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SKOROKHOD PROBLEM – ELEMENTARY PROOF OF THE AZEMA-YOR FORMULA

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Abstract. Let μ be a centered probability measure with the finite second moment. Let the stopping time T for the Brownian motion W be defined as

$$T = \inf\{t \geqslant 0;\ \Psi(W_t) \leqslant \sup_{0 \leqslant s \leqslant t} W_s\},$$

where Ψ is a barycenter function of measure μ. Azema and Yor [1] have shown that W_T has then the distribution μ and $ET = \int x^2 \mu d(x)$. This paper contains an elementary proof of this result.

Introduction. Skorokhod [8] has shown that for the centered probability measure μ with a continuous distribution function there exists the Brownian motion W and the stopping time T so that the distribution of W_T is μ. Moreover, if μ has the finite second moment, then

$$ET = \int_{-\infty}^{+\infty} x^2 \mu(dx).$$

That construction was improved by Monroe [5]. However, the stopping time T, given by the Skorokhod's construction is regarded with respect to the filtration essentially bigger that the natural filtration of Brownian motion. Dubins [4], Rost [7], Chacon–Walsh [3] and Azema–Yor [1] gave new constructions of stopping times with the desired property but which are stopping times with respect to the natural filtration of Brownian motion. Construction given by Azema–Yor is the best one in some respects. It is an explicit formula and not a result of a limit procedure. Pierre [6] gave a new proof of the Azema–Yor formula but with assumptions of regularity of a measure μ. The proof of this formula given in our paper is based on the following

PROPERTY. Let W be the Brownian motion. If $W_0 \equiv 0$, $a < 0 < b$,

$$\tau_x = \inf\{t \geqslant 0 \colon W_t = x\}, \qquad x \in R,$$

then

$$P(\tau_a < \tau_b) = \frac{b}{b-a}.$$

Definition 1. If μ is a probability distribution such that

$$\int_{-\infty}^{\infty} |x|\,\mu(dx) < \infty,$$

then the *barycenter function* Ψ *of the measure* μ is defined as

$$\Psi(a) = \begin{cases} \dfrac{1}{\mu([a,\infty))} \displaystyle\int_{[a,\infty)} x\mu(dx) & \text{if } \mu([a,\infty)) > 0, \\ a & \text{otherwise.} \end{cases}$$

Notation. We write

$$S_t = \sup_{0 \leqslant s \leqslant t} W_s.$$

THEOREM. *If* $T = \inf\{t \geqslant 0;\ \Psi(W_t) \leqslant S_t\}$, *then* W_T *has a distribution* μ. *Moreover, if*

$$\int_{-\infty}^{+\infty} x^2\,\mu(dx) < \infty,$$

then

$$ET = \int_{-\infty}^{+\infty} x^2\,\mu(dx).$$

Remark 1. T is a stopping time with respect to the natural filtration of Brownian motion.

Remark 2. To show that W_T has a distribution μ it is enough to prove the following implication:

$$\mu((a,\infty)) > 0 \Rightarrow \mu([a,\infty)) = P(\{W_T \geqslant a\}).$$

Indeed, if $\mu((a,\infty)) = 0$ and $a_n \nearrow a$ $(n = 1, 2, \ldots)$, then either there exists an N such that $\mu([a_n,\infty)) = 0$ for $n \geqslant N$ and then, for $b \geqslant 0$, $\Psi(a_N + b) =$

a_N+b so that $W_T \leqslant a_N$ and $P(\{W_T \geqslant a\}) = 0$ or, for every $n = 1, 2, \ldots$, $\mu((a_n, \infty)) > 0$ and then

$$\mu([a, \infty)) = \lim_{n\to\infty} \mu([a_n, \infty)) = \lim_{n\to\infty} P(\{W_T \geqslant a_n\}) = P(\{W_T \geqslant a\}).$$

Notation. We write

$$P_a = P(\{W_T \geqslant a\}), \quad \varphi(a) = \inf_{x\leqslant a}\{\Psi(x)-x\}, \quad v(x) = \inf\{y\colon \Psi(y) > x\},$$

$$K_i^n = i\,2^{-n}\,\Psi(a) \quad (0 \leqslant i \leqslant 2^n), \quad K = K_{2^n}^n,$$

$$a_i^n = v(K_i^n) \quad (0 \leqslant i \leqslant 2^n), \quad a_0 = a_0^n, \quad \frac{-\infty}{-\infty} = 1.$$

Remark 3. Since the set where function Ψ is not continuous is at least denumerable, we can make an assumption that a is such that $\Psi(a_i^n) = K_i^n$ for $0 \leqslant i \leqslant 2^n$.

Remark 4. *If $\Psi(a) > a$, then $\varphi(a) > 0$.*

Proof. Suppose that $\varphi(a) = 0$. Since $\varphi(a) = \inf\{\Psi(x)-x;\ \varphi(a) - a \leqslant x \leqslant a\}$, there exists an $(x_n)_{n=0}^{\infty} \in [\varphi(a)-a, a]$ such that

$$\varphi(a) = \lim_{n\to\infty}(\Psi(x_n)-x_n) \quad \text{and} \quad x_n \to x_0.$$

If we can find a subsequence $x_{n_k} \nearrow x_0$, then – by the left continuity of the function φ – we have $\Psi(x_0) = x_0$. Also, if there exist subsequences $x_{n_k} \searrow x_0$, then

$$x_0 \leqslant \Psi(x_0) \leqslant \lim_{k\to\infty} \Psi(x_{n_k}) = x_0.$$

Since $x_0 \leqslant a$, we obtain a contradiction: $\Psi(a) \leqslant a$.

Remark 5. Let

$$\Gamma = \bigcup_{y\in R} y \times [\Psi(y), \lim_{z\searrow y} \Psi(z)]$$

and let $Z_t = (W_t, S_t)$ be the process with values in R^2. Then, by the definition of the stopping time T, we infer that T is a first entrance time of process Z to the closed set Γ.

LEMMA 1. *If $v(x) \leqslant y \leqslant x$, then*

$$\frac{y-v(y)}{x-v(y)} \geqslant P(S_T \geqslant x \mid S_T \geqslant y) \geqslant \frac{y-v(x)}{x-v(x)}.$$

Proof. We have

$$P(S_T \geqslant x \mid S_T \geqslant y)$$
$$= P(\{\text{after exit from}\,(y, y)\ Z \text{ achieves } (x, x) \text{ before it enters } \Gamma\}),$$

$$\frac{y-v(y)}{x-v(y)}$$
$$= P\big(\{\text{after exit from } (y, y)\ Z \text{ achieves } (x, x) \text{ before it enters } \{v(y)\times R\}\}\big),$$

$$\frac{y-v(x)}{x-v(x)}$$
$$= P\big(\{\text{after exit from } (y, y)\ Z \text{ achieves } (x, x) \text{ before it enters } \{v(x)\times R\}\}\big).$$

LEMMA 2. *If* $\Psi(a) > a$, *then*

$$\lim_{n\to\infty} \prod_{i=1}^{2^n} \frac{K_{i-1}^n - a_{i-1}^n}{K_i^n - a_{i-1}^n} \geqslant P_a \geqslant \lim_{n\to\infty} \prod_{i=1}^{2^n} \frac{K_{i-1}^n - a_i^n}{K_i^n - a_i^n}.$$

Proof. Let n be so large that $2^{-n}\Psi(a) < \varphi(a)$. From the definition, $\{W_T \geqslant a\} = \{S_T \geqslant \Psi(a)\}$, whence

$$P(\{W_T \geqslant a\}) = P\big(\{S_T \geqslant \Psi(a)\}\big) = \prod_{i=1}^{2^n} P(S_T \geqslant K_i^n \mid S_T \geqslant K_{i-1}^n).$$

Since

$$K_i^n - a_i^n \geqslant \Psi(a_i^n) - a_i^n \geqslant \varphi(a) > 2^{-n}\Psi(a) = K_i^n - K_{i-1}^n \qquad (0 < i \leqslant 2^n),$$

we have $a_i^n < K_{i-1}^n < K_i^n$ and from Lemma 1 it follows that

$$\prod_{i=1}^{2^n} \frac{K_{i-1}^n - a_{i-1}^n}{K_i^n - a_{i-1}^n} \geqslant P_a \geqslant \prod_{i=1}^{2^n} \frac{K_{i-1}^n - a_i^n}{K_i^n - a_i^n}.$$

LEMMA 3. *We have*

$$\lim_{n\to\infty} \prod_{i=1}^{2^n} \frac{K_{i-1}^n - a_{i-1}^n}{K_i^n - a_{i-1}^n} = \lim_{n\to\infty} \prod_{i=1}^{2^n} \frac{K_{i-1}^n - a_i^n}{K_i^n - a_i^n}.$$

Proof. We can write

$$1 \geqslant \left(\prod_{i=1}^{2^n} \frac{K_{i-1}^n - a_i^n}{K_i^n - a_i^n}\right)\left(\prod_{i=1}^{2^n} \frac{K_{i-1}^n - a_{i-1}^n}{K_i^n - a_{i-1}^n}\right)^{-1}$$
$$= \prod_{i=1}^{2^n-1} \frac{(K_{i-1}^n - a_i^n)(K_{i+1}^n - a_i^n)}{(K_i^n - a_i^n)^2} \, \frac{K_{2^n-1}^n - a_{2^n}^n}{K_{2^n}^n - a_{2^n}^n} \, \frac{K_1^n - a_0^n}{K_0^n - a_0^n}$$
$$= \prod_{i=1}^{2^n-1} \frac{\big(K_i^n - a_i^n - 2^{-n}\Psi(a)\big)\big(K_i^n - a_i^n + 2^{-n}\Psi(a)\big)}{(K_i^n - a_i^n)^2} \times$$

$$\times\frac{\Psi(a)-v(\Psi(a))-2^{-n}\Psi(a)}{\Psi(a)-v(\Psi(a))}\,\frac{2^{-n}\Psi(a)-v(0)}{-v(0)}$$

$$=\prod_{i=1}^{2^n-1}\left[1-\left(\frac{\Psi(a)}{2^n(K_i^n-a_i^n)}\right)^2\right]\frac{\Psi(a)-v(\Psi(a))-2^{-n}\Psi(a)}{\Psi(a)-v(\Psi(a))}\,\frac{2^{-n}\Psi(a)-v(0)}{-v(0)}$$

$$\geqslant\left[1-\left(\frac{\Psi(a)}{2^n\varphi(a)}\right)^2\right]^{2^n-1}\frac{\Psi(a)-v(\Psi(a))-2^{-n}\Psi(a)}{\Psi(a)-v(\Psi(a))}\,\frac{2^{-n}\Psi(a)-v(0)}{-v(0)}\to 1,$$

when $n\to\infty$.

LEMMA 4. *If* $2^{-n}\Psi(a)<\varphi(a)$, *then*

$$\prod_{i=1}^{2^n}\frac{K_{i-1}^n-a_{i-1}^n}{K_i^n-a_{i-1}^n}\geqslant\mu([a,\infty))\geqslant\prod_{i=1}^{2^n}\frac{K_{i-1}^n-a_i^n}{K_i^n-a_i^n}.$$

Proof. Since

$$\mu([a,\infty))=\prod_{i=1}^{2^n}\frac{\mu([a_i^n,\infty))}{\mu([a_{i-1}^n,\infty))},$$

it is enough to show that

$$\frac{K_{i-1}^n-a_{i-1}^n}{K_i^n-a_{i-1}^n}\geqslant\frac{\mu([a_i^n,\infty))}{\mu([a_{i-1}^n,\infty))}\geqslant\frac{K_{i-1}^n-a_i^n}{K_i^n-a_i^n}\quad\text{for } 0<i\leqslant 2^n.$$

From the definition of function Ψ and from Remark 3 we get

$$\frac{K_{i-1}^n-a_{i-1}^n}{K_i^n-a_{i-1}^n}=\frac{\mu([a_i^n,\infty))}{\mu([a_{i-1}^n,\infty))}\,\frac{\int_{[a_{i-1}^n,\infty]}(x-a_{i-1}^n)\,\mu(dx)}{\int_{[a_i^n,\infty)}(x-a_{i-1}^n)\,\mu(dx)}\geqslant\frac{\mu([a_i^n,\infty))}{\mu([a_{i-1}^n,\infty))}$$

and

$$\frac{K_{i-1}^n-a_i^n}{K_i^n-a_i^n}=\frac{\mu([a_i^n,\infty))}{\mu([a_{i-1}^n,\infty))}\,\frac{\int_{[a_{i-1}^n,\infty)}(x-a_i^n)\,\mu(dx)}{\int_{[a_i^n,\infty)}(x-a_i^n)\,\mu(dx)}\leqslant\frac{\mu([a_i^n,\infty))}{\mu([a_{i-1}^n,\infty))}.$$

Lemmas 2, 3, and 4 now easily imply that W_T has the distribution μ.

Remark 6. If T is a stopping time such that $ET<\infty$, then $EW_T=0$ and $EW_T^2=ET$.

To complete the proof of the theorem it must be shown that if

$$\int_{-\infty}^{+\infty}x^2\,\mu(dx)<\infty,$$

then

$$ET = \int_{-\infty}^{+\infty} x^2 \mu(dx).$$

From Remark 6 it follows that it is enough to show that $ET < \infty$. Let Ψ_n and T_n $(n = 1, 2, \ldots)$ be defined as follows:

$$\Psi_n(x) = \begin{cases} 0 & \text{if } x \leqslant -n, \\ \Psi(x) & \text{if } -n < x \leqslant n, \\ \Psi(n) & \text{if } n < x \leqslant \Psi(n), \\ x & \text{if } \Psi(n) < x; \end{cases} \qquad T_n = \inf\{t \geqslant 0;\ \Psi_n(W_t) \leqslant S_t\}.$$

From the definition, $T_n \to T$ and $ET_n < \infty$ ($ET_n \leqslant$ means exit time of Brownian motion from $[-n, \Psi(x)]$). So $ET_n = EW_{T_n}^2$.

To obtain from Fatou Lemma that $ET < \infty$ it is enough to show that

$$\limsup_{n\to\infty} EW_{T_n}^2 < \infty.$$

Let $A = \{W_T \in [n, \Psi(n))\}$. Then

$$\begin{aligned} EW_{T_n}^2 &= E_{\chi_{A^c}} W_{T_n}^2 + E_{\chi_A} W_{T_n}^2 \leqslant E_{\chi_{A^c}} W_T^2 + E_{\chi_A} W_{T_n}^2 \\ &= EW_T^2 + E_{\chi_A}(W_{T_n}^2 - W_T^2) \leqslant EW_T^2 + 2\Psi(n) E_{\chi_A}(W_{T_n} - W_T) \\ &= EW_T^2 + 2\Psi(n) \int_{[n, \Psi(n))} (\Psi(n) - x)\mu(dx) \\ &= EW_T^2 + 2\Psi(n) \int_{[\Psi(n), \infty)} (x - \Psi(n))\mu(dx) \\ &\leqslant EW_T^2 + 2 \int_{[\Psi(n), \infty)} x^2 \mu(dx) \leqslant EW_T^2 + \varepsilon \quad \text{for } n \geqslant n_0, \end{aligned}$$

which completes the proof.

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