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PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 6, Fasc. 1 (1985), p. 1-10

ROBUST TESTS AGAINST DEPENDENCE

BY

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Abstract. When testing simple hypotheses $\bigotimes_{i=1}^{n} P_i$, $\bigotimes_{i=1}^{n} Q_i$ in a robust framework one usually considers neighbourhoods of P_i and Q_i in terms of ε-contamination or total variation, which are describable in terms of capacities. In the present paper we consider neighbourhoods which allow any departure from independence, but retain the marginals P_i, Q_i of the test problem, i.e. we consider the extreme case, where exact measurement of the components is possible but no assumptions can be made about the independence.

1. Introduction. Let $(X_i, \mathfrak{A}_i)$ be measure spaces, let $M^1(X_i, \mathfrak{A}_i)$ denote the set of all probability measures on $(X_i, \mathfrak{A}_i)$, $1 \leqslant i \leqslant n$, and define

$$(X, \mathfrak{A}) = \bigotimes_{i=1}^{n} (X_i, \mathfrak{A}_i).$$

Furthermore, for $P_i, Q_i \in M^1(X_i, \mathfrak{A}_i)$, $1 \leqslant i \leqslant n$, define

$$M_1 = M(P_1, \ldots, P_n) = \{P \in M^1(X, \mathfrak{A});\ \pi_i(P) = P_i,\ 1 \leqslant i \leqslant n\}$$

and

$$M_2 = M(Q_1, \ldots, Q_n) = \{P \in M^1(X, \mathfrak{A});\ \pi_i(P) = Q_i,\ 1 \leqslant i \leqslant n\},$$

where π_i denotes the i-th projection on X and $\pi_i(P)$ denotes the image measure. M_1, M_2 are neighbourhoods of $\bigotimes_{i=1}^{n} P_i$, $\bigotimes_{i=1}^{n} Q_i$ containing all probability measures with i-th marginals P_i, Q_i and arbitrary dependence structure.

The robust test-model M_1, M_2 cannot be described in terms of capacities as the usual ε-contamination or total variation models and there do not exist least favourable pairs in the sense of Huber and Strassen [5]. We shall instead determine least favourable pairs as introduced by Baumann [1] which depend on the level α and on n. It turns out that there is a large number of least favourable pairs and that for the determination of a robust test it is helpful to choose a suitable pair. To do this we develop in Section 2 some tools which seem to be of some independent interest.

It seems possible that similar methods as presented for our model will also be applicable to robust test models which are caused by dependence and which are less extreme as M_1, M_2 are (f.i. considering only positive (negative) dependence or intersections with total variation neighbourhoods, but the author did not succeed in this point so far).

2. Measures with given marginals. Let $M(X, \mathfrak{A})$ be the set of finite measures on $(X, \mathfrak{A})$. For $P \in M(X, \mathfrak{A})$ define $|P| = P(X)$ and for $R_i \in M(X_i, \mathfrak{A}_i)$, $1 \leqslant i \leqslant n$, with $|R_1| = \ldots = |R_n|$ define $M(R_1, \ldots, R_n)$ as in Section 1. Clearly, $M(R_1, \ldots, R_n) \neq \emptyset$.

Let for measures $P, Q \in M(X, \mathfrak{A})$, $P \leqslant Q$, be defined as $P(A) \leqslant Q(A)$, $A \in \mathfrak{A}$. Then the following lemma is trivial:

LEMMA 1. *If* $R_i \in M(X_i, \mathfrak{A}_i)$, $1 \leqslant i \leqslant n$, *and*

$$|R_1| = \min_i |R_i|,$$

then there exists an $R \in M(X, \mathfrak{A})$ *with* $\pi_1(R) = R_1$ *and* $\pi_i(R) \leqslant R_i$ $(2 \leqslant i \leqslant n)$.

For $P, Q \in M(X, \mathfrak{A})$ define $P \wedge Q \in M(X, \mathfrak{A})$ by

$$P \wedge Q(A) = \inf\{P(AB) + Q(AB^c);\ B \in \mathfrak{A}\}$$

(B^c denoting the complement of B) and

$$d_v(P, Q) = \sup\{P(B) - Q(B);\ B \in \mathfrak{A}\}.$$

By a simple calculation $d_v(P, Q) = |P| - |P \wedge Q|$.

Define for $\mathscr{P}_i \subset M(X, \mathfrak{A})$, $i = 1, 2$,

$$d_v(\mathscr{P}_1, \mathscr{P}_2) = \inf\{d_v(P_1, P_2);\ P_i \in \mathscr{P}_i,\ i = 1, 2\}.$$

The following proposition will be important for finding least favourable pairs for the testproblem M_1, M_2.

PROPOSITION 2. *Let* $P_i, Q_i \in M(X_i, \mathfrak{A}_i)$, $1 \leqslant i \leqslant n$, *with* $|P_1| = \ldots = |P_n|$, $|Q_1| = \ldots = |Q_n|$. *Then*

$$d_v\big(M(P_1, \ldots, P_n), M(Q_1, \ldots, Q_n)\big) = \max_{1 \leqslant i \leqslant n} d_v(P_i, Q_i).$$

Proof. The statement of Proposition 2 is equivalent to

$$\sup\{|P \wedge Q|;\ P \in M(P_1, \ldots, P_n),\ Q \in M(Q_1, \ldots, Q_n)\} = \min_{1 \leqslant i \leqslant n} |P_i \wedge Q_i|.$$

Define $S_i = P_i \wedge Q_i$, $1 \leqslant i \leqslant n$, and assume that

$$|S_1| = \min_{1 \leqslant i \leqslant n} |S_i|.$$

By Lemma 1 there exists an $R \in M(X, \mathfrak{A})$ with $\pi_1(R) = S_1$ and $\pi_i(R) \leqslant S_i$, $2 \leqslant i \leqslant n$. Now defining $P_i' = P_i - \pi_i(R)$, $Q_i' = Q_i - \pi_i(R)$, $1 \leqslant i \leqslant n$, we obtain

$$|P_i'| = P_i(X_i) - (\pi_i(R))(X_i) = P_1(X_1) - R(X)$$
$$= |P_1| - (\pi_1(R))(X_1) = |P| - |P_1 \wedge Q_1|$$

and, similarly,

$$|Q_i'| = |Q_1| - |P_1 \wedge Q_1|, \quad 1 \leqslant i \leqslant n.$$

Let $R_1' \in M(P_1', \ldots, P_n')$, $R_2' \in M(Q_1', \ldots, Q_n')$ and define $R_1 = R + R_1'$, $R_2 = R + R_2'$. Clearly, $R \leqslant R_i$ $(i = 1, 2)$, $R_1 \in M(P_1, \ldots, P_n)$, $R_2 \in M(Q_1, \ldots, Q_n)$ and

$$|R_1 \wedge R_2| \geqslant |R| = \min_{1 \leqslant i \leqslant n} |P_i \wedge Q_i|.$$

On the other hand, for $P \in M(P_1, \ldots, P_n)$, $Q \in M(Q_1, \ldots, Q_n)$ the bound

$$|P \wedge Q| \leqslant \min_{1 \leqslant i \leqslant n} |P_i \wedge Q_i|$$

is obvious by definition. Therefore

$$|R_1 \wedge R_2| = |R| = \min_{1 \leqslant i \leqslant n} |P_i \wedge Q_i|$$

(and any pair R_1', R_2' is orthogonal!), which implies Proposition 2.

Remark 1. (a) The proof of Proposition 2 shows that there are many pairs (R_1, R_2) minimizing the distance d_v between $M(P_1, \ldots, P_n)$ and $M(Q_1, \ldots, Q_n)$ and how to construct them.

(b) Assume that $|P_i| = |Q_i| = 1$, $1 \leqslant i \leqslant n$. For the product measures we get the (probably well known) bounds:

$$d_v(\otimes P_i, \otimes Q_i) \leqslant 1 - \prod_{i=1}^{n} \left(1 - d_v(P_i, Q_i)\right), \tag{1}$$

$$d_v(\otimes P_i, \otimes Q_i) \geqslant \prod_{i=1}^{n} P_i(A_i) - \prod_{i=1}^{n} Q_i(A_i), \quad A_i \in \mathfrak{A}_i. \tag{2}$$

For the proof of relation (1) observe that by Fubini's theorem and induction on n one obtains

$$|\otimes P_i \wedge \otimes Q_i| \geqslant \prod_{i=1}^{n} |P_i \wedge Q_i|.$$

From relation (2) we see that the independent case does not typically correspond to a least favourable situation. To be more precise let

$$d_v(P_1, Q_1) = \max_{1 \leqslant j \leqslant n} d_v(P_j, Q_j),$$

let $A_1 = \{dP_1/dQ_1 \geqslant 1\}$ be a Jordan–Hahn set and assume that $Q_1(A_1) > 0$.

If there exists an i, $2 \leqslant i \leqslant n$, *such that* $Q_i(A_i) > 0$, *where* $A_i = \{dQ_i/dP_i > P_1(A_1)/Q_1(A_1)\}$, *then*

$$d_v(\otimes P_j, \otimes Q_j) > \max d_v(P_j, Q_j).$$

Proof. Since

$$\frac{1-Q_i(A_i^c)}{1-P_i(A_i^c)} = \frac{Q_i(A_i)}{P_i(A_i)} > \frac{P_1(A_1)}{Q_1(A_1)},$$

we obtain

$$d_v(\otimes P_j, \otimes Q_j) \geqslant P_1(A_1)P_i(A_i^c) - Q_1(A_1)Q_i(A_i^c)$$
$$> P_1(A_1) - Q_1(A_1) = \max d_v(P_j, Q_j).$$

(c) *If* $n = 2$, X_i $(i = 1, 2)$ *are Polish spaces*, $|P_i| = |Q_i| = 1$, $|P_1 \wedge Q_1| \leqslant |P_2 \wedge Q_2|$ *and* $Q \in M(Q_1, Q_2)$ *such that*

$$Q(A \times B) \geqslant P_1 \wedge Q_1(A) + P_2 \wedge Q_2(B) - |P_2 \wedge Q_2|$$

for all $A \in \mathfrak{A}_1$, $B \in \mathfrak{A}_2$, *then*

$$d_v(M(P_1, P_2), Q) = d_v(M(P_1, P_2), M(Q_1, Q_2)),$$

i.e. one can find a $P \in M(P_1, P_2)$ *such that the pair* P, Q *is a "least favourable" pair in* M_1, M_2.

Proof. By Theorem 4 of Hansel and Troallic [4] our assumption implies the existence of an $R \in M(X, \mathfrak{A})$ with $\pi_1(R) = P_1 \wedge Q_1$, $\pi_2(R) \leqslant P_2 \wedge Q_2$ and $R \leqslant Q$. Therefore, the proof of Proposition 2 implies our statement.

A similar but more complicated sufficient condition can be given for $n \geqslant 2$ using Theorem 1 of Gaffke and Rüschendorf [3].

Let now $(X_1, \mathfrak{A}_1) = \ldots = (X_n, \mathfrak{A}_n)$ and define, for $B \in \mathfrak{A}_1$,

$$\Delta_n(B) = \{(x, \ldots, x) \in X;\ x \in B\}.$$

Assume that $\Delta_n(B)$ is measurable. In the following proposition we construct a measure with given marginals which is maximally concentrated on the diagonal Δ_n.

Proposition 3. *Let* $P_i \in M(X_1, \mathfrak{A}_1)$, $1 \leqslant i \leqslant n$, *with* $|P_1| = \ldots = |P_n|$. *Then:*

(a) $\sup\{P(\Delta_n(B)); P\in M(P_1,\ldots,P_n)\} = P_1 \wedge \ldots \wedge P_n(B)$, *where* $B\in \mathfrak{A}_1$.
(b) *There exists an* $R_1 \in M(P_1,\ldots,P_n)$ *such that*

$$R_1(\Delta_n(B)) = P_1 \wedge \ldots \wedge P_n(B), \quad B\in \mathfrak{A}_1.$$

Proof. Let $(M, \mathfrak{B}, \mu)$ be a measure space, let $f: (M, \mathfrak{B}) \to (X_1, \mathfrak{A}_1)$ be such that $f(\mu) = P_1 \wedge \ldots \wedge P_n$ and define $R\in M(X, \mathfrak{A})$ by $R = (f,\ldots,f)(\mu)$ – the image of μ under $(f,\ldots,f)$. Then,

$$R(\Delta_n(B)) = f(\mu)(B) = P_1 \wedge \ldots \wedge P_n(B)$$

for all $B\in \mathfrak{A}_1$. By Lemma 1 there exists an $R_1\in M(P_1,\ldots,P_n)$ with $R\leqslant R_1$ and, therefore, $R_1(\Delta_n(B)) \geqslant P_1 \wedge \ldots \wedge P_n(B)$. On the other hand, let

$$B = \sum_{i=1}^{n} B_i$$

be a measurable disjoint partition of B and let $P\in M(P_1,\ldots,P_n)$. Then

$$P(\Delta_n(B)) = \sum_{i=1}^{n} P\{(x,\ldots,x)\in X;\ x\in B_i\} \leqslant \sum_{i=1}^{n} P_i(B_i),$$

which implies

$$P(\Delta_n(B)) \leqslant \inf\{\sum_{i=1}^{n} P_i(B_i);\ B = \sum_{i=1}^{n} B_i\} = P_1 \wedge \ldots \wedge P_n(B).$$

Remark 2. (a) If $n=2$ and $B = X_1$, Proposition 3 yields for $P_i\in M^1(X_1, \mathfrak{A}_1)$, $i=1, 2$,

$$\tilde{d}(P_1, P_2) = \inf\{P\{(x, y);\ x\neq y\};\ P\in M(P_1, P_2)\} = d_v(P_1, P_2).$$

This result on the Wasserstein-distance $\tilde{d}$ is due to Dobrushin [2].

(b) Some further optimization problems concerning $M(P_1,\ldots,P_n)$ are considered in Rüschendorf [6].

LEMMA 4. *If* $P_i\in M(X_1, \mathfrak{A}_1)$, $1\leqslant i\leqslant n$, *and*

$$|P_1| = \min_{1\leqslant i\leqslant n} |P_i|,$$

then there exist $P_i'\in M(X_1, \mathfrak{A}_1)$, $1\leqslant i\leqslant n$, *with* $P_i' \leqslant P_i$, $|P_i'| = |P_1|$ *and* $P_i \wedge P_1 = P_i' \wedge P_1$, $2\leqslant i\leqslant n$.

Proof. Define $\lambda_i = P_i - P_1 \wedge P_i$ and $\mu_i = P_1 - P_1 \wedge P_i$, $2\leqslant i\leqslant n$. Then λ_i, μ_i are orthogonal and $|\mu_i| \leqslant |\lambda_i|$. With

$$P_i' = P_1 \wedge P_i + \frac{|\mu_i|}{|\lambda_i|}\lambda_i, \quad 2\leqslant i\leqslant n,$$

the assertion of Lemma 4 holds.

COROLLARY 5. *Let* $P_i, Q_i \in M^1(X_1, \mathfrak{A}_1)$, $1 \leqslant i \leqslant n$,

$$|P_1 \wedge Q_1| = \min_{1 \leqslant i \leqslant n} |P_i \wedge Q_i|$$

and assume that $P_1 \wedge \ldots \wedge P_n \leqslant Q_1 \wedge \ldots \wedge Q_n$.

Then there exists an $R \in M(X, \mathfrak{A})$ *with* $\pi_1(R) = P_1 \wedge Q_1$, $\pi_i(R) \leqslant P_i \wedge Q_i$, $2 \leqslant i \leqslant n$, *and*

$$R(\Delta_n(A)) = \sup\{P(\Delta_n(A));\ P \in M(P_1, \ldots, P_n)\}$$

for all $A \in \mathfrak{A}_1$.

Proof. Let $P_i' \leqslant P_i \wedge Q_i$ be measures with $|P_i'| = |P_1 \wedge Q_1|$ and

$$P_i' \wedge P_1 \wedge Q_1 = P_i \wedge Q_i \wedge P_1 \wedge Q_1, \quad 2 \leqslant i \leqslant n,$$

as in Lemma 4.

Then by Proposition 3 there exists an $R \in M(P_1 \wedge Q_1, P_2', \ldots, P_n')$ with

$$\begin{aligned} R(\Delta_n(A)) &= \sup\{P(\Delta_n(A));\ P \in M(P_1 \wedge Q_1, P_2', \ldots, P_n')\} \\ &= P_1 \wedge Q_1 \wedge P_2' \wedge \ldots \wedge P_n'(A) = \bigwedge_{i=1}^{n} P_i \wedge \bigwedge_{i=1}^{n} Q_i(A) \\ &= \bigwedge_{i=1}^{n} P_i(A) = \sup\{P(\Delta_n(A));\ P \in M(P_1, \ldots, P_n)\}, \end{aligned}$$

where

$$\bigwedge_{i=1}^{n} P_i = P_1 \wedge \ldots \wedge P_n.$$

3. Determination of robust tests. Consider now the test problem M_1, M_2 from Section 1. For subsets $\mathscr{P}_i \subset M^1(X, \mathfrak{A})$, $i = 1, 2$, and $\alpha \in [0, 1]$ let

$$\beta(\alpha, \mathscr{P}_1, \mathscr{P}_2) = \sup_{\varphi \in \Phi_\alpha(\mathscr{P}_1)} \inf_{Q \in \mathscr{P}_2} E_Q \varphi$$

denote the maximin-power at level α, where $\Phi_\alpha(\mathscr{P}_1)$ are the tests of level α. Let $R_i \in M_i$, $i = 1, 2$; then (R_1, R_2) is called *least favourable of level* α if

$$\beta(\alpha, M_1, M_2) = \beta(\alpha, R_1, R_2)$$

(cf. Baumann [1]).

Define, for $k \geqslant 0$,

$$L(M_2, kM_1) = \{(R_1, R_2);\ R_i \in M_i,\ i = 1, 2,\ d_v(M_2, kM_1) = d_v(R_2, kR_1)\}.$$

The proof of Proposition 2 shows that $L(M_2, kM_1) \neq \Phi$ and how to find elements of $L(M_2, kM_1)$. Finally, for $k \geqslant 0$, $\alpha \in [0, 1]$, define

$$h_\alpha(k) = \alpha k + \max_{1 \leqslant i \leqslant n} d_v(Q_i, kP_i).$$

THEOREM 6. *Let* $\alpha \in [0, 1]$.

(a) $\beta(\alpha, M_1, M_2) = \min\{h_\alpha(k); k \geqslant 0\}$.

(b) *Let* $k^* \geqslant 0$ *be a minimum point of* h_α *and let* $(R_1, R_2) \in L(M_2, k^* M_1)$; *then*

(1) (R_1, R_2) *is the least favourable at level* α *for* M_1, M_2.

(2) *There exists an LQ-test* φ^* *for* R_1, R_2 *with critical value* k^* *at level* α *which is a maximin test at level* α *for* M_1, M_2.

Proof. (a) Let $\bar{M}_i$ denote the closure of M_i in $ba(X, \mathfrak{A})$ – the set of finitely additive set functions – w.r.t. weak*-topology, $i = 1, 2$. By Satz 5.3 of Baumann [1]

$$\beta(\alpha, M_1, M_2) = \inf\{\alpha k + d_v(\bar{M}_2, k\bar{M}_1), k \geqslant 0\}$$

(where d_v is defined in $ba(X, \mathfrak{A})$ as in $M(X, \mathfrak{A})$). Clearly, for $P \in \bar{M}_1$ we have $\pi_i(P) = P_i$ $(1 \leqslant i \leqslant n)$ and, for $Q \in \bar{M}_2$, $\pi_i(Q) = Q_i$ $(1 \leqslant i \leqslant n)$. Therefore,

$$d_v(Q, kP) = \sup\{Q(B) - kP(B); B \in \mathfrak{A}\} \geqslant \max_{i \leqslant n} d_v(Q_i, kP_i)$$

which implies, using Proposition 2,

$$d_v(\bar{M}_2, k\bar{M}_1) = d_v(M_2, kM_1) = \max_{1 \leqslant n} d_v(Q_i, kP_i)$$

and, therefore,

$$\beta(\alpha, M_1, M_2) = \min_{k \geqslant 0} h_\alpha(k).$$

(b) If $(R_1, R_2) \in L(M_2, kM_1)$, then

$$d_v(R_2, k^* R_1) = \max_i d_v(Q_i, k^* P_i)$$

and, therefore,

$$\begin{aligned} \beta(\alpha, M_1, M_2) &= h_\alpha(k^*) = \alpha k^* + \max_i d_v(Q_i, k^* P_i) \\ &= \alpha k^* + d_v(R_2, k^* R_1) \\ &\geqslant \inf\{\alpha k + d_v(R_2, kR_1); k \geqslant 0\} = \beta(\alpha, R_1, R_2). \end{aligned}$$

Since trivially $\beta(\alpha, M_1, M_2) \leqslant \beta(\alpha, R_1, R_2)$, (R_1, R_2) is least favourable at level α. Point (b) of (2) is well-known from the duality treatment of test problems (cf. f.i. Baumann [1]).

Remark 3. (a) As is clear from Theorem 6, there are many least favourable pairs at level α and the least favourable pairs generally depend on α (cf. the following example). Therefore, there are no least favourable pairs in the sense of Huber and Strassen [5].

(b) If there is a component, say $i = 1$, such that

$$d_v(Q_1, kP_1) = \max_j d_v(Q_j, kP_j) \quad \text{for all } k \geqslant 0,$$

then there is a least favourable pair independent of α and a maximin-test can be chosen depending only on the first component of the observation. This is especially true if $P_1 = \ldots = P_n$ and $Q_1 = \ldots = Q_n$. But in the other case the maximin-power is strictly larger (for some α) than the maximum power of the tests concerning the individual components only.

(c) For the determination of a maximin-test it is useful to choose a suitable least favourable pair (R_1, R_2) in order to have a randomization region as small as possible. The following example shows how to choose (R_1, R_2) in certain cases and how to manage the necessary optimization problems concerning the randomization region.

4. An example. Consider the case $n = 2$ and measures P_i, Q_i on $[0, 1]$ determined by $P_i = f_i \lambda^1$ $(i = 1, 2)$, $f_1(x) = 2x$ $(x \in [0, 1])$, $f_2(x) = 1$ $(x \in [0, 1])$ and $Q_1 = P_2$, $Q_2 = P_1$; i.e. we consider $M_1 = M(P_1, P_2)$, $M_2 = M(P_2, P_1)$. Then

$$\tilde{f}_1(k) = \alpha k + d_v(Q_1, kP_1) = \begin{cases} 1-(1-\alpha)k, & k \leqslant \frac{1}{2}, \\ \alpha k + \frac{1}{4k}, & k > \frac{1}{2}, \end{cases}$$

and

$$\tilde{f}_2(k) = \alpha k + d_v(Q_2, kP_2) = \begin{cases} 1-(1-\alpha)k + \frac{k^2}{4}, & k < 2, \\ \alpha k, & k \geqslant 2. \end{cases}$$

Furthermore,

$$\inf_k \tilde{f}_1(k) = \tilde{f}_1\left(\frac{1}{2\sqrt{\alpha}}\right) = \sqrt{\alpha}$$

and

$$\inf_k \tilde{f}_2(k) = \tilde{f}_2(2(1-\alpha)) = 1-(1-\alpha)^2.$$

Finally,

$$h_\alpha(k) = \begin{cases} \tilde{f}_2(k), & k \leqslant 1, \\ \tilde{f}_1(k), & k \geqslant 1, \end{cases}$$

and

$$\inf_{k \leqslant 1} \tilde{f}_2(k) = \begin{cases} \tilde{f}_2(2(1-\alpha)), & \alpha \geqslant \frac{1}{2}, \\ \tilde{f}_2(1), & \alpha < \frac{1}{2}, \end{cases} \qquad \inf_{k \geqslant 1} \tilde{f}_1(k) = \begin{cases} \tilde{f}_1\left(\frac{1}{2\sqrt{\alpha}}\right), & \alpha < \frac{1}{4}, \\ \tilde{f}_1(1), & \alpha \geqslant \frac{1}{4}. \end{cases}$$

Therefore,

$$\inf_{k\geq 0} h_\alpha(k) = \begin{cases} \bar{f}_1\left(\dfrac{1}{2\sqrt{\alpha}}\right) = \sqrt{\alpha}, & \alpha < \dfrac{1}{4}, \\ \bar{f}_1(1) = \bar{f}_2(1) = \alpha + \dfrac{1}{4}, & \dfrac{1}{4} \leqslant \alpha \leqslant \dfrac{1}{2}, \\ \bar{f}_2(2(1-\alpha)) = 1-(1-\alpha)^2, & \alpha > \dfrac{1}{2}. \end{cases}$$

Therefore, for $\alpha < \frac{1}{4}$ the maximin-test is based only on the first component of the observation (x_1, x_2) while for $\alpha > \frac{1}{2}$ it is based only on the second component. For $\alpha \in [\frac{1}{4}, \frac{1}{2}]$ the maximin-power is strictly larger than the power of the marginal tests and we can choose $k^* = 1$ independent of $\alpha \in [\frac{1}{4}, \frac{1}{2}]$.

Since we have $P_1 \wedge Q_1 = P_2 \wedge Q_2$, there exists, by Proposition 3(a), $R \in M(P_1 \wedge P_1, P_2 \wedge Q_2)$ which is concentrated on the diagonal Δ_2. Since $P_1 \wedge Q_1 = g\lambda_1$ with

$$g(x) = \begin{cases} 2x, & 0 \leqslant x \leqslant \frac{1}{2}, \\ 1, & \frac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

we have $|P_1 \wedge Q_1| = |R| = \frac{3}{4}$. To determine a least favourable pair, define $g_1 = f_1 - g$, $g_2 = f_2 - g$; then $g_2(x) = g_1(1-x)$ and $\int g_1\, d\lambda_1 = \int g_2\, d\lambda_1 = \frac{1}{4}$.

Define:

$$h_1(x, y) = 4g_1(x)g_2(y) = \begin{cases} 4(2x-1)(1-2y), & x \geqslant \dfrac{1}{2},\ y \leqslant \dfrac{1}{2}, \\ 0 & \text{else}; \end{cases}$$

$$h_2(x, y) = 4g_2(x)g_1(y) = \begin{cases} 4(1-2x)(2y-1), & x \leqslant \dfrac{1}{2},\ y \geqslant \dfrac{1}{2}, \\ 0 & \text{else}. \end{cases}$$

Then $R_i' = h_i \lambda_2$ $(i = 1, 2)$ define elements of $M(P_1 - P_1 \wedge Q_1, P_2 - P_2 \wedge Q_2)$, resp. $M(Q_1 - P_1 \wedge Q_1, Q_2 - P_2 \wedge Q_2)$, λ_2 denoting Lebesgue measure on $[0, 1]^2$. Our choice yields measures R_1', R_2' with maximal support. Now define, as in the proof of Proposition 2, $R_1 = R + R_1'$, $R_2 = R + R_2'$ and define the LQ-test

$$\varphi(x, y) = \begin{cases} 1, & x \leqslant \frac{1}{2}, \\ \gamma, & x \geqslant \frac{1}{2},\ y \geqslant \frac{1}{2}, \\ 0 & \text{else}, \end{cases}$$

with $\gamma = 4(\alpha - \frac{1}{4})$. Then

$$\text{(1)} \qquad \sup\{E_P\varphi;\ P \in M(P_1, P_2)\} = E_{R_1}\varphi = \frac{1}{4} + \frac{\gamma}{4} = \alpha,$$

(2) $$\inf\{E_Q\varphi;\ Q\in M(Q_1, Q_2)\} = E_{R_2}\varphi = \frac{1}{2}+\frac{\gamma}{4} = \alpha+\frac{1}{4}.$$

To prove (1) and (2) observe that, for $P\in M(P_1, P_2)$,

$$P\left\{x\leqslant\frac{1}{2},\ y\leqslant\frac{1}{2}\right\}\leqslant P\left\{x\leqslant\frac{1}{2}\right\} = \frac{1}{4}$$

and, therefore,

$$P\left\{x\leqslant\frac{1}{2},\ y\geqslant\frac{1}{2}\right\}\geqslant\frac{1}{4} \quad\text{and}\quad P\left\{x\geqslant\frac{1}{2},\ y\geqslant\frac{1}{2}\right\}\leqslant\frac{1}{4}.$$

The sup in (1) is, therefore, bounded by $\frac{1}{4}\cdot 1+\gamma\cdot\frac{1}{4}$, which is attained for $P = R_1$. The inf in (2) is attained for a $Q\in M(Q_1, Q_2)$ such that $Q\{x\leqslant\frac{1}{2}, y \geqslant\frac{1}{2}\} = \frac{1}{3}$ and, therefore, $Q\{x\leqslant\frac{1}{2}, y\leqslant\frac{1}{2}\} = 0$, $Q\{x\geqslant\frac{1}{2}, y\geqslant\frac{1}{2}\} = \frac{1}{4}$. $Q = R_2$ is an element with these properties. (1) and (2) imply that φ is a maximin-test at level α.

Using Corollary 5, this example could be discussed in greater generality. Generally, the measures R_1', R_2' should be chosen as the product measure in order to obtain the smallest possible randomization region.

REFERENCES

[1] V. Baumann, *Eine parameterfreie Theorie der ungünstigsten Verteilungen*, Z. Wahrscheinlichkeitstheorie verw. Geb. 11 (1968), p. 40-60.

[2] R. L. Dobrushin, *Prescribing a system of random variables by conditional distributions*, Theory Prob. Applications 15 (1970), p. 458-486.

[3] N. Gaffke and L. Rüschendorf, *On a class of extremal problems in statistics*, Math. Operationsforsch. Statist., Ser. Optimization, 12 (1981), p. 123-136.

[4] G. Hansel and J. P. Troallic, *Mesures marginales et théorème de Ford-Fulkerson*, Z. Wahrscheinlichkeitstheorie verw. Geb. 43 (1978), p. 245-251.

[5] P. J. Huber and V. Strassen, *Minimax-tests and the Neymann–Pearson lemma for capacities*, Ann. Statist. 1 (1973), p. 251-263.

[6] L. Rüschendorf, *Sharpness of Fréchet bounds*, Z. Wahrscheinlichkeitstheorie verw. Geb. 57 (1981), p. 293-302.

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Received on 10. 1. 1983