PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 4, Fasc. 1 (1984), p. 47-56

CHARACTERIZATION OF THE ELLIPTICALLY CONTOURED MEASURES ON INFINITE-DIMENSIONAL BANACH SPACES

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Abstract. This paper gives a characterization of the elliptically contoured measures on infinite-dimensional Banach spaces. The main results are Theorems 1 and 2. This characterization is not valid in the finite-dimensional Banach spaces.

The elliptically contoured measures were studied by Das Gupta et al. [2] in the finite-dimensional case and by Crawford [1] in the case of the measures with the strong second order on infinite-dimensional spaces. In this paper we omit this assumption.

By E we denote a real separable Banach space and by E^* its dual. $\mathscr{B}(E)$ denotes the family of all Borel sets in E. If μ is a measure on E, then by $\mu_{x_1^*,\ldots,x_n^*}$ we denote the n-dimensional measure defined by

$$\mu_{x_1^*,\ldots,x_n^*} = \mu\{x \in E \mid (x_1^*(x), \ldots, x_n^*(x)) \in A\},$$

where $A \in \mathscr{B}(R^n)$ and $x_1^*, \ldots, x_n^* \in E^*$ are linearly independent.

Definition 1. We say that μ is a *cylindrical measure* on the Banach space E if μ is a finite additive measure on cylinder sets in E and for every linearly independent set $x_1^*, \ldots, x_n^* \in E^*$ the measure $\mu_{x_1^*,\ldots,x_n^*}$ is σ-additive on R^n.

Definition 2. We say that the measure μ on R^n is an *elliptically contoured n-dimensional measure* if there exist a function $f\colon [0, \infty) \to [0, \infty)$ and a positive definite, symmetric $(n \times n)$-matrix Σ such that

(i) $\int_0^\infty r^{n-1} f(r^2)\, dr < \infty$ (which is equivalent to $\mu(R^n) < \infty$);

(ii) $|\Sigma|^{-1/2} f(\bar{x}\Sigma^{-1}\bar{x}^T)$ is the density of the measure μ.

We will use the notation $\mu = \mathscr{E}(f, \Sigma, n)$.

Definition 3. A measure μ on the Banach space E is *elliptically contoured* if for every natural number n and every linearly independent set $x_1^*, \ldots, x_n^* \in E^*$ the measure $\mu_{x_1^*, \ldots, x_n^*}$ is an elliptically contoured n-dimensional measure.

In the paper we will consider only elliptically contoured probability measures.

Examples. 1. The Gaussian measure on R^n with the density

$$|\Sigma|^{-1/2}(2\pi)^{-n/2}\exp\{\tfrac{1}{2}\bar{x}\Sigma^{-1}\bar{x}^{\mathrm{T}}\}$$

is obviously an elliptically contoured n-dimensional measure with

$$f(r^2) = (2\pi)^{-n/2}\exp\{-\tfrac{1}{2}r^2\}.$$

2. If the measure μ on R^n is invariant on the rotations and has a density, then the density must be constant on the sets

$$\{\bar{x} \in R^n \mid \sum_{i=1}^{n} x_i^2 = r^2\}, \quad r \geqslant 0.$$

Then μ is an elliptically contoured measure on R^n with the matrix I and some function $f\colon [0, \infty) \to [0, \infty)$.

3. The n-dimensional Student distribution has a density

$$|T|^{-1/2}C(n)\left[1+\frac{1}{n}\bar{x}T^{-1}\bar{x}^{\mathrm{T}}\right]^{-(n+k)/2},$$

where k is the number of degrees of fredom. Then it is an elliptically contoured n-dimensional measure.

Let us recall some known properties of elliptically contoured measures on R^n. For details see [1] and [6]

Property 1. *Let* $\mu = \mathscr{E}(f, \Sigma, n)$, $\Sigma = B^{\mathrm{T}}B$, *and let* C *be a nonsingular* $(n \times n)$-*matrix. Then*:

(i) *if* $\lambda(A) = \mu(AC^{-1}B)$ *for every* $A \in \mathscr{B}(R^n)$, *then* $\lambda = \mathscr{E}(f, C^{\mathrm{T}}C, n)$;

(ii) *if* $\lambda(A) = \mu(AC^{-1})$ *for every* $A \in \mathscr{B}(R^n)$, *then* $\lambda = \mathscr{E}(f, C^{\mathrm{T}}\Sigma C, n)$.

Property 2. *Let* $\mu = \mathscr{E}(f, \Sigma, n)$. *Then the characteristic function of* μ *is of the form*

$$\hat{\mu}(\bar{x}) = \int_{-\infty}^{\infty} \exp\{i(\bar{x}\Sigma\bar{x}^{\mathrm{T}})^{1/2}y\} f_1(y^2)\,dy := \psi(\bar{x}\Sigma\bar{x}^{\mathrm{T}}),$$

where

$$f_1(y^2) = \int_{R^{n-1}}\!\!\ldots\int f\Big(y^2 + \sum_{i=1}^{n-1} x_i^2\Big)dx_1 \ldots dx_{n-1}.$$

PROPERTY 3. *Let* $\mu = \mathscr{E}(f, \Sigma, n)$ *and let the measure* ν *on* R^{n-1} *be defined by*

$$\nu(A) = \mu\{x \in R^n \mid (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in A\}, \quad A \in \mathscr{B}(R^{n-1}),\ 1 \leqslant k \leqslant n.$$

Then $\nu = \mathscr{E}(g, S, n-1)$, *where* S *is a positive definite, symmetric matrix obtained from* Σ *by removing the k-th row and the k-th column, and*

$$g(r^2) = \int_{-\infty}^{\infty} f(r^2 + x^2)\, dx.$$

PROPERTY 4. *If two representations of the elliptically contoured n-dimensional measure* μ *are given, i.e., if* $\mu = \mathscr{E}(f, \Sigma, n) = \mathscr{E}(g, S, n)$, *then there exists a number* $a > 0$ *such that*

$$a^2 S = \Sigma \quad \textit{and} \quad g(r^2) = a^{-n} f(r^2/a^2).$$

Sketch of the proof. Assume that $\Sigma = I$ and put $S = (s_{ij})_{i,j=1}^n$. Now we define

$$\tilde{S} = s_{11}^{-1} S \quad \text{and} \quad \tilde{g}(r^2) = s_{11}^{-n/2} g(r^2 s_{11}^{-1}).$$

It is easy to see that $\mu = \mathscr{E}(\tilde{g}, \tilde{S}, n)$. Property 2 implies that for every $\bar{x} \in R^n$

$$\hat{\mu}(\bar{x}) = \int_{-\infty}^{\infty} \exp\{i(\sum_{i=1}^{n} x_i^2)^{1/2} y_1\} \int_{R^{n-1}} \ldots \int f(\sum_{j=1}^{n} y_j^2)\, dy_2 \ldots dy_n\, dy_1$$

and, on the other hand,

$$\hat{\mu}(\bar{x}) = \int_{-\infty}^{\infty} \exp\{i(\bar{x}\tilde{S}\bar{x}^{\mathrm{T}})^{1/2} y_1\} \int_{R^{n-1}} \ldots \int \tilde{g}(\sum_{j=1}^{n} y_j^2)\, dy_2 \ldots dy_n\, dy_1.$$

Recall that $\tilde{s}_{11} = 1$. If $\bar{x} = (x, 0, \ldots, 0)$, then $\hat{\mu}(\bar{x})$ is the characteristic function of a probability measure on R. Consequently, we obtain the equality

$$\int_{R^{n-1}} \ldots \int f(\sum_{j=1}^{n} y_j^2)\, dy_2 \ldots dy_n = \int_{R^{n-1}} \ldots \int \tilde{g}(\sum_{j=1}^{n} y_j^2)\, dy_2 \ldots dy_n$$

almost everywhere with respect to the Lebesgue measure. Now, it is easy to see that $I = \tilde{S} = s_{11}^{-1} S$, and then $g(r^2) = a^{-n} f(r^2/a^2)$, where $a > 0$ and $a^2 = s_{11}^{-1}$. If $\Sigma \neq I$, then there exists a nonsingular matrix B such that $\Sigma = B^{\mathrm{T}} B$ and we can use the above argumentation to the measure

$$\lambda = \mathscr{E}(f, I, n) = \mathscr{E}(g, (B^{-1})^{\mathrm{T}} S(B^{-1}), n)$$

(see Property 1).

THEOREM 1. *If* μ *is a cylindrical measure on an infinite-dimensional Banach space E, then the following conditions are equivalent*:

(1) μ *is elliptically contoured.*

(2) *There exists* $Q: E^* \times E^* \to R$, *an inner product on* E^*, *such that the*

characteristic function of μ *is of the form* $\hat{\mu}(x^*) = \psi(Q(x^*, x^*))$, *where* $\psi\colon R^+ \to R$ *is a function.*

(3) *There exist a Gaussian, cylindrical, symmetric measure* γ *on E and a probability measure* λ *on* $(0, \infty)$ *such that*

$$\mu(A) = \int_0^\infty \gamma(A/\sqrt{t})\,\lambda(dt)$$

for every cylinder set $A \in E$.

Proof. (1) $\Rightarrow$ (2). First we prove that if μ is an elliptically contoured measure on an infinite-dimensional Banach space E and for some linearly independent set $x_1^*, \ldots, x_n^* \in E^*$ we have $\mu_{x_1^*,\ldots,x_n^*} = \mathscr{E}(f, \Sigma, n)$, then for every linearly independent set $y_1^*, \ldots, y_n^* \in E^*$ there exists a positive-definite, symmetric matrix S such that $\mu_{y_1^*,\ldots,y_n^*} = \mathscr{E}(f, S, n)$.

Assume that $x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_n^*$ are linearly independent. From the definition we know that there exist functions g, h and matrices T, P such that

$$\mu_{y_1^*,\ldots,y_n^*} = \mathscr{E}(g, T, n) \quad\text{and}\quad \mu_{x_1^*,\ldots,x_n^*, y_1^*,\ldots,y_n^*} = \mathscr{E}(h, P, 2n).$$

Now, from Property 3 we obtain

$$\begin{aligned}|\Sigma|^{-1/2} f(\bar{x}\Sigma^{-1}\bar{x}^{\mathrm{T}}) &= \int_{R^n}\!\!\ldots\int |P|^{-1/2} h(\bar{y}P^{-1}\bar{y}^{\mathrm{T}})\,dz_1 \ldots dz_n \\ &= |P_{11}|^{-1/2} \int_{R^n}\!\!\ldots\int h\Big(\bar{x}P_{11}^{-1}\bar{x}^{\mathrm{T}} + \sum_{i=1}^n z_i^2\Big)dz_1 \ldots dz_n\end{aligned}$$

and

$$\begin{aligned}|T|^{-1/2} g(\bar{z}T^{-1}\bar{z}^{\mathrm{T}}) &= \int_{R^n}\!\!\ldots\int |P|^{-1/2} h(\bar{y}P^{-1}\bar{y}^{\mathrm{T}})\,dx_1 \ldots dx_n \\ &= |P_{22}|^{-1/2} \int_{R^n}\!\!\ldots\int h\Big(zP_{22}^{-1}z + \sum_{i=1}^n x_i^2\Big)dx_1 \ldots dx_n,\end{aligned}$$

where

$$\bar{y} = (x_1, \ldots, x_n, z_1, \ldots, z_n), \qquad P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

and P_{ij} is an $(n \times n)$-matrix. Now, by Property 4, there exists a number $a > 0$ such that $\Sigma = a^2 P_{11}$ and

$$\int_{R^n}\!\!\ldots\int h\Big(r^2 + \sum_{i=1}^n y_i^2\Big)dy_1 \ldots dy_n = a^{-n} f(r^2/a^2)$$

and there exists a number $b > 0$ such that $T = b^2 P_{22}$ and

$$\int_{R^n}\!\!\ldots\int h\Big(r^2 + \sum_{i=1}^n y_i^2\Big)dy_1 \ldots dy_n = b^{-n} g(r^2/b^2).$$

This means that $g(r^2) = c^{-n} f(r^2/c^2)$ for $c = a/b$. It is easy to see that

$$\mu_{y_1^*,\ldots,y_n^*} = \mathscr{E}(f, c^2 T, n).$$

If $x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_n^* \in E^*$ are not linearly independent, then we can choose $z_1^*, \ldots, z_n^* \in E^*$ such that $x_1^*, \ldots, x_n^*, z_1^*, \ldots, z_n^*$ are linearly independent and $y_1^*, \ldots, y_n^*, z_1^*, \ldots, z_n^*$ are linearly independent.

Since $\mu_{x_1^*,\ldots,x_n^*} = \mathscr{E}(f, \Sigma, n)$ and $x_1^*, \ldots, x_n^*, z_1^*, \ldots, z_n^*$ are linearly independent, according to the preceding considerations there exists an $(n \times n)$-matrix T such that

$$\mu_{z_1^*,\ldots,z_n^*} = \mathscr{E}(f, T, n),$$

and since $y_1^*, \ldots, y_n^*, z_1^*, \ldots, z_n^*$ are linearly independent, there exists an $(n \times n)$-matrix S such that

$$\mu_{y_1^*,\ldots,y_n^*} = \mathscr{E}(f, S, n).$$

Now, we can construct by induction the sequence $\{f_n\}$ such that for every $x_1^*, \ldots, x_n^*$ there exists a positive definite, symmetric matrix Σ such that

$$\mu_{x_1^*,\ldots,x_n^*} = \mathscr{E}(f_n, \Sigma, n) \quad \text{and} \quad \int_{-\infty}^{\infty} f_{n+1}(x^2+r^2)\,dx = f_n(r^2).$$

It is enough to fix an x^* $(0 \neq x^* \in E^*)$ and fix one of the representations of the measure $\mu_{x*} = \mathscr{E}(f_1, \sigma_{x*}^2, 1)$. We know that for every $y^* \in E^*$ there exists $\sigma_{y*} > 0$ such that $\mu_{y*} = \mathscr{E}(f_1, \sigma_{y*}^2, 1)$.

If $f_1, \ldots, f_n$ are given and $x_1^*, \ldots, x_n^* \in E^*$ are linearly independent, then we take some representation $\mu_{x_1^*,\ldots,x_{n+1}^*} = \mathscr{E}(g, \Sigma, n+1)$. By assumption there exists a matrix T such that $\mu_{x_1^*,\ldots,x_n^*} = \mathscr{E}(f_n, T, n)$. Then

$$\begin{aligned} |T|^{-1/2} f_n(\bar{x}T^{-1}\bar{x}^{\mathrm{T}}) &= \int_{-\infty}^{\infty} |\Sigma|^{-1/2} g(\bar{x}\Sigma^{-1}\bar{x}^{\mathrm{T}})\,dx_{n+1} \\ &= |S|^{-1/2} \int_{-\infty}^{\infty} g(\bar{x}S^{-1}\bar{x}^{\mathrm{T}}+y^2)\,dy, \end{aligned}$$

where (see Property 3) the matrix S is obtained from Σ by removing the $(n+1)$-st row and the $(n+1)$-st column. By Property 4 there exists a number $a > 0$ such that

$$S = a^2 T \quad \text{and} \quad f_n(r^2) = a^{-n} \int_{-\infty}^{\infty} g(r^2 a^{-2} + y^2)\,dy.$$

Now, it is enough to put

$$f_{n+1}(r^2) = a^{-n-1} g(r^2 a^{-2}) \quad \text{and} \quad \Sigma' = a^{-2}\Sigma.$$

Let $x^*, y^* \in E^*$. We define $Q(x^*, y^*)$ by

$$Q(x^*, y^*) = a\sigma^2 \quad \text{if } y^* = ax^* \text{ and } \mu_{x*} = \mathscr{E}(f_1, \sigma^2, 1)$$

and by

$$Q(x^*, y^*) = \sigma_{12}$$

if x^*, y^* are linearly independent and

$$\mu_{x^*,y^*} = \mathscr{E}\left(f_2, \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}, 2\right).$$

From Property 2 it follows that the characteristic function of the measure μ is of the from

$$\hat{\mu}(x^*) = \int_{-\infty}^{\infty} \exp\{i(Q(x^*, x^*))^{1/2} y\} f_1(y^2)\,dy := \psi(Q(x^*, x^*)).$$

(2) $\Rightarrow$(3). From the Kuelbs theorem (see [4]) it follows that the function ψ in the above formula must be absolutely monotonic. Then there exists a finite measure λ on $[0, \infty)$ such that

$$\hat{\mu}(x^*) = \psi(Q(x^*, x^*)) = \int_0^{\infty} \exp\{-\tfrac{1}{2}Q(x^*, x^*)t\}\,\lambda(dt)$$

if $Q(x^*, x^*) \neq 0$. Obviously, λ must be a probability measure and $\lambda(\{0\}) = 0$.

Now, we construct a Gaussian, symmetric, cylindrical measure γ on E. Let $x_1^*, \ldots, x_n^* \in E^*$ be linearly independent and write $\Sigma = (Q(x_i^*, x_j^*))_{i,j=1}^n$. We define $\gamma_{x_1^*,\ldots,x_n^*}$ as the symmetric Gaussian measure on R^n with the covariance matrix Σ. It is easy to see that the family

$$\{\gamma_{x_1^*,\ldots,x_n^*} \mid n \in N,\ x_i^* \in E^* \text{ are linearly independent}\}$$

defines a symmetric, Gaussian, cylindrical measure on E such that

$$\gamma_{x_1^*,\ldots,x_n^*}(A) = \gamma\{x \in E \mid (x_1^*(x), \ldots, x_n^*(x)) \in A\} \quad \text{for every } A \in \mathscr{B}(R^n).$$

We now construct a cylindrical measure ν on E by the formula

$$\nu(A) = \int_0^{\infty} \gamma(A/\sqrt{t})\,\lambda(dt)$$

for every cylinder set $A \subset E$. For every $x^* \in E^*$ we have

$$\begin{aligned}\hat{\nu}(x^*) &= \int_E \exp\{ix^*(x)\}\,\nu(dx)\\ &= \int_{-\infty}^{\infty} e^{iy} \int_0^{\infty} [2\pi Q(x^*, x^*)t]^{-1/2} \exp\left\{-\frac{y^2}{2tQ(x^*, x^*)}\right\}\lambda(dt)\,dy\\ &= \int_0^{\infty} \exp\{-\tfrac{1}{2}Q(x^*, x^*)t\}\,\lambda(dt) = \hat{\mu}(x^*).\end{aligned}$$

Then, obviously, $\mu = \nu$ on every cylinder set in E.

(3) $\Rightarrow$(1). We have only to show that the finite-dimensional distributions

of μ are elliptically contoured n-dimensional measures. Let $x_1^*, \ldots, x_n^* \in E^*$ be linearly independent. Then for every $A \in \mathscr{B}(R^n)$ we have

$$\begin{aligned}\mu_{x_1^*,\ldots,x_n^*}(A) &= \int_0^\infty \gamma_{x_1^*,\ldots,x_n^*}(A/\sqrt{t})\,\lambda(dt)\\ &= \int_0^\infty \int_{At^{-1/2}}\!\!\ldots\int |\Sigma|^{-1/2}(2\pi)^{-n/2}\exp\left\{-\frac{1}{2}\bar{x}\Sigma^{-1}\bar{x}^{\mathrm{T}}\right\}dx\,\lambda(dt)\\ &= \int_A\!\!\ldots\int |\Sigma|^{-1/2}\int_0^\infty (2\pi t)^{-n/2}\exp\left\{-\frac{\bar{x}\Sigma^{-1}\bar{x}^{\mathrm{T}}}{2t}\right\}\lambda(dt)\,dx,\end{aligned}$$

where Σ is the covariance matrix of $\gamma_{x_1^*,\ldots,x_n^*}$. Then $\mu_{x_1^*,\ldots,x_n^*}$ is elliptically contoured since its density is of the form

$$|\Sigma|^{-1/2}\int_0^\infty (2\pi t)^{-n/2}\exp\left\{-\frac{\bar{x}\Sigma^{-1}\bar{x}^{\mathrm{T}}}{2t}\right\}\lambda(dt).$$

LEMMA. *Let μ and ν be cylindrical measures on the Banach space E and let λ be a probability measure on R such that $\lambda(\{0\}) = 0$; let*

$$\mu(A) = \int_{-\infty}^{\infty} \nu(At)\,\lambda(dt)$$

for every cylinder set $A \subset E$. Then μ can be extended to a Radon measure if and only if ν can be extended to a Radon measure.

Proof. If ν can be extended to a Radon measure, then so can μ since E is a Polish space.

Assume that μ can be extended to a Radon measure and suppose that ν cannot be extended to a Radon measure. Then there exists $\varepsilon > 0$ such that for every compact set $K \subset E$ there exists an n-dimensional projection π: $E \to R^n$ such that

$$\nu\big(\pi^{-1}(\pi K)\big) < 1-\varepsilon.$$

For the measure λ we can find $T > 0$ such that $\lambda(\{-T, T\}) = 0$ and $\lambda(-T, T) > 1-\varepsilon/2$. Since μ can be extended to a Radon measure, there exists a compact set $K_0 \subset E$ such that $aK_0 \subset K_0$ for every $|a| \leqslant 1$ and $\mu(K_0) > 1-\varepsilon/4$. If we take now such a projection π: $E \to R^n$ for which $\nu\left[\pi^{-1}\big(\pi(K_0 T)\big)\right] < 1-\varepsilon$, then we obtain the following contradiction:

$$\begin{aligned}1-\varepsilon/4 < \mu(K_0) \leqslant \mu\big(\pi^{-1}(\pi K_0)\big) &= \int_{-\infty}^{\infty} \nu\left[\pi^{-1}\big(\pi(K_0 t)\big)\right]\lambda(dt)\\ &< \varepsilon/2 + \int_{-T}^{T} \nu\left[\pi^{-1}\big(\pi(tT^{-1}K_0 T)\big)\right]\lambda(dt)\\ &< \varepsilon/2 + \nu\left[\pi^{-1}\big(\pi(K_0 T)\big)\right]\lambda(-T, T) < 1-\varepsilon/2.\end{aligned}$$

Therefore, ν must be a Radon measure.

THEOREM 2. *If μ is a Radon measure on an infinite-dimensional Banach space E, then the following conditions are equivalent:*

(1) *μ is elliptically contoured.*

(2) *There exist a Gaussian, symmetric, Radon measure γ on E and a probability measure λ on $(0, \infty)$ such that, for every $A \in \mathscr{B}(E)$,*

$$\mu(A) = \int_0^\infty \gamma(A/\sqrt{t})\,\lambda(dt).$$

(3) *There exist a Gaussian, zero mean, random vector X on E and a real positive random variable Θ on $(0, \infty)$ such that X and Θ are independent and μ is the law of the random vector $Y = X\sqrt{\Theta}$.*

Proof. The equivalence of conditions (1) and (2) follows easily from Theorem 1 and the Lemma.

(2) $\Rightarrow$ (3). For the measures γ and λ we can construct a Gaussian, zero mean, random vector X on E and a positive random variable Θ such that X and Θ are independent, γ is the distribution of X, and λ is the distribution of Θ. Then we have only to prove that μ is the distribution of $Y = X\sqrt{\Theta}$. Let

$$C = \{x \in E \mid (x_1^*(x), \ldots, x_n^*(x)) \in A\},$$

where $x_1^*, \ldots, x_n^* \in E^*$ are linearly independent and $A \in \mathscr{B}(R^n)$. Then

$$\begin{aligned} P\{Y \in C\} &= P\{(x_1^*(Y), \ldots, x_n^*(Y)) \in A\} \\ &= \int_0^\infty P\{(x_1^*(X), \ldots, x_n^*(X)) \in A/\sqrt{t}\}\,\lambda(dt) \\ &= \int_0^\infty P\{X \in C/\sqrt{t}\}\,\lambda(dt) = \int_0^\infty \gamma(C/\sqrt{t})\,\lambda(dt) = \mu(C). \end{aligned}$$

(3) $\Rightarrow$ (1). It is sufficient to prove that the finite-dimensional distributions of the random vector Y are elliptically contoured. Let C be defined as above. Then we obtain

$$\begin{aligned} P\{Y \in C\} &= \int_0^\infty P\{X \in C/\sqrt{t}\}\,\lambda(dt) \\ &= \int_0^\infty \underset{At^{-1/2}}{\int \ldots \int} (2\pi)^{-n/2} |\Sigma|^{-1/2} \exp\left\{-\frac{1}{2}\bar{x}\Sigma^{-1}\bar{x}^{\mathrm{T}}\right\} dx\,\lambda(dt) \\ &= \underset{A}{\int \ldots \int} |\Sigma|^{-1/2} \int_0^\infty (2\pi t)^{-n/2} \exp\left\{-\frac{\bar{x}\Sigma^{-1}\bar{x}^{\mathrm{T}}}{2t}\right\} \lambda(dt)\,dx, \end{aligned}$$

where Σ is the covariance matrix of the random vector $(x_1^*(X), \ldots, x_n^*(X))$. This means that $(x_1^*(Y), \ldots, x_n^*(Y))$ has the distribution $\mathscr{E}(f_n, \Sigma, n)$, where

$$f_n(r^2) = \int_0^\infty (2\pi t)^{-n/2} \exp\left\{-\frac{r^2}{2t}\right\} \lambda(dt).$$

From now on we denote the elliptically contoured measures on infinite-dimensional Banach spaces by $\mathscr{E}(\gamma, \lambda)$ or $\mathscr{E}(X, \Theta)$.

THEOREM 3. *If F is a linear measurable subspace of the Banach space E and $\mu = \mathscr{E}(\gamma, \lambda)$ is the measure on E, then μ is concentrated on F if and only if so is γ.*

This theorem follows easily from the equality $F/\sqrt{t} = F$ for every $t > 0$.

THEOREM 4. *For every elliptically contoured measure $\mathscr{E}(\gamma, \lambda)$ on E the following conditions are equivalent for every $p > 0$:*

(i) $\mathrm{E}_\mu \|x\|^p < \infty$.

(ii) $\mathrm{E}_\mu |x^*(x)|^p < \infty$ *for every* $x^* \in E^*$.

(iii) $\int_0^\infty t^{p/2} \lambda(dt) < \infty$.

This follows immediately from Theorem 2 (3).

Remarks. 1. Let us note that Theorems 1 and 2 are not valid in a finite-dimensional space (cf. [6]). But for every symmetric Gaussian measure γ on R^n and every probability measure λ on $(0, \infty)$ the measure $\mu = \mathscr{E}(\gamma, \lambda)$ is elliptically contoured on R^n.

2. Crawford (cf. [1]) proved Theorem 1 for the elliptically contoured measures with the strong second order. Our representation coincides in this case with that of Crawford if

$$\int_0^\infty t\lambda(dt) = 1.$$

3. It is easy to see that for each symmetric Gaussian measure γ on E and for every $(p/2)$-stable measure λ on $(0, \infty)$, where $p < 2$, the measure $\mu = \mathscr{E}(\gamma, \lambda)$ is p-stable on E ([3], Chapter VI, p. 167). Moreover, elliptically contoured measures occur in the Random Central Limit Theorem:

Let $\{X_n\}$ be independent symmetric, identically distributed, random variables such that $(X_1 + \ldots + X_n)/\sqrt{n}$ converges in distribution to a symmetric Gaussian variable X. If $\Theta_n/a_n \xrightarrow{P} \Theta$, where Θ_n are nonnegative random variables, $a_n \to \infty$, and Θ is a nonnegative random variable, X and Θ being independent, then $(X_1 + \ldots + X_{\Theta_n})/\sqrt{a_n}$ converges in distribution to $X\sqrt{\Theta}$.

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Received on 22. 1. 1981;
revised version on 17. 9. 1981