PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 5, Fasc. 2 (1985), p. 197-209

JUMPS OF STOCHASTIC PROCESSES WITH VALUES IN A TOPOLOGICAL GROUP

BY

EBERHARD SIEBERT (TÜBINGEN)

Abstract. We consider a stochastic process X taking its values in a Polish group G and having independent increments. First we investigate the jump measures ν_t on G associated with the process X. Then we identify the measures ν_t with the Lévy measures of certain convolution semigroups on G closely connected with X. Finally we show that for a submultiplicative function φ on G the integrability with respect to the process X is essentially equivalent with the integrability of φ with respect to the jump measures ν_t of X.

Introduction. Let $(X_t)_{t \geqslant 0}$ be a stochastic process without discontinuities of the second kind taking its values in a Polish group G. Then the jumps of the process on the time interval $[0, t]$ define a measure ν_t on the Borel sets in G not containing the identity. In the present paper we investigate some properties of these jump measures. In Section 1 we are concerned with the definition (Theorem 1) and the approximation (Theorem 2) of jump measures. Essentially we follow here the classical approach. In Section 2 we specialize our considerations to processes with independent increments taking their values in a locally compact group. We prove that the jump measures coincide with the Levy measures of certain convolution semigroups connected with the process (Theorem 3). Finally in Section 3 we show that for a submultiplicative function φ on the Polish group G the variables $\sup\{\varphi(X_s)\colon 0 \leqslant s \leqslant t\}$, $t \geqslant 0$, are integrable if the process $(X_t)_{t \geqslant 0}$ has independent increments and if its jumps are uniformly bounded (Theorem 4). Moreover, for a homogeneous process $(X_t)_{t \geqslant 0}$ the integrability of the variables $\varphi(X_t)$ (and even of the variables $\sup\{\varphi(X_s)\colon 0 \leqslant s \leqslant t)\}$ is equivalent to the integrability of φ with respect to the jump measure ν_1 outside some neighbourhood of the identity of G (Theorem 5).

Measures constructed from the jumps of a process with independent increments have always played an important role in the investigation of these processes: If the process takes its values in a Euclidean space or, more generally, in a Banach space (see for example [5] and [6]); if the process takes its values in a locally compact group we recommend [9], R 4.6, R 5.6, and [10] for rather complete lists of relevant references. The integrability of submultiplicative functions with respect to infinitely divisible probability measures has also been investigated in various papers [1, 11, 12, 14, 15, 16]. In fact on (non-commutative) locally compact groups the existing results for these topics were not so satisfactory as on Banach spaces. In the present paper we have tried to handle these two classes of groups simultaneously. Therefore as far as possible we have considered processes with values in a Polish group.

The author would like to thank Professor J. Kisyński of the University at Warsaw for several stimulating discussions.

Preliminaries. Let N and R denote the sets of positive integers and of real numbers, respectively. We put $R_+ = \{r \in R: r \geqslant 0\}$ and $R_+^* = \{r \in R: r > 0\}$. A decomposition $\Delta = \{c_0, c_1, \ldots, c_n\}$ of a real interval $[s, t]$ is a subset of R such that $s = c_0 < c_1 < \ldots < c_n = t$. We put $|\Delta| = \max\{c_{j-1}: j = 1, \ldots, n\}$.

Let T be a set and S a subset of T. Then by 1_S we denote the indicator function of S. If T is a topological space we denote by $\bar{S}$, $\mathring{S}$, ∂S the closure, interior, and boundary of S, respectively. Moreover, by $\mathscr{C}^b(T)$ we denote the space of real valued bounded continuous functions on T; $\operatorname{supp}(f)$ is the support of $f \in \mathscr{C}^b(T)$. $\mathfrak{B}(T)$ is the σ-algebra of Borel subsets of T i.e. the σ-algebra generated by the open subsets of T. Measures on T are always understood as abstract measures on $\mathfrak{B}(T)$. Weak convergence of finite measures on T means pointwise convergence on $\mathscr{C}^b(T)$. By ε_x we denote the unit mass in $x \in T$.

By G we always denote a Polish group, i.e. G is a topological group with a countable basis of its topology and with a complete left invariant metric ϱ which induces the topology. For $x \in G$ and $\varepsilon \in R_+^*$ let $K(x, \varepsilon) = \{y \in G: \varrho(x, y) < \varepsilon\}$. $\mathfrak{V}(G)$ denotes the system of neighbourhoods U of the identity e of G such that $U \in \mathfrak{B}(G)$. For abbreviation we put $K(\varepsilon) = K(e, \varepsilon)$ and $G^* = G \setminus \{e\}$. $\mathscr{M}^1(G)$ denotes the convolution semigroup of probability measures on G. A (continuous) convolution semigroup $(\mu_t)_{t \geqslant 0}$ in $\mathscr{M}^1(G)$ is a family of probability measures μ_t on G such that $\mu_0 = \varepsilon_e$, $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \in R_+$, and $\lim_{t \downarrow 0} \mu_t = \mu_0$ weakly.

Let I be a subinterval of R_+. A stochastic process $(\Omega, \mathfrak{A}, P, (X_t)_{t \in I})$ with values in G consists of a probability space $(\Omega, \mathfrak{A}, P)$ and of a family $(X_t)_{t \in I}$ of measurable mappings X_t of $(\Omega, \mathfrak{A})$ into $(G, \mathfrak{B}(G))$. Usually we shall denote the process by $(X_t)_{t \in I}$ only.

1. JUMP MEASURES OF STOCHASTIC PROCESSES

Let $(\Omega, \mathfrak{A}, P, (X_t)_{t \in I})$ be a stochastic process with values in the Polish group G where either $I = [0, 1]$ or $I = R_+$. We assume that the paths of the process are right continuous and admit left-hand limits. Thus for every $t \in I$, $t > 0$, there exists $X_{t-} = \lim_{s\uparrow t} X_s$; and $X_{t-}^{-1} X_t$ is the *jump* of the process at time t. Let $X(s, t) = X_s^{-1} X_t$ denote the left increment of the process on the time interval $[s, t]$ and let $\mu(s, t)$ denote the distribution of $X(s, t)$ $(0 \leqslant s \leqslant t \in I)$. For every $B \in \mathfrak{B}(G^*)$ and for every $t \in I$ the number of discontinuities of the process up to time t whose jumps lie in B is given by

$$Z_t^B = \sum_{0 < s \leqslant t} \mathbf{1}_B(X_{s-}^{-1} X_s).$$

If $e \notin \bar{B}$, then it is well known that Z_t^B is finite everywhere. If $(B(n))_{n\geqslant 1}$ is a pairwise disjoint sequence in $\mathfrak{B}(G^*)$ with union B, we obviously have

$$Z_t^B = \sum_{n\geqslant 1} Z_t^{B(n)}.$$

For some fixed $t \in I$, $t > 0$, let now $(\Delta_n)_{n\geqslant 1}$ be a sequence of decompositions $\Delta_n = \{c_{n0}, c_{n1}, \ldots, c_{nk(n)}\}$ of the interval $[0, t]$ such that $\lim |\Delta_n| = 0$. For every $B \in \mathfrak{B}(G^*)$ we put

$$S_n^B = \sum_{1 \leqslant j \leqslant k(n)} \mathbf{1}_B\big(X(c_{n,j-1}, c_{nj})\big).$$

THEOREM 1. (i) *For every $B \in \mathfrak{B}(G^*)$ with $e \notin \bar{B}$ we have*

$$Z_t^{\mathring{B}} \leqslant \varliminf S_n^B \leqslant \varlimsup S_n^B \leqslant Z_t^{\bar{B}} = Z_t^{\mathring{B}} + Z_t^{\partial B}.$$

(ii) *For every $B \in \mathfrak{B}(G^*)$ the function Z_t^B of Ω into $[0, \infty]$ is measurable.*

Proof. (i) We fix some $\omega \in \Omega$. Let $s \in]0, t]$ such that

$$X_{s-}(\omega)^{-1} X_s(\omega) \in \mathring{B}.$$

But $Z_t^B(\omega) < \infty$. Hence for $n \in N$ sufficiently large there exists $j \in \{1, \ldots, k(n)\}$ such that s is the only discontinuity of $(X_r(\omega))_{0\leqslant r\leqslant t}$ on $]c_{n,j-1}, c_{nj}]$ and such that $X(c_{n,j-1}, c_{nj})(\omega) \in \mathring{B}$. Consequently,

$$\varliminf S_n^B(\omega) \geqslant Z_t^{\mathring{B}}(\omega).$$

Let us now assume $S_n^B(\omega) \geqslant Z_t^{\bar{B}}(\omega) + 1$ for infinitely many $n \in N$. After passing to an appropriate subsequence we may assume without loss of generality that for every $n \in N$ there exists some $j(n) \in \{1, \ldots, k(n)\}$ with the following properties:

(1) $X(c_{n,j(n)-1}, c_{nj(n)})(\omega) \in \bar{B}$;

(2) on $]c_{n,j(n)-1}, c_{nj(n)}]$ there is no discontinuity of $(X_r(\omega))_{0\leqslant r\leqslant t}$ whose jump lies in $\bar{B}$;

(3) the sequences $(c_{n,j(n)-1})_{n\geqslant 1}$ and $(c_{nj(n)})_{n\geqslant 1}$ converge to s.

Now $c_{nj(n)} < s$ for infinitely many $n \in N$ would imply (in view of (1)) $e = X_{s-}(\omega)^{-1} X_{s-}(\omega) = \lim X(c_{n,j(n)-1}, c_{nj(n)})(\omega) \in \bar{B}$, hence a contradiction. Thus $s \leqslant c_{nj(n)}$ for almost all $n \in N$ and analogously $c_{n,j(n)-1} < s$ for almost all $n \in N$. Thus (by (1) and (3)), $X_{s-}(\omega)^{-1} X_s(\omega) = \lim X(c_{n,j(n)-1}, c_{nj(n)})(\omega) \in \bar{B}$. But this contradicts (2). Hence (i) is proved.

(ii) 1. Let $0 < \delta < \varepsilon$. Then there exists some $\gamma \in]\delta, \varepsilon[$ such that $Z_t^{\partial K(\gamma)} = 0$.

(Since $Z_t^{CK(\delta)}$ is finite there exist finite subsets F_k of $]\delta, \varepsilon[$ such that $Z_t^{\partial K(\gamma)}$ is zero on $[Z_t^{CK(\delta)} = k]$ if $\gamma \in]\delta, \varepsilon[\backslash F_k$ (all $k \in N$). Every $\gamma \in]\delta, \varepsilon[$ not in $F_1 \cup F_2 \cup \dots$ satisfies $Z_t^{\partial K(\gamma)} = 0$.)

2. Let some γ as in 1 be fixed and let $\mathfrak{L} = \{B \in \mathfrak{B}(G)\colon B \subset \complement K(\gamma)\}$, $\mathscr{E} = \{B \in \mathfrak{L}\colon Z_t^{\partial B} = 0\}$. Then $\mathscr{E}$ is closed with respect to finite intersections and a generator for the σ-algebra $\mathfrak{L}$ in $\complement K(\gamma)$.

(As in 1 one shows that for every $x \in G$ and for every $\varepsilon \in \boldsymbol{R}_+^*$ with $K(x, \varepsilon) \cap K(\gamma) = \emptyset$ there exist $\delta \in]0, \varepsilon[$ arbitrarily close to ε such that $Z_t^{\partial K(x,\delta)} = 0$. Hence $K(x, \delta) \in \mathscr{E}$ such that $\mathscr{E}$ is a generator of $\mathfrak{L}$. Moreover, if B, $C \in \mathscr{E}$ then $\partial(B \cap C) \subset \partial B \cup \partial C$ implies $Z_t^{\partial(B \cup C)} \leqslant Z_t^{\partial B} + z_t^{\partial C} = 0$; hence $B \bigcap C \in \mathscr{E}$.)

3. Let $\mathfrak{D}$ denote the system of sets $B \in \mathfrak{L}$ such that Z_t^B is measurable. Then $\mathfrak{D}$ is a Dynkin system in $\complement K(\gamma)$ that contains $\mathscr{E}$.

(Let $B \in \mathscr{E}$. Then in view of (i) we have $Z_t^B = \lim S_n^B$, hence $B \in \mathfrak{D}$. In particular, we have $\complement K(\gamma) \in \mathfrak{D}$ since $\partial(\complement K(\gamma)) = \partial K(\gamma)$ and $Z_t^{\partial K(\gamma)} = 0$. The remaining assertions are now obvious).

4. In view of 2. and 3. we have $\mathfrak{D} = \mathfrak{L}$. This proves the measurability of Z_t^B if $B \cap K(\varepsilon) = \emptyset$ for some $\varepsilon \in \boldsymbol{R}_+^*$. If, finally, $B \in \mathfrak{L}(G^*)$ is arbitrary, the assertion follows from $Z_t^B = \lim Z_t^{B \cap \complement K(1/n)}$.

Definition. Let $t \in I$. For every $B \in \mathfrak{B}(G^*)$ we put $\nu_t(B) = \mathrm{E}(Z_t^B)$ (where E denotes the mathematical expectation with respect to P). Obviously ν_t is a measure on $\mathfrak{B}(G^*)$. We call ν_t the *jump measure* of the process $(X_s)_{0 \leqslant s \leqslant t}$.

Example. Let G be the locally compact group of complex $d \times d$-matrices M with $|\det M| = 1$ and let λ be the (normed) Haar measure on the compact subgroup of unitary matrices in G. Let $(U_n)_{n \geqslant 1}$ be a sequence of independent random variables with values in G such that every U_n is distributed according to λ. Moreover, let $(N_t)_{t \geqslant 0}$ be a homogeneous Poisson process with intensity $\alpha \in \boldsymbol{R}_+^*$ starting in 0 and independent of $(U_n)_{n \geqslant 1}$. We put $T_0 = 0$ and $T_n = \inf\{t \in \boldsymbol{R}_+^*\colon N_t = n\}$ for all $n \in N$. Finally, let H be a Hermitian $d \times d$-matrix with trace 0. We put

$$X_t = \Big[\prod_{j=1}^{N_t} \exp\{(T_j - T_{j-1}) H\}\, U_j\Big] \exp\{(t - T_{N_t}) H\}$$

for all $t \in \boldsymbol{R}_+$. Then $(X_t)_{t \geqslant 0}$ is a process with values in G that has all the desired properties. Obviously we have $X_{t-}^{-1} X_t = U_{N_t}$ if $t = T_{N_t}$ and $=$ identity matrix in G if $t > T_{N_t}$. Hence $Z_t^B = 1_B(U_1) + \dots + 1_B(U_{N_t})$ for $B \in \mathfrak{B}(G^*)$. Thus Wald's identity yields:

$$\nu_t(B) = \mathrm{E}(Z_t^B) = \mathrm{E}(N_t)\,\mathrm{E}(1_B(U_1)) = \alpha t \lambda(B).$$

Thus ν_t is the restriction of $\alpha t \lambda$ to G^*.

The process $(X_t)_{t \geqslant 0}$ has been considered in connection with a problem in atomic physics [7].

Let now $(\Omega, \mathfrak{A}, P, (X_t)_{t\in I})$ be a separable stochastic process with values in G. By $\mathfrak{A}_t$ we denote the sub-σ-algebra of $\mathfrak{A}$ generated by $\{X_s;\ 0 \leqslant s \leqslant t\}$. For all $t\in I$ and δ, $\varepsilon \in R_+^*$ we define

$$\alpha_t(\varepsilon, \delta) = \inf\sup\{P(\varrho(X_r, X_s) \geqslant \varepsilon \mid \mathfrak{A}_r)(\omega):\ \omega\in\Omega',\ 0 \leqslant r \leqslant s \leqslant r+\delta \leqslant t\}$$

where the infimum is taken over all $\Omega' \in \mathfrak{A}$ with $P(\Omega') = 1$.

The following facts are well known (cf. [5], IV.4):

If the condition

$$\text{(C)} \qquad \lim_{\delta\to 0} \alpha_t(\varepsilon, \delta) = 0 \quad \text{for all } \varepsilon \in R_+^*$$

is fulfilled, then the process $(X_s)_{0\leqslant s\leqslant t}$ has no discontinuities of the second kind and is stochastically continuous. Moreover, the process can be modified in such a way that its paths are right continuous and admit left-hand limits. We shall therefore tacitely assume that a process satisfying (C) already enjoys these properties. Finally, if $(X_t)_{t\in I}$ is a separable stochastically continuous process with independent left increments, then it obviously has property (C) for all $t\in I$.

THEOREM 2. *Let the process $(X_t)_{t\in I}$ have the property (C) for all $t\in I$. For some fixed $t\in I$, $t>0$, let the sequences $(\Delta_n)_{n\geqslant 1}$ and $(S_n^B)_{n\geqslant 1}$, $B\in\mathfrak{B}(G^*)$, be defined as before Theorem 1 and let v_t be the jump measure of $(X_s)_{0\leqslant s\leqslant t}$.*

Then for $B\in\mathfrak{B}(G^*)$ such that $e\notin \bar{B}$ and $v_t(\partial B) = 0$ we have

$$\lim_{n\geqslant 1} \sum_{j=1}^{k(n)} \mu(c_{n,j-1}, c_{nj})(B) = v_t(B) \quad \textit{and} \quad v_t(B) < \infty.$$

Proof. First of all let $\varepsilon\in R_+^*$ such that $B \cap K(\varepsilon) = \emptyset$. In view of (C) we may then choose $t\in]0, 1[$ such that $\beta = 2\alpha_1(\varepsilon/4, t) < 1$. Now for every $k\in N$ we denote by $A^k(\varepsilon, \Delta_n)$ the event in $\mathfrak{A}$ that the process $(X_s)_{s\in I}$ has at least k ε-oscillations on Δ_n (i.e. there exist integers $j(1), \ldots, j(k)$ with $1 \leqslant j(1) < \ldots < j(k) \leqslant k(n)$ such that $\varrho(X(c_{n,j(i)-1}, c_{nj(i)}), e) > \varepsilon$ for $i = 1, \ldots, k$) (all $n\in N$). Then we have $P(A^k(\varepsilon, \Delta_n)) \leqslant \beta^k$ ([5], IV. 4, Lemma 3).

Obviously, $[S_n^B \geqslant k] \subset A^k(\varepsilon, \Delta_n)$, hence $P(S_n^B \geqslant k) \leqslant \beta^k$ and consequently

$$\mathrm{E}((S_n^B)^r) \leqslant \sum_{k\geqslant 0} P(S_n^B \geqslant k)(k+1)^r \leqslant \sum_{k\geqslant 0} \beta^k (k+1)^r < \infty$$

for all n, $r\in N$. Thus we have

$$(*) \qquad \sup\{\mathrm{E}((S_n^B)^r):\ n\in N\} < \infty \quad \text{for all } r\in N.$$

If now $t\in I$ is arbitrary, we decompose $[0, t]$ in a finite number of sufficiently small intervals and apply the conclusions above to each of them. Hence $(*)$ holds generally.

Finally, $\mathrm{E}(Z_t^{\partial B}) = v_t(\partial B) = 0$ yields $Z_t^{\partial B} = 0$ almost everywhere. Hence taking Theorem 1 (i) into account we conclude $\lim S_n^B = Z_t^B$ almost everywhere.

Consequently, with the aid of $(*)$ we obtain $\lim \mathrm{E}(S_n^B) = \mathrm{E}(Z_t^B) < \infty$. Hence the assertion.

COROLLARY 1. *Let* $U \in \mathfrak{V}(G)$ *such that* $\nu_t(\partial U) = 0$. *(By the proof of Theorem* 1 *(ii), these neighbourhoods* U *constitute a basis of* $\mathfrak{V}(G)$.*) Then the sequence*

$$\Big\{ \sum_{1 \leqslant j \leqslant k(n)} \mu(c_{n,j-1}, c_{nj}) \Big\}_{n \geqslant 1}$$

converges weakly to ν_t *on* $G \backslash U$.

Proof. This follows readily from the characterization of weak convergence in terms of sets of continuity (cf. [5], IX. 1).

COROLLARY 2. *The process* $(X_s)_{0 \leqslant s \leqslant t}$ *has continuous paths with probability* 1 *if and only if for all* $U \in \mathfrak{V}(G)$:

$$\lim_{n \geqslant 1} \sum_{1 \leqslant j \leqslant k(n)} \mu(c_{n,j-1}, c_{nj})(\complement U) = 0.$$

2. JUMPS OF PROCESSES WITH INDEPENDENT INCREMENTS

By I we either denote the interval $[0, 1]$ or the interval $\boldsymbol{R}_+$ and, accordingly, by S either the set $\{(s, t) \in \boldsymbol{R}^2 \colon 0 \leqslant s \leqslant t \leqslant 1\}$ or the set $\{(s, t) \in \boldsymbol{R}^2 \colon 0 \leqslant s \leqslant t\}$. Let $X = (\Omega, \mathfrak{A}, P, (X_t)_{t \in I})$ be a separable stochastically continuous process with values in the Polish group G such that $X_0 = e$. Let us assume that the process X has independent left increments: for every finite sequence $0 < t_1 < t_2 < \ldots < t_n$ in I the random variables $X(0, t_1), X(t_1, t_2), \ldots, X(t_{n-1}, t_n)$ are independent. Let us call the process X *additive* if in addition its paths are right continuous and admit left-hand limits. (As mentioned in Section 1 the process X can always be modified as to fulfill this last condition.) As usual let us call the additive process X *homogeneous* if for every $(s, t) \in S$ the distribution of $X(s, t)$ only depends on the difference $t - s$.

The distributions $\mu(s, t)$ of the increments $X(s, t)$, $(s, t) \in S$, of an additive process $(X_t)_{t \in I}$ with values in G form a (continuous) convolution hemigroup in $\mathscr{M}^1(G)$ (cf. [9, 10, 13]). Conversely, with every continuous convolution hemigroup $(\mu(s, t))_{(s,t) \in S}$ in $\mathscr{M}^1(G)$ there is associated an additive process $(X_t)_{t \in I}$ with values in G such that $\mu(s, t)$ is the distribution of $X_s^{-1} X_t$. (This follows from general results in the theory of stochastic processes. Cf. [3].)

Remark. Let $(\mu_t)_{t \geqslant 0}$ be a continuous convolution semigroup in $\mathscr{M}^1(G)$. Then by $\mu(s, t) = \mu_{t-s}$, $0 \leqslant s \leqslant t$, there is given a continuous convolution hemigroup in $\mathscr{M}^1(G)$. Let $(X_t)_{t \geqslant 0}$ be the (homogeneous) additive process associated with this hemigroup. Finally, let ν_t denote the jump measure of the process $(X_s)_{0 \leqslant s \leqslant t}$. Taking into account Corollary 1 of Theorem 2 we observe ν_t

$= t\nu_1$ for all $t \in \boldsymbol{R}_+$ and

$$\int f d\nu_1 = \lim_{s \downarrow 0} \frac{1}{s} \int f d\mu_s$$

for all $f \in \mathscr{C}^b(G)$ *with* $e \notin \operatorname{supp}(f)$.

If G is a locally compact group having a countable basis of its topology or a separable Fréchet space, then with $(\mu_t)_{t \geqslant 0}$ there is associated a so called Lévy measure η (via the Lévy-Khinchine formula; cf. [9] resp. [2]); and it holds

$$\int f d\eta = \lim_{s \downarrow 0} \frac{1}{s} \int f d\mu_s$$

for all $f \in \mathscr{C}^b(G)$ with $e \notin \operatorname{supp}(f)$. Thus η and ν_1 coincide: If $B \in \mathfrak{B}(G^*)$ then $\eta(B)$ is just the expected number of discontinuities of $(X_t)_{0 \leqslant t \leqslant 1}$ whose jumps lie in B.

We are now going to extend this observation to hemigroups on locally compact groups. Thus let G be a *locally compact group* having a countable basis of its topology. By $\mathscr{D}(G)$ we denote the space of real-valued infinitely differentiable functions with compact support on G (cf. [9], 4.4.2). If $(\mu_t)_{t \geqslant 0}$ is a continuous convolution semigroup in $\mathscr{M}^1(G)$, its generating functional A is defined by

$$A(f) = \lim_{t \downarrow 0} \frac{1}{t} \int (f - f(e))\, d\mu_t \qquad \text{for all } f \in \mathscr{D}(G).$$

The *Lévy measure* of A (or of $(\mu_t)_{t \geqslant 0}$) is the unique measure η on G^* such that $\int f d\eta = A(f)$ for all $f \in \mathscr{D}(G)$ with $e \notin \operatorname{supp}(f)$ (cf. [9], 4.4). By $A(G)$ let us denote the set of generating functionals of convolution semigroups in $\mathscr{M}^1(G)$.

A continuous convolution hemigroup $(\mu(s,t))_{(s,t) \in S}$ in $\mathscr{M}^1(G)$ is said to be of *bounded variation* (cf. [13]) if there exists a mapping $B(\cdot)$ of I into $A(G)$ with $B(0) = 0$ and $B(t) - B(s) \in A(G)$, $(s,t) \in S$, and a continuous isotone mapping v of I into itself such that the following holds:

For every $f \in \mathscr{D}(G)$ there exists a $c(f) \in \boldsymbol{R}_+^*$ such that for all $(s,t) \in S$

$$\text{(E)} \qquad \left| \left\{ \int f d\mu(s,t) - f(e) \right\} - \left\{ B(t)(f) - B(s)(f) \right\} \right| \leqslant c(f) \big(v(t) - v(s)\big)^2.$$

If $(\mu_t)_{t \geqslant 0}$ is a continuous convolution semigroup in $\mathscr{M}^1(G)$ with generating functional A, then the associated hemigroup is of bounded variation: one only has to define $B(t) = tA$ and $v(t) = t$ for all $t \in \boldsymbol{R}_+$. In general not every hemigroup $(\mu(s,t))_{(s,t) \in S}$ is of bounded variation. But if G is a Lie group, then, according to [4], there always exists a continuous mapping $x(\cdot)$ of I into G such that

$$(\varepsilon_{x(s)} * \mu(s,t) * \varepsilon_{x(t)^{-1}})_{(s,t) \in S}$$

is a hemigroup of bounded variation.

THEOREM 3. *Let $(\mu(s,t))_{(s,t)\in S}$ be a continuous convolution hemigroup of bounded variation in $\mathscr{M}^1(G)$. Let $B(\cdot)$ be the mapping of I into $A(G)$ associated with the hemigroup by* (E) *and let η_t denote the Lévy measure of $B(t)$, $t\in I$. Finally, let $(X_t)_{t\in I}$ denote the additive process associated with the hemigroup, and let ν_t denote the jump measure of the process $(X_s)_{0\leqslant s\leqslant t}$, $t\in I$.*

Then for every $t\in I$ the measures η_t and ν_t coincide.

Proof. We fix some $t\in I$, $t>0$. Let $(\Delta_n)_{n\geqslant 1}$ be a sequence of decompositions $\Delta_n=\{c_{n0}, c_{n1}, \ldots, c_{nk(n)}\}$ of $[0,t]$ such that $\lim|\Delta_n|=0$. Now let $f\in\mathscr{D}(G)$ with $e\notin \operatorname{supp}(f)$. Then there exists a $U\in\mathfrak{B}(G)$ such that $U\cap\operatorname{supp}(f)=\emptyset$ and $\nu_t(\partial U)=0$. Thus in view of Corollary 1 of Theorem 2 we have

$$\lim_{n\geqslant 1}\sum\int f\,d\mu(c_{n,j-1}, c_{nj})=\int f\,d\nu_t.$$

On the other hand, in view of (E) we have the following chain of inequalities:

$$\begin{aligned}
\Big|\sum\int f\,d\mu(c_{n,j-1}, c_{nj})-\int f\,d\eta_t\Big| &= \Big|\sum\int f\,d\mu(c_{n,j-1}, c_{nj})-B(t)(f)\Big|\\
&=\Big|\sum\int f\,d\mu(c_{n,j-1}, c_{nj})-\sum\{B(c_{nj})(f)-B(c_{n,j-1})(f)\}\Big|\\
&\leqslant\sum\Big|\Big\{\int f\,d\mu(c_{n,j-1}, c_{nj})-f(e)\Big\}-\{B(c_{nj})(f)-B(c_{n,j-1})(f)\}\Big|\\
&\leqslant c(f)\sum\{v(c_{nj})-v(c_{n,j-1})\}^2\\
&\leqslant c(f)\,v(t)\sup\{v(c_{nj})-v(c_{n,j-1}):\ 1\leqslant j\leqslant k(n)\}.
\end{aligned}$$

(The summation is always to be extended over $j=1,\ldots,k(n)$.)

In view of $\lim|\Delta_n|=0$ and of the continuity of v, this yields

$$\lim_{n\geqslant 1}\sum\int f\,d\mu(c_{n,j-1}, c_{nj})=\int f\,d\eta_t.$$

Consequently, $\int f\,d\nu_t=\int f\,d\eta_t$ for all $f\in\mathscr{D}(G)$ with $e\notin\operatorname{supp}(f)$. This yields $\nu_t=\eta_t$ since the system of these functions is sufficiently rich.

COROLLARY. *Let $(\mu(s,t))_{(s,t)\in S}$ be a continuous convolution hemigroup in $\mathscr{M}^1(G)$ and $(X_t)_{t\in I}$ the associated additive process. Let us consider the following assertions:*

(i) $$\lim_{t-s\to 0}\frac{1}{t-s}\mu(s,t)(\complement U)=0 \textit{ for every } U\in\mathfrak{B}(G),\qquad (s,t)\in S.$$

(ii) *The paths of the process $(X_t)_{t\in I}$ are continuous with probability* 1.

Then (i) *implies* (ii).

If the hemigroup $(\mu(s,t))_{(s,t)\in S}$ is of bounded variation with respect to $v(t)=t$, $t\in I$, then (i) *and* (ii) *are equivalent.*

Proof. (i)$\Rightarrow$(ii). Let $\varepsilon\in\boldsymbol{R}_+^*$ and $U\in\mathfrak{B}(G)$ be given. Then there exists a $\delta\in\boldsymbol{R}_+^*$ such that $\dfrac{1}{t-s}\mu(s,t)(\complement U)\leqslant\varepsilon$ for all $(s,t)\in S$ with $0<t-s<\delta$. Consequently, for every decomposition $\Delta = \{c_1, \ldots, c_n\}$ of $[0, r]$, $r \in I$, with $|\Delta| < \delta$, we have

$$\sum_{1 \leqslant j \leqslant n} \mu(c_{j-1}, c_j)(CU) \leqslant \varepsilon.$$

Hence the assertion by Corollary 2 of Theorem 2.

(ii) $\Rightarrow$ (i). Let the hemigroup $(\mu(s, t))_{(s,t)\in S}$ be of bounded variation with respect to $B(\cdot)$, and $v(t) = t$, $t \in I$. In view of Theorem 3 the Lévy measure η_t of $B(t)$ is 0. Given $U \in \mathfrak{B}(G)$ there exist $V \in \mathfrak{B}(G)$ and $f \in \mathscr{D}(G)$ such that $1_V \leqslant f \leqslant f(e) = 1$ and $f(e) - f \geqslant 1_{CU}$. Consequently, $B(t)(f) = 0$ for all $t \in I$ (cf. [9], 4.5.9). Hence, in view of (E), we have, for all $(s, t) \in S$,

$$\mu(s, t)(CU) \leqslant \left|\int f\, d\mu(s, t) - f(e)\right| \leqslant c(f)(t-s)^2.$$

Hence the assertion.

Remark. Let $(X_t)_{t\in I}$ be an additive process taking its values in a separable Banach space. Then the distribution of X_t is infinitely divisible, hence admits a Lévy-Khinchine representation with Lévy measure η_t $(t \in I)$. It is well known that η_t coincides with the jump measure ν_t of $(X_s)_{0\leqslant s\leqslant t}$ (cf. [5], Chapter 6; [6], Chapter 4).

3. JUMP MEASURES AND SUBMULTIPLICATIVE FUNCTIONS

Let G be a Polish group and φ a continuous *submultiplicative* function on G, i.e. φ is a continuous function of G into $\boldsymbol{R}_+^*$ such that $\varphi(xy) \leqslant \varphi(x)\varphi(y)$ for all $x, y \in G$ (cf. [14]).

Now let $X = (\Omega, \mathfrak{A}, P, (X_t)_{t\geqslant 0})$ be an additive process with values in G. Again let $\mathfrak{A}_t$ denote the sub-σ-algebra of $\mathfrak{A}$ generated by $\{X_s; s \leqslant t\}$. A function T of Ω into $[0, \infty]$ is said to be a Markov time for X if $[T \leqslant t] \in \mathfrak{A}_t$ for all $t \in \boldsymbol{R}_+$. As usual we define

$$\mathfrak{A}_T = \{A \in \mathfrak{A}: A \cap [T \leqslant t] \in \mathfrak{A}_t \text{ for all } t \in \boldsymbol{R}_+\}.$$

The process X possesses the *strong Markov property* in the following sense: For every product measurable subset B of $G^{\boldsymbol{R}_+}$ the random variable

$$\omega \to 1_{[T<\infty]}(\omega)\, P\big((X_{T(\omega)}(\omega)^{-1} X_t)_{t\geqslant 0} \in B\big)$$

is a version of the conditional probability

$$P\big([T < \infty] \cap [(X_{t+T})_{t\geqslant 0} \in B] \mid \mathfrak{A}_T\big).$$

THEOREM 4. *Let φ be a continuous submultiplicative function on G and put $\tau = \ln \varphi$. Moreover, let there exist some $a \in \boldsymbol{R}_+^*$ such that $\sup\{\tau(X_{t-}^{-1} X_t): t \in \boldsymbol{R}_+\} \leqslant a$ with probability* 1.

Then we have

$$\int (\sup\{\varphi(X_s):\ 0 \leqslant s \leqslant \alpha\})\, dP < \infty$$

for all $\alpha \in \boldsymbol{R}_+^*$.

Proof. Let us fix $\alpha \in \boldsymbol{R}_+^*$ and $r \in [0, \alpha]$. We put $Y(t) = X_r^{-1} X_{r+t}$ for all $t \in \boldsymbol{R}_+$. Then $Y = (Y(t))_{t \geqslant 0}$ is a process with the same properties as $(X_t)_{t \geqslant 0}$. For this proof let the σ-algebras $\mathfrak{A}_t$ and $\mathfrak{A}_T$ be defined with respect to the process Y.

Let now some $c \in \boldsymbol{R}_+^*$ be chosen such that $(0 \leqslant)\ \tau(e) < c - a$. Then for all $t \in \boldsymbol{R}_+^*$ and $n \in \boldsymbol{N}$ we define

$$A_n^t = [\sup\{\tau(Y(s)):\ 0 \leqslant s < t\} > nc]$$

and

$$T_n = \inf\{t \in \boldsymbol{R}_+:\ \tau(Y(t)) > nc\}.$$

Obviously, every T_n is a Markov time for Y, and $A_n^t = [T_n < t] \in \mathfrak{A}_{T_n}$. Taking into account $\tau(Y(T_n -)) \leqslant nc$, our assumption yields:

$$\tau(Y(T_n)) \leqslant \tau(Y(T_n -)) + \tau(Y(T_n -)^{-1} Y(T_n)) \leqslant nc + a.$$

Consequently,

$$A_n^t \subset A_{n-1}^t \cap [\sup\{\tau(Y(T_{n-1})^{-1} Y(s)):\ T_{n-1} \leqslant s < t\} > c - a].$$

Let

$$A^t = [\sup\{\tau(X_u^{-1} X_{u+s}):\ 0 \leqslant u \leqslant \alpha,\ 0 \leqslant s \leqslant t\} > c - a].$$

Then the strong Markov property yields:

$$\begin{aligned}
P(A_n^t) &\leqslant P\big(A_{n-1}^t \cap [\sup\{\tau(Y(T_{n-1})^{-1} Y(s)):\ T_{n-1} \leqslant s < t\} > c - a]\big)\\
&= \int 1(A_{n-1}^t)\, P\big([\sup\{\tau(Y(T_{n-1})^{-1} Y(s)):\ T_{n-1} \leqslant s < t\}\\
&\qquad\qquad > c - a] \,|\, \mathfrak{A}_{T_{n-1}}\big)\, dP\\
&\leqslant \int 1(A_{n-1}^t)\, P\big([T_{n-1} < \infty] \cap [\sup\{\tau(Y(T_{n-1})^{-1} Y(T_{n-1} + s)):\\
&\qquad\qquad 0 \leqslant s < t\} > c - a] \,|\, \mathfrak{A}_{T_{n-1}}\big)\, dP\\
&= \int 1(A_{n-1}^t)\, P\big([T_{n-1} < \infty] \cap [\sup\{\tau(Y(s)):\ 0 \leqslant s < t\} > c - a]\big)\, dP\\
&\leqslant P(A_{n-1}^t)\, P(A^t).
\end{aligned}$$

(For typographical reasons we have here the indicator function of A_{n-1}^t denoted by $1(A_{n-1}^t)$.)

Hence, by induction, $P(A_n^t) \leqslant (P(A^t))^n$ for all $n \in \boldsymbol{N}$. By the uniform stochastic continuity on compact intervals of the process X (cf. [5], IV.2, Theorem 4) we have $\lim_{t\downarrow 0} P(A^t) = 0$. Hence there exist $\delta > 1$ and $t > 0$ such that $\delta P(A^t) < 1$ and $c < -\ln(\delta P(A^t))$. Thus, for all $r \in [0, \alpha]$,

$$\begin{aligned} &\int(\sup\{\varphi(X_r^{-1} X_{r+s})\colon\ 0 \leqslant s < t\})\, dP \\ &\quad = \int(\exp(\sup\{\tau(Y(s))\colon\ 0 \leqslant s < t\}))\, dP \\ &\quad \leqslant \sum_{n \geqslant 0} e^{nc}(P(A^t))^n = \beta < \infty. \end{aligned}$$

Moreover, by the stochastic independence of increments, we have:

$$\begin{aligned} &\int(\sup\{\varphi(X_{r+s})\colon\ 0 \leqslant s < t\})\, dP \\ &\quad \leqslant \left(\int \varphi(X_r)\, dP\right) \int(\sup\{\varphi(X_r^{-1} X_{r+s})\colon\ 0 \leqslant s < t\})\, dP \\ &\quad \leqslant \beta\left(\int \varphi(X_r)\, dP\right) \quad \text{for all } r \in [0, \alpha]. \end{aligned}$$

Hence the assertion by covering $[0, \alpha]$ with finitely many intervals of length $t/2$.

Remarks.

1. The proof of this theorem has been strongly influenced by the proof of a related (but weaker) result due to Tortrat [16]. In fact, the basic idea already appears in Skorohod [15].

2. Let ν_t denote the jump measure of the process $(X_s)_{0 \leqslant s \leqslant t}$, $t \in \boldsymbol{R}_+$, and let $B(\tau, a) = \{x \in G\colon\ \tau(x) \leqslant a\}$. In view of the definition of jump measures the assumption

$$\sup\{X_{t-}^{-1} X_t)\colon\ t \in \boldsymbol{R}_+\} \leqslant a \quad \text{with probability } 1$$

(of Theorem 4) is equivalent with $\nu_t(\mathbf{C}B(\tau, a)) = 0$ for all $t \in \boldsymbol{R}_+$.

THEOREM 5. *Let G be a locally compact group having a countable basis of its topology or a separable Fréchet space. Let $(\mu_t)_{t \geqslant 0}$ be a continuous convolution semigroup in $\mathscr{M}^1(G)$ with Lévy measure η. Finally, let φ be a continuous submultiplicative function on G. Then the following assertions are equivalent*:

(i) $\int \varphi\, d\mu_t < \infty$ *for some* $t \in \boldsymbol{R}_+^*$.

(ii) $\int 1_{\mathbf{C}U}\, \varphi d\eta < \infty$ *for all* $U \in \mathfrak{B}(G)$.

Proof. Let $U \in \mathfrak{B}(G)$. Without loss of generality we may assume $U = [\tau < a]$ for some $a \in \boldsymbol{R}_+^*$ (since $\eta(\mathbf{C}V) < \infty$ for all $V \in \mathfrak{B}(G)$). Let $\varkappa$ denote the restriction of η to $\mathbf{C}U$. Now $\varkappa$ is finite. Hence for every $t \in \boldsymbol{R}_+$ there exists $e(t\varkappa) = \varepsilon_e + t\varkappa + (t^2/2)\varkappa * \varkappa + \ldots$, and $(e(t\varkappa))_{t \geqslant 0}$ is a continuous convolution semigroup in $\mathscr{M}^1(G)$ (Poisson semigroup with exponent $\varkappa$). By the Lévy - Khinchine formula (which in both cases is available; cf. [9] resp. [2]) there exists a continuous convolution semigroup $(\nu_t)_{t \geqslant 0}$ in $\mathscr{M}^1(G)$ such that the generating functional of $(\mu_t)_{t \geqslant 0}$ is the sum of the generating functionals of $(\nu_t)_{t \geqslant 0}$ and

$(e(t\varkappa))_{t\geqslant 0}$. Moreover, every μ_t can be represented by a norm convergent perturbation series ([8], p. 11, p. 61):

$$\mu_t = \exp\{-t\varkappa(G)\} \sum_{k\geqslant 0} u_k(t),$$

$$u_0(t) = \nu_t, \qquad u_k(t) = \int_0^t \nu_r * \varkappa * u_{k-1}(t-r)\,dr \qquad (k\in N).$$

(i) $\Rightarrow$ (ii) From

$$\mu_t \geqslant \exp\{-t\varkappa(G)\} \int_0^t \nu_r * \varkappa * \nu_{t-r}\,dr$$

we have $\int \varphi d\varkappa < \infty$ (cf. [14], Proof of Theorem 1).

(ii) $\Rightarrow$ (i) Let $(Y_t)_{t\geqslant 0}$ denote the additive process associated with $(\nu_t)_{t\geqslant 0}$. Since the Lévy measure of $(\nu_t)_{t\geqslant 0}$ is the restriction of η to $U\backslash\{e\}$, in view of the first remark in Section 2 we have:

$$\sup\{\tau Y_{t-}^{-1} Y_t): t\in \boldsymbol{R}_+\} \leqslant a \quad \text{with probability } 1.$$

Hence Theorem 4 yields $a(t) = \sup\{\int \varphi d\nu_s:\ 0\leqslant s\leqslant t\} < \infty$ for every $t\in \boldsymbol{R}_+$. Now with the aid of the perturbation series we obtain:

$$\int \varphi d\mu_t \leqslant a(t)\exp\{t(a(t)\int \varphi d\varkappa - \varkappa(G))\} < \infty.$$

Remarks.

1. For a locally compact group (not necessarily having a countable basis of its topology) Theorem 5 has been proved in [14] by purely analytical methods. Let us mention that in [14] the result has been established for a slightly larger class of submultiplicative functions.

For a separable Hilbert space Theorem 5 has been proved in [12]. For a separable Banach space proofs for this result have been given in [1] and [11].

Our proof has the advantage that it works in both cases and, more generally, on every Polish group where one has a Lévy-Khinchine formula for the continuous convolution semigroups.

2. Let G be a Polish group and let $(X_t)_{t\geqslant 0}$ be the homogeneous additive process associated with the continuous convolution semigroup $(\mu_t)_{t\geqslant 0}$ in $\mathscr{M}^1(G)$. If φ is a continuous submultiplicative function on G, then $\int \varphi d\mu_t < \infty$ for some $t\in \boldsymbol{R}_+^*$ is equivalent with

$$\int \sup\{\varphi(X_s):\ 0\leqslant s\leqslant t\}\,dP < \infty$$

for all $t\in \boldsymbol{R}_+^*$ ([14], Theorem 2).

REFERENCES

[1] A. de Acosta, *Exponential moments of vector valued random series and triangular arrays*, Ann. Probab. 8 (1980), p. 381-389.

[2] E. Dettweiler, *Grenzwertsätze für Wahrscheinlichkeitsmasse auf Badrikianschen Räumen*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 34 (1976), p. 285-311.

[3] E. B. Dynkin, *Die Grundlagen der Theorie der Markoffschen Prozesse*, Springer, Berlin-Göttingen-Heidelberg 1961.

[4] Ph. Feinsilver, *Processes with independent increments on a Lie group*, Trans. Amer. Math. Soc. 242 (1978), p. 73-121.

[5] I. I. Gihman and A. V. Skorohod, *Introduction to the Theory of Random Processes*, Saunders, Philadelphia-London-Toronto 1969.

[6] — *The Theory of Stochastic Processes II*, Springer, Berlin-Heidelberg-New York 1975.

[7] L. Hantsch and W. von Waldenfels, *A random walk on the general linear group related to a problem in atomic physics*, in: *Probability Measures on Groups. Proceedings*, Oberwolfach 1978, p. 131-143. Lecture Notes in Math. 706, Springer, Berlin-Heidelberg-New York, 1979.

[8] W. Hazod, *Stetige Faltungshalbgruppen von Wahrscheinlichkeitsmassen und erzeugende Distributionen*, Lecture Notes in Math. 595, Springer, Berlin-Heidelberg-New York 1977.

[9] H. Heyer, *Probability Measures on Locally Compact Groups*, Springer, Berlin-Heidelberg-New York 1977.

[10] — *Stetige Hemigruppen von Wahrscheinlichkeitmassen und additive Prozesse auf einer lokalkompakten Gruppe*, Nieuw Arch. Wiskunde 27 (1979), p. 287-340.

[11] Z. J. Jurek and J. Smalara, *On integrability with respect to infinitely divisible measures*, Bull. Acad. Polon. Sci. 29 (1981), p. 179-185.

[12] V. M. Kruglov, *On unboundedly divisible distributions in Hilbert space*, Math. Notes Acad. Sci. USSR 16 (1974), p. 940-946.

[13] E. Siebert, *Continuous hemigroups of probability measures on a Lie group*, in: *Probability Measures on Groups*, Proceedings, Oberwolfach 1981, p. 362-402. Lecture Notes in Math. 928, Springer, Berlin-Heidelberg-New York 1982.

[14] — *Continuous convolution semigroups integrating a submultiplicative function*, to appear in: *Manuscripta math.* 37 (1982), p. 383-391.

[15] A. V. Skorohod, *A note on Gaussian measures in a Banach space*, Theory Probab. Appl. 15 (1970), p. 508.

[16] A. Tortrat, *Lois indéfiniment divisibles et théorèmes des Ito-Nisio er Yuriskii*, Ann Inst. H. Poincaré 15 (1979), p. 85-92.

Mathematisches Institut
der Universität Tübingen
Auf der Morgenstelle 10
D-7400 Tubingen 1, Bundesrepublik Deutschland

Received on 29. 4. 1982