 in [5] (see also [29]), there exists a so-called *tail process*

$$Y^\delta(i), \quad i \in \mathbb{Z},$$

of the stationary process X , which was introduced in [5]. It turns out that for any $m \leq n, m, n \in \mathbb{Z}$, we have the stochastic representation

$$(1.9) \quad (Y^\delta(m), \dots, Y^\delta(n)) \stackrel{d}{=} (\mathcal{P}X^\delta(m), \dots, \mathcal{P}X^\delta(n)),$$

where $X^\delta(i) := e^{W(\delta i)}$, $i \in \mathbb{Z}$, with \mathcal{P} a unit Pareto random variable with survival function $1/x$, $x > 1$, being independent of the process X .

Under the *finite mean cluster size condition* (see Condition 2.1 below) and condition $\mathcal{A}(a_n)$ (see [5], [4], [32]) it follows that θ_W^δ is positive, see the seminal contribution [5].

We shall show the positivity of the extremal index under a weaker condition, namely supposing that

$$(1.10) \quad \lim_{|z| \rightarrow \infty, z \in \mathbb{Z}} W(z\delta) = -\infty$$

holds almost surely for $\delta \in (0, \infty)$. In our derivations the following simple result is crucial:

LEMMA 1.1. *If $r_n, n \geq 1$, are positive integers such that*

$$\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} n/r_n = \infty,$$

then for any $\delta \in (0, \infty)$ we have

$$(1.11) \quad \widetilde{\theta}_W^\delta := \lim_{n \rightarrow \infty} \frac{n}{r_n} \mathbb{P}\left\{ \max_{i \in \{0, \delta, \dots, \delta r_n\}} \xi_W(t) > n \right\} = \theta_W^\delta \in [0, \infty).$$

In the next section we shall show that the new expression for the extremal index in (1.11) is positive under (1.10). Using the explicit form of the *tail process*, we shall derive several new interesting formulas for \mathcal{H}_W^δ .

Brief outline of the rest of the paper is the following. In Section 2 we give our main results which establish the positivity of the Pickands-type constants and some new formulas. In Section 3 we shall discuss the connection with mixed moving maxima (M3) representation of Brown–Resnick processes. Then we derive some explicit lower bounds for \mathcal{H}_W^δ in case that B in (1.2) is a Gaussian or a Lévy process, and discuss the relation between \mathcal{H}_W^δ and the *mean cluster index*. Further, we shall show that the classical Pickands constants are related to a small ball problem. All the proofs are relegated to Section 4.

2. MAIN RESULTS

We keep the same setup as in the Introduction, and denote additionally by \mathcal{E} a unit exponential random variable which is independent of everything else. According to [5] a candidate for the extremal index is given by the formula

$$(2.1) \quad \widehat{\theta}_W^\delta = \lim_{m \rightarrow \infty} \mathbb{P}\left\{ \max_{1 \leq i \leq m} Y^\delta(i) \leq 1 \right\},$$

where $Y^\delta(i)$, $i \in \mathbb{Z}$, is the *tail process* of ξ_W^δ , see [5]. As in the aforementioned paper we shall impose the *finite mean cluster size condition* of [5], Condition 4.1:

CONDITION 2.1. Given $\delta > 0$, there exists a sequence of positive integers $r_n, n \in \mathbb{N}$, satisfying $\lim_{n \rightarrow \infty} r_n/n = 1/\lim_{n \rightarrow \infty} r_n = 0$ such that

$$(2.2) \quad \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left\{ \max_{m \leq |k| \leq r_n} \xi_W(k\delta) > nx \mid \xi_W(0) > nx \right\} = 0$$

holds for any $x > 0$.

By Proposition 4.2 in [5] we infer that $\widehat{\theta}_W^\delta > 0$ follows from Condition 2.1. Our main result below establishes new formulas for \mathcal{H}_W^δ . Moreover, from the above-mentioned reference, Condition 2.1 together with well-known $\mathcal{A}(a_n)$ conditions of Hsing and Davis implies that the candidate for the extremal index is equal to the extremal index, i.e., $\widehat{\theta}_W^\delta = \theta_W^\delta > 0$. It is well known that $\mathcal{A}(a_n)$ is implied by the strong mixing of ξ_W^δ . However, our results derived below do not require strong mixing but just mixing of ξ_W^δ .

THEOREM 2.1. Let $X(t) = e^{W(t)}, t \in \mathbb{R}$, with W as in (1.2) be such that (1.3) holds and $\xi_W(t), t \in \mathbb{R}$, is max-stable and stationary. Then (1.10) holds for $\delta > 0$ if and only if Condition 2.1 holds. Moreover, if (1.10) holds for $\delta > 0$, then we have consecutively

$$(2.3) \quad \mathcal{H}_W^\delta = \frac{1}{\delta} \mathbb{P}\left\{ \sup_{i < 0} W^\delta(i) < 0 = \sup_{i \in \mathbb{Z}} W^\delta(i) \right\},$$

$$(2.4) \quad \mathcal{H}_W^\delta = \frac{1}{\delta} \mathbb{P}\left\{ \sup_{i \geq 1} (\mathcal{E} + W^\delta(i)) \leq 0 \right\},$$

$$(2.5) \quad \mathcal{H}_W^\delta = \frac{1}{\delta} [\mathbb{E}\{\sup_{i \geq 0} e^{W^\delta(i)}\} - \mathbb{E}\{\sup_{i \geq 1} e^{W^\delta(i)}\}] \in (0, 1/\delta),$$

where $W^\delta(t) = W(t\delta), t \in \mathbb{Z}$, and \mathcal{E} is a unit exponential random variable independent of W .

REMARK 2.1. (a) If $\mathbb{P}\{W^\delta(i) = 0\} = 0$ for any negative integer i , then

$$\mathbb{P}\left\{ \sup_{-m \leq i < 0} W^\delta(i) < 0 = \sup_{-m \leq j \leq m} W^\delta(j) \right\} = \mathbb{P}\left\{ \sup_{-m \leq j \leq m} W^\delta(j) = 0 \right\}$$

holds for any integer $m > 1$. Consequently, by (2.3) we have

$$(2.6) \quad \begin{aligned} \mathcal{H}_W^\delta &= \frac{1}{\delta} \lim_{m \rightarrow \infty} \mathbb{P}\left\{ \sup_{-m \leq i < 0} W^\delta(i) < 0 = \sup_{-m \leq j \leq m} W^\delta(j) \right\} \\ &= \frac{1}{\delta} \mathbb{P}\left\{ \sup_{i \in \mathbb{Z}} W^\delta(i) = 0 \right\} > 0, \end{aligned}$$

which has been shown in [19] for the case where B is a standard fractional Brownian motion. The assumption $W(0) = 0$ can be removed (see [27]).

(b) We assumed above that ξ_W has càdlàg sample paths in order to define H_W^0 . For the results of Theorem 2.1, this assumption is not needed.

(c) In [12] it is shown that under the assumptions of Theorem 2.1 we have

$$(2.7) \quad \mathcal{H}_W^\delta = \mathbb{E} \left\{ \frac{\sup_{t \in \delta\mathbb{Z}} e^{W(t)}}{\delta \sum_{t \in \delta\mathbb{Z}} e^{W(t)}} \right\}.$$

According to (2.5), for calculation of \mathcal{H}_W^δ it suffices to know $W(t), t \in \delta\mathbb{Z}, t > 0$, i.e., only the values of W for positive t matter. This is not the case for formula (2.7). Both (2.7) and (2.5) are given in terms of expectations and not as limits, which is a great advantage for simulations. To this end, we mention that simulation of Pickands constants has been the topic of many works (see, e.g., [9], [36], [19]).

(d) If $X(t) = e^{W(t)}, t \in \mathbb{R}$, is Brown–Resnick stationary, i.e., the associated max-stable process with ζ_W is max-stable and stationary, then the time reversed process $V(t) = W(-t), t \in \mathbb{R}$, also defines Brown–Resnick stationary processes. Moreover, for any $\delta \geq 0$

$$\mathcal{H}_W^\delta = \mathcal{H}_V^\delta.$$

Consequently, the formulas in Theorem 2.1 can be stated with V instead of W , for instance we have

$$(2.8) \quad \begin{aligned} \mathcal{H}_W^\delta &= \frac{1}{\delta} \mathbb{P} \left\{ \sup_{i \leq -1} (\mathcal{E} + W^\delta(i)) \leq 0 \right\} \\ &= \frac{1}{\delta} \mathbb{P} \{ W^\delta(i) < 0, i \in \mathbb{N}; W^\delta(i) \leq 0, i \in \mathbb{Z} \}. \end{aligned}$$

(e) If $W(t) = \sqrt{2}tL - t^2$ with L an $N(0, 1)$ random variable with distribution Φ and probability density function φ , by (2.4) we have

$$(2.9) \quad \begin{aligned} \mathcal{H}_W^\delta &= \frac{1}{\delta} \int_0^\infty \mathbb{P} \left\{ \mathcal{E} + \sup_{i \geq 1} (\sqrt{2}\delta ib - (\delta i)^2) \leq 0 \right\} \varphi(b) db \\ &= \frac{1}{\delta} \int_{-\delta/\sqrt{2}}^{\delta/\sqrt{2}} \varphi(b) db = \frac{1}{\delta} [\Phi(\delta/\sqrt{2}) - \Phi(-\delta/\sqrt{2})] \end{aligned}$$

for any $\delta > 0$. Consequently, letting $\delta \rightarrow 0$ we obtain the well-known result

$$\mathcal{H}_W^0 = \sqrt{2}\varphi(0) = \frac{1}{\sqrt{\pi}}.$$

A canonical example for W with representation (1.2) is the case when B is a centered Gaussian process with stationary increments, continuous sample paths, and variance function σ^2 . Then the max-stable process ξ_W is stationary, see [40]. Using a direct argument, we establish in the next theorem the positivity of \mathcal{H}_W^0 .

THEOREM 2.2. *If*

$$\liminf_{t \rightarrow \infty} \frac{\sigma^2(t)}{\ln t} > 8,$$

then $\mathcal{H}_W^0 > 0$.

Since (2.2) implies (1.10) (see Corollary 2.4 in [34] or [30]), using $\mathcal{H}_W^0 \geq \mathcal{H}_W^\delta$ for any $\delta > 0$ we immediately establish the positivity of \mathcal{H}_W^0 .

Indeed, the positivity of \mathcal{H}_W^0 is crucial for the study of extremes of Gaussian processes. Condition (2.2) can be easily checked, e.g., if $W(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha$. Consequently, the classical Pickands constants \mathcal{H}_W^δ are positive for any $\delta \geq 0$. This fact is highly non-trivial; after announced in Pickands' pioneering work [41], correct proofs were obtained later by Pickands himself, and in [7], [43], see, e.g., Theorem B3 in [8]. We note in passing that under general conditions on σ^2 the positivity of \mathcal{H}_W^0 is established in [11].

Apart from the alternative proof for the positiveness of the original Pickands constants, Theorem 2.1 extends to non-Gaussian processes W . For the above Gaussian setup, direct calculations show the positivity of \mathcal{H}_W^δ under a slightly weaker condition than (2.2).

3. DISCUSSIONS AND EXTENSIONS

3.1. Relation with lower tail probabilities. For the classical case of Piterbarg constants \mathcal{H}_{B_α} , i.e., for $W(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha, t \in \mathbb{R}, \alpha \in (0, 2]$, we show below that (2.6) implies a nice relation with a small ball problem.

PROPOSITION 3.1. *For any $\alpha \in (0, 2]$ we have*

$$\lim_{\eta \rightarrow 0} \eta^{-2/\alpha} \mathbb{P}\{\forall_{k \in \mathbb{Z} \setminus \{0\}} B_\alpha(1/k) \leq \eta\} = 2^{1/\alpha} \mathcal{H}_{B_\alpha}.$$

The above result strongly relates to the self-similarity property of fractional Brownian motion. In case of a general Gaussian W , ξ_W is still stationary if W has stationary increments. However, fBm is the only centered Gaussian process with stationary increments being further self-similar. Hence, no obvious extensions of the above relation with lower tails can be derived for general W .

3.2. Non-Gaussian W . The classical Pickands constants are defined for $W(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha$ with B_α a standard fBm with Hurst index $\alpha/2 \in (0, 1]$. The more general case where B_α is substituted by a centered Gaussian process with stationary increments is discussed in detail in [11].

Our setup clearly allows for any random process W , not necessarily Gaussian, which is Brown–Resnick stationary. Along with the Gaussian case of W , the Lévy one has also been dealt already in the literature. In view of [23], [49], if $B(t), t \geq 0$,

is a Lévy process such that

$$(3.1) \quad \Phi(1) < \infty, \quad \Phi(\theta) := \ln \mathbb{E}\{e^{\theta B(1)}\},$$

then $W(t) = B(t) - \Phi(1)t$, $t \geq 0$, is Brown–Resnick stationary, i.e., $\xi_W(t)$, $t \geq 0$, is max-stable stationary with unit Gumbel marginals.

In [31] an important constant appears in the asymptotic analysis of the maximum of standardised increments of random walks, which in fact is the Pickands constant \mathcal{H}_W^δ , $\delta > 0$, introduced here for W as above. In [31], Lemma 5.16, a new formula for \mathcal{H}_W^δ is derived, which is identical with our formula in (2.8). Another instance of the Pickands constant given by formula (2.3) is displayed in [44], Theorem 5.3. With the notation of that theorem, for $\delta = 1$ we have

$$W(i) = \sum_{j=1}^i A_i,$$

where A_i 's are i.i.d. with the same distribution as $Z\mathbb{1}(U \leq e^{-\eta Z})$ for some $\eta > 0$ with U uniformly distributed on $(0, 1)$ being independent of Z which has some pdf symmetric around zero.

Pickands constants appear also in the context of semi-min-stable processes, see [51]. In view of the aforementioned paper, several results derived here for max-stable processes are extendable to semi-min-stable processes.

3.3. Finite mean cluster size condition. As noted in [45], Condition 2.1 is implied by the following so-called *short-lasting exceedance condition*:

CONDITION 3.1. *Given $\delta > 0$, there exists a sequence of integers r_n , $n \in \mathbb{N}$, satisfying $\lim_{n \rightarrow \infty} r_n/n = 1/\lim_{n \rightarrow \infty} r_n = 0$ such that*

$$(3.2) \quad \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{k=m}^{r_n} \mathbb{P}\{\xi_W(k\delta) > nx \mid \xi_W(0) > nx\} = 0$$

is valid for any $x > 0$.

This latter condition is a rephrasing of the so-called B condition (see, e.g., [1], [13], [2]) which was formulated by discretising the original Berman's condition, see [6]. Condition 3.1 is weaker than the $D'(xn)$ condition of Leadbetter as discussed in [22], Section 5.3.2.

Commonly, Condition 2.1 assumed for $x = 1$ is referred to as the *anti-clustering condition*, see, e.g., [46], [47]. Clearly, the finite mean cluster size condition is stronger than the anti-clustering condition. The latter appears in various contexts related to extremes of stationary processes, see, e.g., [3], [37], [46], [5], [47] and the references therein.

3.4. M3 representation. Since we assume that ξ_W is max-stable stationary with càdlàg sample paths, and W with representation (1.2) is such that B satisfies (1.3), it follows that assuming the almost sure convergence

$$W(t) \rightarrow -\infty$$

as $|t| \rightarrow \infty$ is equivalent to the fact that ξ_W has a *mixed moving maxima representation* (for short, M3), see [20], Theorem 3, and [52]. More specifically, under (3.4) we have the equality of finite-dimensional distributions

$$(3.3) \quad \xi_W(t) \stackrel{d}{=} \max_{i \geq 1} P_i e^{F_i(t-T_i)}, \quad t \in \mathbb{R},$$

between the right-hand side and the left-hand side in (3.3), where the F_i 's are independent copies of a measurable càdlàg process $F_W(t), t \in \mathbb{R}$, satisfying almost surely

$$(3.4) \quad \sup_{t \in \mathbb{R}} F_W(t) = F_W(0) = 0,$$

and $\sum_{i=1}^{\infty} \varepsilon_{(P_i, T_i)}$ is a PPP in $(0, \infty) \times \mathbb{R}$ with intensity $C_W \cdot p^{-2} dp \cdot dt$ with

$$(3.5) \quad C_W = \left(\mathbb{E} \left\{ \int_{\mathbb{R}} e^{F_W(t)} dt \right\} \right)^{-1} \in (0, \infty).$$

Moreover, ξ_W^δ , the restriction of ξ_W on $\delta\mathbb{Z}$, has an M3 representation for any $\delta > 0$, see [12] for more details. Denote the corresponding constant in the intensity of this PPP by $C_W^\delta > 0$ (and thus C_W^0 is just C_W given in (3.5)).

In view of [12], Proposition 1, if $\xi_W^\delta, \delta > 0$, admits an M3 representation as mentioned above, then

$$(3.6) \quad \mathcal{H}_W^\delta = C_W^\delta,$$

provided that (1.10) holds. Hence Theorem 2.1 presents new formulas for C_W^δ . Note in passing that (3.6) has been shown in [40]. Therein it is proved that C_W^δ is given by the right-hand side of (2.6) assuming further that $W(t) = B(t) - \mathbb{E}\{e^{\ln B(t)}\}, t \in \mathbb{R}$, with B a centered Gaussian process with stationary increments satisfying $W(0) = 0$ almost surely.

In view of [12], Theorem 1, if (1.10) holds, then we have

$$(3.7) \quad \mathcal{H}_W^\delta = \mathbb{E} \left\{ \frac{M^\delta}{S^\delta} \right\} = C_W^\delta,$$

with $M^\delta := \max_{i \in \mathbb{Z}} e^{W(i\delta)}$ and $S^\delta := \delta \sum_{t \in \delta\mathbb{Z}} e^{W(t)}$. Thus $\mathcal{H}_W^\delta > 0$.

The representation of \mathcal{H}_W^δ as an expectation of the ratio M^δ/S^δ is crucial for its simulation. Such a representation has initially been shown in [19] for classical Pickands constants.

3.5. Lower bounds. In Theorem 2.1 we present new formulas for \mathcal{H}_W^δ , which in turn establish the positivity of \mathcal{H}_W^δ and thus the positivity for the extremal index of ξ_W^δ . If only the positivity of \mathcal{H}_W^δ is of primary interest, then the conditions of Theorem 2.1 can be relaxed. Next, we consider two important classes of processes for B , that is, centered Gaussian processes with stationary increments and Lévy processes. Results for the Lévy case have already been given in [12].

For particular values of δ , we show that it is possible to derive a positive lower bound for \mathcal{H}_W^δ and thus establishing the positivity of \mathcal{H}_W^δ . Let $x_+ := \max(x, 0)$.

THEOREM 3.1. (i) *Let $W(t) = B(t) - \sigma^2(t)/2$, $t \geq 0$, where $B(t)$ is a centered Gaussian process with stationary increments and variance function σ^2 such that $\sigma(0) = 0$. Then for any $\delta > 0$*

$$(3.8) \quad \mathcal{H}_W^\delta \geq \frac{1}{\delta} \max \left(0, 1 - \sum_{k=1}^{\infty} \exp \left(-\frac{\sigma^2(\delta k)}{8} \right) \right).$$

(ii) *Let $W(t) = B(t) - \Phi(1)t$, $t \geq 0$, where $B(t)$ is a Lévy process satisfying the condition (3.1). Then for any $\delta > 0$*

$$(3.9) \quad \mathcal{H}_W^\delta \geq \frac{1}{\delta} \frac{\max(0, 1 - 2 \exp((\Phi(1/2) - \frac{1}{2}\Phi(1))\delta))}{1 - \exp((\Phi(1/2) - \frac{1}{2}\Phi(1))\delta)}.$$

REMARK 3.1. (a) *It follows from (i) of Theorem 3.1 that if $\sigma(\delta k) \geq C(\delta k)^{\kappa/2}$ for all $k \in \mathbb{N}$ and some $\kappa > 0$, then*

$$(3.10) \quad \mathcal{H}_W^\delta \geq \frac{1}{\delta} \left(1 - \frac{1}{\delta} \frac{\Gamma(1/\kappa)}{\kappa (C^2/8)^{1/\kappa}} \right).$$

Since $\mathcal{H}_W^0 \geq \mathcal{H}_W^\delta$ for any $\delta > 0$, the above implies $\mathcal{H}_W^0 > 0$.

(b) *If B is a Lévy process as in (ii) of Theorem 3.1, then (see the proof in Section 4)*

$$(3.11) \quad \mathcal{H}_W^0 \geq \frac{1}{8} [\Phi(1) - 2\Phi(1/2)] > 0.$$

3.6. Case $\delta = 0$. Since (1.7) holds also for $\delta = 0$ and $\mathcal{H}_W^0 \geq \mathcal{H}_W^\delta$, it follows that the extremal index of the continuous process ξ_W is

$$\tilde{\theta}_W = \mathcal{H}_W^0 \geq 0,$$

which is positive provided that (1.10) holds. In the special case where $W(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha$ we have

$$(3.12) \quad \lim_{\delta \downarrow 0} \mathcal{H}_W^\delta = \mathcal{H}_W^0 =: \mathcal{H}_W,$$

hence for such W and for any $\alpha \in (0, 2]$

$$(3.13) \quad \tilde{\theta}_W = \lim_{\delta \downarrow 0} \frac{\theta_W^\delta}{\delta}.$$

Recall that we denote by $\theta_W^\delta, \delta > 0$, the extremal index of ξ_W^δ . Using the terminology of [28] we refer to \bar{H}_W defined by (assuming that the limit exists)

$$\lim_{\delta \downarrow 0} \frac{\theta_W^\delta}{\delta} = \lim_{\delta \downarrow 0} \mathcal{H}_W^\delta = \bar{H}_W$$

as the *mean cluster index of the process W* . Since for any $T > 0$ and $\delta > 0$

$$0 \leq \mathbb{E} \left\{ \sup_{t \in \delta\mathbb{Z} \cap [0, T]} e^{W(t)} \right\} =: \mathcal{H}_W^0([0, T]),$$

we have clearly $\bar{H}_W \in [0, \mathcal{H}_W]$.

We show next that if ξ_W has an M3 representation, then \bar{H}_W is positive.

PROPOSITION 3.2. *Suppose ξ_W is max-stable and stationary with $W(0) = 0$. If ξ_W has an M3 representation and \bar{H}_W exists, then*

$$(3.14) \quad \bar{H}_W \geq \mathbb{E} \left\{ \frac{\sup_{t \in \mathbb{R}} e^{W(t)}}{\eta \sum_{t \in \eta\mathbb{Z}} e^{W(t)}} \right\} > 0$$

holds for any $\eta > 0$.

REMARK 3.2. (a) *In view of Theorems 2 and 3 in [12] we have for some general W as in (1.2), with B being a Gaussian or Lévy process,*

$$(3.15) \quad \mathcal{H}_W^0 = \mathbb{E} \left\{ \frac{\sup_{t \in \mathbb{R}} e^{W(t)}}{\eta \sum_{t \in \eta\mathbb{Z}} e^{W(t)}} \right\} = \mathbb{E} \left\{ \frac{\sup_{t \in \mathbb{R}} e^{W(t)}}{\int_{t \in \mathbb{R}} e^{W(t)} dt} \right\}$$

for any $\eta > 0$. Consequently, under these conditions and the setup of Proposition 3.2,

$$(3.16) \quad \mathcal{H}_W^0 = \bar{H}_W.$$

(b) *If $W(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha, t \in \mathbb{R}$, by (3.12) and (2.4) for any $\alpha \in (0, 2]$*

$$(3.17) \quad \mathcal{H}_W^0 = \bar{H}_W = \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{P} \left\{ \sup_{i \geq 1} (\mathcal{E} + W^\delta(i)) \leq 0 \right\},$$

with \mathcal{E} a unit exponential random variable independent of W . Expression (3.17) of the classical Pickands constant was initially derived in [1] for some general W , (see also a recent contribution [2]). In [28], Proposition 3, or the formula in [24], p. 44, the classical Pickands constant is the limit of a cluster index.

4. PROOFS

Proof of Lemma 1.1. Since $\lim_{n \rightarrow \infty} r_n = \infty$, by (1.7) and (1.8) we obtain

$$\lim_{n \rightarrow \infty} r_n^{-1} \mathbb{E} \left\{ \max_{i \in \{0, \delta, \dots, \delta r_n\}} e^{W(i)} \right\} = \delta \mathcal{H}_W^\delta = \theta_W^\delta.$$

For any $n \in \mathbb{N}$ we have

$$\begin{aligned} \frac{\mathbb{P}\{\max_{i \in \{0, \delta, \dots, \delta r_n\}} \xi_W(i) > n\}}{r_n \mathbb{P}\{\xi_W(0) > n\}} &= \frac{\mathbb{P}\{\max_{i \in \{0, \delta, \dots, \delta r_n\}} \xi_W(i) > n\}}{r_n [1 - e^{-1/n}]} \\ &\sim nr_n^{-1} [1 - \mathbb{P}\{\max_{i \in \{0, \delta, \dots, \delta r_n\}} \xi_W(i) \leq n\}] \\ &= nr_n^{-1} [1 - e^{-c_n/n}], \quad c_n := \mathbb{E} \left\{ \max_{i \in \{0, \delta, \dots, \delta r_n\}} e^{W(i)} \right\}, \end{aligned}$$

where the last equality follows from (1.5). The assumption $\lim_{n \rightarrow \infty} n/r_n = \infty$ and $\mathbb{E}\{e^{W(i)}\} = 1, i \in \delta\mathbb{Z}$, imply

$$(4.1) \quad \frac{c_n}{n} \leq \frac{1}{n} \mathbb{E} \left\{ \sum_{i \in \{0, \delta, \dots, \delta r_n\}} e^{W(i)} \right\} = \frac{r_n + 1}{n} \rightarrow 0, \quad n \rightarrow \infty.$$

Consequently,

$$\frac{\mathbb{P}\{\max_{i \in \{0, \delta, \dots, \delta r_n\}} \xi_W(i) > n\}}{r_n \mathbb{P}\{\xi_W(0) > n\}} \sim r_n^{-1} \mathbb{E} \left\{ \max_{i \in \{0, \delta, \dots, \delta r_n\}} e^{W(i)} \right\} \sim \theta_W^\delta, \quad n \rightarrow \infty,$$

hence the claim follows. ■

Proof of Theorem 2.1. We show first the stochastic representation (1.9). Recall that $X(t) = e^{W(t)}$ and for $\delta > 0$ we set

$$W^\delta(t) = W(\delta t), \quad X^\delta(t) = e^{W^\delta(t)}, \quad t \in \mathbb{Z}.$$

By (1.5), the fact that $\mathbb{P}\{\xi_W(0) \leq x\} = e^{-1/x}, x > 0$, and the assumption that $X(0) = 1$ almost surely, for any y_1, \dots, y_n positive and $y_0 > 1$ we have

$$\begin{aligned} &\mathbb{P}\{\xi_W^\delta(i) \leq Ty_i, i = 0, \dots, n \mid \xi_W^\delta(0) > T\} \\ &= \frac{1 - \mathbb{P}\{\xi_W^\delta(0) \leq T, \xi_W^\delta(i) \leq Ty_i, i \in \{0, \dots, n\}\}}{\mathbb{P}\{\xi_W^\delta(0) > T\}} \\ &\quad - \frac{1 - \mathbb{P}\{\xi_W^\delta(i) \leq Ty_i, i \in \{0, \dots, n\}\}}{\mathbb{P}\{\xi_W^\delta(0) > T\}} \\ &= \frac{1 - \exp(-\mathbb{E}\{\max(X^\delta(0), \max_{i \in \{1, \dots, n\}} X^\delta(i)/y_i)\} T^{-1})}{1 - e^{-1/T}} \end{aligned}$$

$$\begin{aligned} & \frac{1 - \exp(-\mathbb{E}\{\max_{i \in \{1, \dots, n\}} X^\delta(i)/y_i\}T^{-1})}{1 - e^{-1/T}} \\ & \sim T \left[1 - \left[1 - \frac{1}{T} \mathbb{E}\{\max(1, \max_{i \in \{0, \dots, n\}} X^\delta(i)/y_i)\} \right] \right. \\ & \quad \left. - \left(1 - \left[1 - \frac{1}{T} \mathbb{E}\{\max_{i \in \{0, \dots, n\}} X^\delta(i)/y_i\} \right] \right) \right] \\ & \rightarrow \mathbb{E}\left\{ \left(1 - \max_{i \in \{0, \dots, n\}} X^\delta(i)/y_i \right)_+ \right\}, \quad T \rightarrow \infty, \\ & = \mathbb{P}\{\mathcal{P} \leq y_0, \mathcal{P}X^\delta(i) \leq y_i, \forall i \in \{1, \dots, n\}\}, \end{aligned}$$

where \mathcal{P} is a unit Pareto random variable with survival function $1/s, s > 1$, independent of the process X . Hence the claim in (1.9) follows by Theorem 2.1 (ii) in [5]. Next, by the above derivations for any sequence of integers $r_n > m \in \mathbb{N}$ for any $x > 0$ (recall $X^\delta(0) = 1$ almost surely), we have

$$\begin{aligned} & 1 - \mathbb{P}\left\{ \max_{m \leq |i| \leq r_n} \xi_W^\delta(i) > nx \mid \xi_W^\delta(0) > nx \right\} \\ & = \frac{\mathbb{P}\{\max_{m \leq |i| \leq r_n} \xi_W^\delta(i) \leq nx, \xi_W^\delta(0) > nx\}}{\mathbb{P}\{\xi_W^\delta(0) > nx\}} \\ & = \frac{1 - \exp(-\mathbb{E}\{\max(X^\delta(0), \max_{|i| \in \{m, \dots, r_n\}} X^\delta(i))\} (nx)^{-1})}{1 - e^{-1/(nx)}} \\ & = \frac{1 - \exp(-\mathbb{E}\{\max_{|i| \in \{m, \dots, r_n\}} X^\delta(i)\} (nx)^{-1})}{1 - e^{-1/(nx)}} \\ & \sim nx \left[1 - \left[1 - \frac{1}{nx} \mathbb{E}\{\max(1, \max_{|i| \in \{m, \dots, r_n\}} X^\delta(i))\} \right] \right. \\ & \quad \left. - \left(1 - \left[1 - \frac{1}{nx} \mathbb{E}\{\max_{|i| \in \{m, \dots, r_n\}} X^\delta(i)\} \right] \right) \right] \\ & \sim \mathbb{E}\left\{ \left(1 - \max_{|i| \in \{m, \dots, r_n\}} X^\delta(i) \right)_+ \right\}, \end{aligned}$$

where we used the fact that, as in (4.1), the condition $\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} \frac{n}{r_n} = \infty$ implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}\left\{ \max(X^\delta(0), \max_{|i| \in \{m, \dots, r_n\}} X^\delta(i)) \right\} = 0,$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}\left\{ \max_{|i| \in \{m, \dots, r_n\}} X^\delta(i) \right\} = 0.$$

Consequently,

$$\begin{aligned} \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left\{ \max_{m \leq |i| \leq r_n} \xi_W^\delta(i) > nx \mid \xi_W^\delta(0) > nx \right\} \\ = \lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left[1 - \mathbb{E} \left\{ \left(1 - \max_{|i| \in \{m, \dots, r_n\}} X^\delta(i) \right)_+ \right\} \right] \\ = 1 - \lim_{m \rightarrow \infty} \mathbb{E} \left\{ \left(1 - \max_{|i| \in \mathbb{Z}, i \geq m} X^\delta(i) \right)_+ \right\} = 0, \end{aligned}$$

where we used the assumption (1.10). Hence Condition 2.1 holds.

By Proposition 4.2 in [5], we see that Condition 2.1 implies (1.10). Moreover, since

$$\mathbb{P} \{ \xi_W(0) > n \} = 1 - e^{-1/n} \sim \frac{1}{n}, \quad n \rightarrow \infty,$$

Proposition 4.2 in [5] and Lemma 1.1 imply

$$\theta_W^\delta = \widetilde{\theta}_W^\delta = \widehat{\theta}_W^\delta > 0.$$

Consequently,

$$(4.2) \quad \widehat{\theta}_W^\delta = \mathbb{P} \{ \sup_{i \geq 1} Y^\delta(i) \leq 1 \} = \lim_{n \rightarrow \infty} \mathbb{P} \{ \mathcal{P} \sup_{n \geq i \geq 1} X^\delta(i) \leq 1 \},$$

and so

$$\begin{aligned} \widehat{\theta}_W^\delta &= \lim_{n \rightarrow \infty} \mathbb{E} \left\{ \left(1 - \sup_{n \geq i \geq 1} X^\delta(i) \right)_+ \right\} = \mathbb{E} \left\{ \left(1 - \sup_{i \geq 1} X^\delta(i) \right)_+ \right\} \\ &= \mathbb{E} \left\{ \sup_{i \geq 0} X^\delta(i) - \sup_{i \geq 1} X^\delta(i) \right\} \in (0, 1], \end{aligned}$$

where the second last equality follows from the monotone convergence theorem. In fact, the above claim readily follows also from Remark 4.7 in [5]. Further from (4.2) we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P} \{ \mathcal{P} \sup_{n \geq i \geq 1} X^\delta(i) \leq 1 \} &= \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{n \geq i \geq 1} (\ln \mathcal{P} + \ln X^\delta(i)) \leq 0 \right\} \\ &= \lim_{n \rightarrow \infty} \mathbb{P} \left\{ \sup_{n \geq i \geq 1} (\mathcal{E} + W^\delta(i)) \leq 0 \right\} = \mathbb{P} \left\{ \sup_{i \geq 1} (\mathcal{E} + W^\delta(i)) \leq 0 \right\}, \end{aligned}$$

with $\mathcal{E} = \ln \mathcal{P}$ a unit exponential random variable independent of X .

Next, (2.3) follows from eq. (16) in [45]. Since further we assume (1.2), we infer that (2.3) implies

$$(4.3) \quad \mathcal{H}_W^\delta \in (0, 1/\delta)$$

for any $\delta > 0$, which completes the proof. ■

Proof of Theorem 2.2. By our assumption, for all large k we get

$$\frac{\sigma^2(\delta k)}{8} > \ln(\delta k)^a.$$

Consequently, by (3.8) we have for all δ large and some $a > 1$

$$\mathcal{H}_W^0 \geq \mathcal{H}_W^\delta \geq \frac{1}{\delta} \left(1 - \sum_{k=1}^{\infty} \exp\left(-\frac{\sigma^2(\delta k)}{8}\right) \right) \geq \frac{1}{\delta} \left(1 - \frac{1}{\delta^a} \sum_{k=1}^{\infty} \frac{1}{k^a} \right) > 0.$$

Hence the proof is complete. ■

Proof of Proposition 3.1. Since $B_\alpha(0) = 0$ almost surely, in view of (2.6) (see also [19], Proposition 4) we obtain

$$\lim_{\delta \rightarrow 0} \delta^{-1} \mathbb{P}\{\forall_{k \in \mathbb{Z} \setminus \{0\}} B_\alpha(\delta k) \leq |\delta k|^\alpha / \sqrt{2}\} = \mathcal{H}_{B_\alpha}.$$

Moreover, by the self-similarity of B_α , we have

$$\begin{aligned} \mathbb{P}\{\forall_{k \in \mathbb{Z} \setminus \{0\}} B_\alpha(\delta k) \leq |\delta k|^\alpha / \sqrt{2}\} &= \mathbb{P}\left\{\forall_{k \in \mathbb{Z} \setminus \{0\}} |\delta k|^\alpha B_\alpha\left(\frac{1}{\delta k}\right) \leq |\delta k|^\alpha / \sqrt{2}\right\} \\ &= \mathbb{P}\left\{\forall_{k \in \mathbb{Z} \setminus \{0\}} B_\alpha\left(\frac{1}{\delta k}\right) \leq 1 / \sqrt{2}\right\} = \mathbb{P}\left\{\forall_{k \in \mathbb{Z} \setminus \{0\}} B_\alpha\left(\frac{1}{k}\right) \leq \delta^{\alpha/2} / \sqrt{2}\right\}, \end{aligned}$$

hence the proof follows easily. ■

Proof of Theorem 3.1. (i) The proof is based on a technique developed in Lemma 16 and Corollary 17 in [16] and in Lemma 7 in [48], therefore we omit some details. For any $\delta > 0$ and T a positive integer, using Bonferroni's inequality, we have for any process W such that $\mathbb{E}\{e^{W(k\delta)}\} = 1, k \geq 1$,

$$\begin{aligned} &\mathbb{E}\left\{\sup_{t \in \delta\mathbb{Z} \cap [0, \delta T]} e^{W(t)}\right\} \\ &= \int_{\mathbb{R}} e^s \mathbb{P}\left\{\sup_{t \in \delta\mathbb{Z} \cap [0, \delta T]} W(t) > s\right\} ds \geq \int_{\mathbb{R}} e^s \mathbb{P}\{\exists_{1 \leq k \leq T} W(k\delta) > s\} ds \\ &\geq \sum_{k=1}^T \int_{\mathbb{R}} e^s \mathbb{P}\{W(k\delta) > s\} ds - \sum_{k=1}^{T-1} \sum_{l=k+1}^T \int_{\mathbb{R}} e^s \mathbb{P}\{W(k\delta) > s, W(l\delta) > s\} ds \\ &\geq \sum_{k=1}^T \mathbb{E}\{e^{W(k\delta)}\} \sum_{k=1}^{T-1} \sum_{l=k+1}^T \int_{\mathbb{R}} e^s \mathbb{P}\{W(k\delta) + W(l\delta) > 2s\} ds, \end{aligned}$$

and so

$$(4.4) \quad \mathbb{E}\left\{\sup_{t \in \delta\mathbb{Z} \cap [0, \delta T]} e^{W(t)}\right\} = T - \sum_{k=1}^{T-1} \sum_{l=k+1}^T \int_{\mathbb{R}} e^s \mathbb{P}\{W(k\delta) + W(l\delta) > 2s\} ds$$

$$\begin{aligned}
&= T - \sum_{k=1}^{T-1} \sum_{l=k+1}^T \mathbb{E} \left\{ \exp \left(\frac{W(k\delta) + W(l\delta)}{2} \right) \right\} \\
&= T - \sum_{k=1}^{T-1} \sum_{l=k+1}^T \exp \left(-\frac{\sigma^2(\delta |k-l|)}{8} \right) \\
&\geq T - T \sum_{k=1}^T \exp \left(-\frac{\sigma^2(\delta k)}{8} \right),
\end{aligned}$$

where the last equality follows by the stationarity of increments of the random process B . Along the lines of the proof in [14] we obtain

$$\begin{aligned}
\mathcal{H}_W^\delta &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left\{ \sup_{t \in \delta\mathbb{Z} \cap [0, T]} e^{W(t)} \right\} \geq \lim_{T \rightarrow \infty} \frac{1}{T} \lfloor T/\delta \rfloor \left(1 - \sum_{k=1}^{\infty} \exp \left(-\frac{\sigma^2(\delta k)}{8} \right) \right)_+ \\
&= \frac{1}{\delta} \left(1 - \sum_{k=1}^{\infty} \exp \left(-\frac{\sigma^2(\delta k)}{8} \right) \right)_+.
\end{aligned}$$

(ii) In view of (4.4), in order to establish the proof we need to calculate

$$a_{kl} = \int_{\mathbb{R}} e^s \mathbb{P} \{ W(\delta k) + W(\delta l) > 2s \} ds.$$

By independence of the increments and the fact that $W(\delta l) - W(\delta k) \stackrel{d}{=} W(\delta(l-k))$, we have

$$\begin{aligned}
a_{kl} &= \mathbb{E} \left\{ \exp \left(\frac{W(\delta k) + W(\delta l)}{2} \right) \right\} = \mathbb{E} \{ e^{W(\delta k)} \} \mathbb{E} \left\{ \exp \left(\frac{W(\delta l) - W(\delta k)}{2} \right) \right\} \\
&= \mathbb{E} \{ e^{W(\delta k)} \} \mathbb{E} \left\{ \exp \left(\frac{W(\delta(l-k))}{2} \right) \right\} \\
&= \mathbb{E} \left\{ \exp \left(\frac{B(\delta(l-k)) - \Phi(1)\delta(l-k)}{2} \right) \right\} \\
&= \exp(-\delta(l-k)\lambda),
\end{aligned}$$

where $\lambda := \frac{1}{2}\Phi(1) - \Phi(1/2) > 0$ by Jensen's inequality and the independence and stationarity of increments of the Lévy process B . Consequently, for $N \in \mathbb{N}$ we obtain

$$\begin{aligned}
(4.5) \quad \int_{\mathbb{R}} e^s \mathbb{P} \left\{ \sup_{t \in \delta\mathbb{Z} \cap [0, N]} W(t) > s \right\} ds &\geq \frac{N}{\delta} \left(1 - \sum_{k=1}^{\infty} e^{-\delta k \lambda} \right) \\
&= \frac{N}{\delta} \frac{1 - 2 \exp(-\delta \lambda)}{1 - \exp(-\delta \lambda)},
\end{aligned}$$

which leads to

$$\mathcal{H}_W^\delta \geq \frac{1}{\delta} \frac{1 - 2 \exp(-\delta \lambda)}{1 - \exp(-\delta \lambda)},$$

and thus the proof is complete. ■

Proof of (3.11). By (4.5) and letting $\lambda = \frac{1}{2}\Phi(1) - \Phi(1/2) > 0$, we have

$$\begin{aligned} \mathcal{H}_W^0 &\geq \lim_{N \rightarrow \infty} \frac{1}{N} \int_{\mathbb{R}} e^s \mathbb{P}\left\{ \sup_{t \in \delta\mathbb{Z} \cap [0, N]} W(t) > s \right\} ds \\ &\geq \frac{1}{\delta} \left(1 - \sum_{k=1}^{\infty} e^{-\delta k \lambda} \right) \geq \frac{1}{\delta} \left(1 - \int_0^{\infty} e^{-\delta x \lambda} dx \right) = \frac{1}{\delta} \left(1 - \frac{1}{\delta \lambda} \right) \geq \frac{\lambda}{4} > 0, \end{aligned}$$

establishing the proof. ■

Proof of Proposition 3.2. By [12], for any $\delta > 0$ and any integer $k \in \mathbb{N}$ we have

$$\mathcal{H}_W^\delta \geq \mathbb{E} \left\{ \frac{\sup_{t \in \delta\mathbb{Z}} e^{W(t)}}{k\delta \sum_{t \in k\delta\mathbb{Z}} e^{W(t)}} \right\},$$

hence choosing $\delta_n = \eta l^{-n}$ with $\eta > 0$ and $l > 1$ some integer and for $k = l^n$ which is clearly integer for any $n \geq 1$, we have

$$\begin{aligned} \mathcal{H}_W^{\delta_n} &\geq \mathbb{E} \left\{ \frac{\sup_{t \in \delta_n\mathbb{Z}} e^{W(t)}}{k\delta_n \sum_{t \in k\delta_n\mathbb{Z}} e^{W(t)}} \right\} \\ &= \mathbb{E} \left\{ \frac{\sup_{t \in \delta_n\mathbb{Z}} e^{W(t)}}{\eta \sum_{t \in \eta\mathbb{Z}} e^{W(t)}} \right\} \rightarrow \mathbb{E} \left\{ \frac{\sup_{t \in \mathbb{R}} e^{W(t)}}{\eta \sum_{t \in \eta\mathbb{Z}} e^{W(t)}} \right\}, \quad n \rightarrow \infty, \end{aligned}$$

where the last limit follows by the monotone convergence theorem and the fact that W has continuous sample paths. Since, by construction, $\mathcal{H}_W^{\delta_n}$ is non-decreasing in n , and we assume that $\lim_{\delta \downarrow 0} \mathcal{H}_W^\delta = \overline{H}_W$, the claim follows. ■

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