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LAW OF THE ITERATED LOGARITHM FOR SUBSEQUENCES

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Abstract. Let $\{S_n\}_{n=1}^{\infty}$ denote the partial sums of i.i.d. random variables with mean 0. The present paper investigates the quantity

$$\limsup_{k\to\infty} S_{n_k}/\sqrt{n_k \log\log n_k},$$

where $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing subsequence of the positive integers. The first results are that if $EX_1^2 < \infty$, then the limit superior equals $\sigma\sqrt{2}$ a.s. for subsequences which increase "at most geometrically", and $\sigma\varepsilon^*$, where

$$\varepsilon^* = \inf\{\varepsilon > 0; \sum_k (\log n_k)^{-\varepsilon^2/2} < \infty\},$$

for subsequences which increase "at least geometrically". We also perform a refined analysis for the latter case and finally present criteria for the finiteness of

$$E\sup_k (S_{n_k}/\sqrt{n_k \log\log n_k})^2$$

in both cases.

1. Introduction. Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. random variables and let $\{S_n\}_{n=1}^{\infty}$ denote their partial sums. The purpose of this paper is to study the law of the iterated logarithm (LIL) for subsequences. Thus, let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing subsequence of the positive integers. Then, what can be said about

$$\limsup_{k\to\infty} \frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} ?$$

If $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$, then, clearly, the limit superior is *at most* equal to $\sigma\sqrt{2}$. But, is it always equal to $\sigma\sqrt{2}$? Can it be smaller?

If $\{n_k\}$ is not increasing "too rapidly", one would guess that the answer is $\sigma\sqrt{2}$, for example if $n_k = k^2$. On the other hand, if n_k increases "very rapidly", it is conceivable that the answer could be something strictly smaller than $\sigma\sqrt{2}$.

Huggins [12], Lemma 1, proves that if the ratio n_{k+1}/n_k has a finite limit as $k \to \infty$, then the limit superior equals $\sigma\sqrt{2}$ a.s. Strictly speaking, the result is proved as a functional LIL for Brownian motion (as a first step in proving a functional LIL for time changed Brownian motion and for randomly indexed partial sums) but, by combining this with well-known strong approximation results, the result is valid also for partial sums.

Our first result, Theorem 2.1, states that the limit superior equals $\sigma\sqrt{2}$ a.s. for sequences such that

$$\liminf_{k\to\infty} n_k/n_{k+1} > 0,$$

i.e. for sequences which increase *at most geometrically.*

The second result, Theorem 2.2, states that the limit superior equals $\sigma\varepsilon^*$ a.s., where

$$\varepsilon^* = \inf\{\varepsilon > 0;\ \sum_{k=3}^{\infty} (\log n_k)^{-\varepsilon^2/2} < \infty\},$$

for sequences such that

$$\limsup_{k\to\infty} n_k/n_{k+1} < 1,$$

i.e. for sequences which increase *at least geometrically.* An easy estimate shows that $0 \leqslant \varepsilon^* \leqslant \sqrt{2}$. In particular (see below), for "very rapidly" increasing sequences, like for example when $n_k = 2^{2^k}$ one has $\varepsilon^* = 0$. After a section with some technical results, proofs of these results are given in Sections 4 and 5.

In proving necessities, i.e. that a finite limit superior implies $EX_1^2 < \infty$ and $EX_1 = 0$, it turns out that such a result is true when

$$\liminf_{k\to\infty} n_k/n_{k+1} > 0 \quad \text{and} \quad \limsup_{k\to\infty} n_k/n_{k+1} < 1$$

with $\varepsilon^* > 0$ (see Section 6). When $\varepsilon^* = 0$, however, one can obtain a positive result without the variance being finite. This situation is dealt with in Sections 7 and 8.

In Sections 9 and 10 we state and prove a dominated ergodic theorem, i.e. a result on the finiteness of

$$E\sup_k (S_{n_k}/\sqrt{n_k \log\log n_k})^2.$$

For the case $n_k = k$, see [14].

The necessary and sufficient integrability condition is the same as that of

Siegmund [14] when

$$\liminf_{k\to\infty} n_k/n_{k+1} > 0,$$

i.e. the same as when $n_k = k$, and weaker when

$$\limsup_{k\to\infty} n_k/n_{k+1} < 1.$$

For "very rapidly" increasing sequences $EX_1^2 < \infty$ is necessary and sufficient.

Throughout, very irregular sequences $\{n_k\}_{k=1}^{\infty}$ are excluded, where "very irregular" means sequences such that

$$\liminf_{k\to\infty} n_k/n_{k+1} = 0 \quad \text{and} \quad \limsup_{k\to\infty} n_k/n_{k+1} = 1.$$

Some examples, however, are given.

The last section contains some further remarks and results. Some comments on the convergence/divergence of

$$\sum_{k=3}^{\infty} P(|S_{n_k}| > \varepsilon\sqrt{n_k \log\log n_k}) \quad \text{and} \quad \sum_{n=3}^{\infty} \frac{1}{n} P(|S_n| > \varepsilon\sqrt{n \log\log n})$$

are given. Also, the number of boundary crossings, i.e.

$$\sum_{k=1}^{\infty} I\{|S_{n_k}| > \varepsilon\sqrt{n_k \log^+ \log^+ n_k}\}$$

is investigated. Contrary to the case $n_k = k$, where this quantity has no moments of positive order, see [15] (only a logarithmic moment for $\varepsilon > 2\sigma$ if a little more than finite variance is assumed; see [8], Corollary 8.3), it turns out that the expected number of boundary crossings is always finite for $\varepsilon > \sigma\varepsilon^*$ when

$$\limsup_{k\to\infty} n_k/n_{k+1} < 1$$

(provided, of course, that $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$) and "sometimes" finite when $\liminf_{k\to\infty} n_k/n_{k+1} > 0$.

The final result in Section 11 deals with the case

$$\limsup_{k\to\infty} n_k/n_{k+1} < 1,$$

but a slightly different normalization will be used. For example, when $\varepsilon^* = 0$, Theorem 2.3 tells us that the fluctuations of the sequence $\{S_{n_k}\}$ are of a smaller order of magnitude than $\sqrt{n_k \log\log n_k}$, and in Theorem 11.1 we show that

$$\limsup_{k\to\infty} S_{n_k}/\sqrt{n_k \log k} = \sigma\sqrt{2} \text{ a.s.},$$

provided $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$.

2. LIL – results and examples. Our starting point is (a) the classical Hartman-Wintner-Strassen LIL (see e.g. [16], Chapter 5) and (b) Lemma 1 of Huggins [12], which is used as a first step for proving a functional LIL for time-changed Brownian motion and thus, by using strong approximation results, also for randomly indexed summation processes. In [12], Lemma 1, increasing subsequences $\{n_k\}_{k=1}^{\infty}$ are used for which the ratio n_{k+1}/n_k has a finite limit as $k \to \infty$.

Our first result is an extension of the validity of (the one-dimensional version of) this result to more general subsequences of partial sums of i.i.d. random variables.

THEOREM 2.1. *Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing subsequence of the positive integers such that*

$$\liminf_{k \to \infty} \frac{n_k}{n_{k+1}} > 0. \tag{2.1}$$

Further, let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. random variables, set $S_n = \sum_{k=1}^{n} X_k$ and suppose that $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$. Then

$$\limsup_{k \to \infty} (\liminf_{k \to \infty}) \frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} = {}_{(\pm)}\sigma\sqrt{2} \quad a.s. \tag{2.2}$$

Conversely, if

$$P\left(\limsup_{k \to \infty} \frac{|S_{n_k}|}{\sqrt{n_k \log\log n_k}} < \infty\right) > 0,$$

then $EX_1^2 < \infty$ and $EX_1 = 0$.

Now, suppose that (2.1) does not hold. What kind of results are then possible?

Consider an example; let $n_k = 2^{2^k}$. Then $n_k/n_{k+1} = 2^{-2^k} \to 0$, in particular, (2.1) does not hold. It follows from the proofs below ((3.6) and Lemma 3.3) that

$$\sum_{k=3}^{\infty} P(|S_{n_k}| > \varepsilon\sqrt{n_k \log\log n_k}) < \infty \quad \text{for all } \varepsilon > 0$$

and thus, from the Borel-Cantelli lemma, that

$$\frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} \to 0 \text{ a.s.} \quad \text{as } k \to \infty.$$

Note that the limsup (being 0) is strictly smaller than $\sigma\sqrt{2}$.

This example also raises the question whether it is possible to select the subsequence in such a way that one obtains a lim sup which is strictly between 0 and $\sigma\sqrt{2}$. The next theorem (together with Example 4 below) gives an answer to this question.

THEOREM 2.2. *Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing subsequence of the positive integers such that*

$$\limsup_{k\to\infty} \frac{n_k}{n_{k+1}} < 1 \tag{2.3}$$

and let

$$\varepsilon^* = \inf\{\varepsilon > 0;\ \sum_{k=3}^{\infty} (\log n_k)^{-\varepsilon^2/2} < \infty\}. \tag{2.4}$$

Further, let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. random variables. Set $S_n = \sum_{k=1}^{n} X_k$ and suppose that $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$. Then

$$\limsup_{k\to\infty} (\liminf_{k\to\infty}) \frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} = {}_{(-)}^{+} \sigma\varepsilon^* \quad a.s. \tag{2.5}$$

In particular, if $\varepsilon^* = 0$, then

$$\frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} \to 0 \quad a.s. \qquad as\ k \to \infty. \tag{2.6}$$

For the converse, suppose that $\varepsilon^* > 0$. If

$$P\left(\limsup_{k\to\infty} \frac{|S_{n_k}|}{\sqrt{n_k \log\log n_k}} < \infty\right) > 0,$$

then $EX_1^2 < \infty$ and $EX_1 = 0$.

Remark 2.1. Condition (2.3) implies that there exists a $\lambda > 1$ such that $n_{k+1} \geqslant \lambda n_k$, from which it easily follows that $\varepsilon^* \leqslant \sqrt{2}$; in particular, ε^* does always exist, finite.

Note that there is no converse in Theorem 2.2 when $\varepsilon^* = 0$. It turns out that, in fact, one can do with a little less than finite variance. We shall return to this case in Section 7.

Examples. The first observation (cf. [7]) is that a sequence such as $n_k = 2^k$ satisfies (2.1) as well as (2.3). Since $\varepsilon^* = \sqrt{2}$, in that case there is no contradiction.

1. $n_k = 2^k$. Then $n_k/n_{k+1} = \frac{1}{2}$, i.e. both theorems apply and, since $\varepsilon^* = \sqrt{2}$, they yield the same result.

2. $n_k = k^d$, where $d = 1, 2, \ldots$ Then $n_k/n_{k+1} = (k/k+1)^d \to 1$, i.e. Theorem 2.1 applies.

3. $n_k = 2^{2^k}$. Then $n_k/n_{k+1} = 2^{-2^k} \to 0$, i.e. Theorem 2.2 applies and, since $\varepsilon^* = 0$, we have an example where (2.6) holds (cf. also above).

4. $n_k = [2^{k^\beta}]$, where $\beta > 1$. This is the "typical" example which yields a lim sup strictly between 0 and $\sigma\sqrt{2}$. We have $n_k/n_{k+1} \sim 2^{-\beta k^{\beta-1}} \to 0$ as $k \to \infty$, i.e. Theorem 2.2 applies and it is easy to see that $\varepsilon^* = \sqrt{2/\beta}$. Thus

$$\limsup_{k\to\infty} \frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} = \sigma\sqrt{\frac{2}{\beta}} \quad \text{a.s.} \qquad \text{as } k \to \infty.$$

For $\beta = 1$ we have a geometric increase and for $\beta < 1$ Theorem 2.1 applies.

5. $n_k = k!$. Theorem 2.2 applies with $\varepsilon^* = \sqrt{2}$.

6. $n_k = k^k$. The same.

Just like in [7], sequences for which

$$\liminf_{k\to\infty} \frac{n_k}{n_{k+1}} = 0 \quad \text{and} \quad \limsup_{k\to\infty} \frac{n_k}{n_{k+1}} = 1$$

both hold have been excluded and, just like there, we shall mention two examples of this kind, such that Theorem 2.2 can be applied to one of them with $\varepsilon^* = 0$ and such that the limit superior equals $\sigma\sqrt{2}$ and there is no finite ε^* in the other one.

7a. Let $n_{2k} = 2^{2^k}$ and $n_{2k+1} = 2^{2^k} + 1$ $(k = 1, 2, \ldots)$. Since $\{n_{2k}\}_{k=1}^\infty$ and $\{n_{2k+1}\}_{k=1}^\infty$ both satisfy (2.3) with $\varepsilon^* = 0$, it follows by applying Theorem 2.2 twice that $S_{n_k}/\sqrt{n_k \log\log n_k} \to 0$ a.s. as $k \to \infty$. Also,

$$\sum_k (\log n_k)^{-\varepsilon^2/2} < \infty \qquad \text{for all } \varepsilon > 0,$$

i.e. $\varepsilon^* = 0$.

7b. Let $I_k = \{2^{2^k}+1, 2^{2^k}+2, \ldots, 2^{2^{k+1}}\}$, $k = 1, 2, \ldots$, and set

$$B_1 = \bigcup_{k=1}^\infty I_{2k} \quad \text{and} \quad B_2 = \bigcup_{k=0}^\infty I_{2k+1}.$$

Then

$$\sum_{n\in B_1} (\log n)^{-\varepsilon^2/2} \quad \text{and} \quad \sum_{n\in B_2} (\log n)^{-\varepsilon^2/2}$$

are both infinite. Furthermore, since $P(S_n > \varepsilon\sqrt{n\log\log n} \text{ i.o.}) = 1$ when $\varepsilon < \sigma\sqrt{2}$, it follows that at least one (in fact, both) of

$$P(S_n > \varepsilon\sqrt{n\log\log n} \text{ i.o. } n\in B_1) \quad \text{and} \quad P(S_n > \varepsilon\sqrt{n\log\log n} \text{ i.o. } n\in B_2)$$

are 1 for $\varepsilon < \sigma\sqrt{2}$, i.e. if $\{n_k\}$ is the (one of the) sequence(s) such that the probability equals 1 we have

$$\limsup_{k\to\infty} \frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} = \sigma\sqrt{2} \text{ a.s.}$$

and no finite ε^*.

As a final remark we point out the fact that Theorem 2.1 provides a proof of the LIL for random variables with index set Z_+^d, $d \geq 2$, i.e. the positive integer d-dimensional lattice points and T_θ^d, the d-dimensional sector in Z_+^d, for the case where the summation index tends to infinity along a ray (see [17], Theorem 1, and [9], Section 4). Note also that Theorem 2.1 covers the case where the index tends to infinity, not only along a ray but also along any increasing path.

3. Some preparatory lemmas. In this section we collect some results of technical character which will be used later.

LEMMA 3.1. *Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing subsequence of the positive integers, let*

$$M(x) = \sum_{k=1}^{[x]} n_k, \qquad x > 0$$

and let ψ be the inverse of the subsequence, i.e. $\psi(x) = \operatorname{Card}\{k;\ n_k \leq x\}$. Then, for any random variable X,

$$\sum_{k=1}^{\infty} n_k P(|X| \geq n_k) = EM(\psi(|X|)) < \infty. \tag{3.1}$$

Furthermore, if (2.3) *holds, i.e.*

$$\limsup_{k\to\infty} \frac{n_k}{n_{k+1}} < 1, \tag{3.2}$$

then

$$E|X| < \infty \Rightarrow \sum_{k=1}^{\infty} n_k P(|X| \geq n_k) < \infty. \tag{3.3}$$

Proof. Since $\{|X| \geq n_k\} = \{\psi(|X|) \geq k\}$, (3.1) follows by partial summation.

As for (3.3), (3.2) implies that there exists a $\lambda > 1$ such that

$$n_{k+1} \geq \lambda n_k, \qquad k = 1, 2, \ldots \tag{3.4}$$

Now, set

$$\Sigma = \sum_{k=1}^{\infty} n_k P(|X| \geq n_k).$$

Then, by (3.4),

$$\begin{aligned}\Sigma &= \sum_{k=1}^{\infty} \left(n_{k-1} + (n_k - n_{k-1})\right) P(|X| \geqslant n_k) \\ &\leqslant \sum_{k=1}^{\infty} \lambda^{-1} n_k P(|X| \geqslant n_k) + \sum_{k=1}^{\infty} \sum_{i=n_{k-1}+1}^{n_k} P(|X| \geqslant i) \\ &\leqslant \lambda^{-1} \Sigma + E|X|,\end{aligned}$$

which proves (3.3); in fact, this, together with (3.1) shows that

$$EM\left(\psi(|X|)\right) = \sum_{k=1}^{\infty} n_k P(|X| \geqslant n_k) \leqslant \frac{\lambda}{\lambda - 1} E|X|. \tag{3.5}$$

Remark 3.1. If there exists a C, $0 < C < \infty$, such that $M\left(\psi(x)\right) \geqslant Cx$ (for large x), then, clearly, $EM\left(\psi(|X|)\right) < \infty \Rightarrow E|X| < \infty$, and if, in addition, (3.2) holds, then $EM\left(\psi(|X|)\right) < \infty \Leftrightarrow E|X| < \infty$.

Next some tail probabilities are estimated. Set

$$b_n = \frac{2\delta\sigma^2}{\varepsilon} \sqrt{\frac{n}{\log^+ \log^+ n}}, \quad n = 1, 2, \ldots,$$

with $0 < \delta < 1/3$, and define, for $k \leqslant n$,

$$X'_{k,n} = X_k I\left\{|X_k| < \frac{1}{2} b_n\right\}, \quad X''_{k,n} = X_k I\left\{|X_k| > \sqrt{n}\right\}, \quad X'''_{k,n} = X_k - X'_{k,n} - X''_{k,n}$$

and set

$$S'_n = \sum_{k=1}^{n} X'_{k,n}, \quad S''_n = \sum_{k=1}^{n} X''_{k,n}, \quad S'''_n = \sum_{k=1}^{n} X''_{k,n}.$$

LEMMA 3.2. *Suppose that* $\{X_n\}_{n=1}^{\infty}$ *are i.i.d. with* $EX_1 = 0$ *and* $EX_1^2 = \sigma^2 < \infty$. *Then, for large* n,

LEMMA 3.2. *Suppose that* $\{X_n\}_{n=1}^{\infty}$ *are i.i.d. with* $EX_1 = 0$ *and* $EX_1^2 = \sigma^2 < \infty$. *Then, for large* n,

$$P(|S'_n| > \varepsilon\sqrt{n \log\log n}) \leqslant 2(\log n)^{-\varepsilon^2(1-3\delta)/2\sigma^2}, \quad 0 < \delta < 1/3, \tag{3.6}$$

$$P(S'_n > \varepsilon\sqrt{n \log\log n}) \geqslant (\log n)^{-\varepsilon^2(1+\gamma)/2\sigma^2(1-\delta)}, \quad \gamma > 0,\ 0 < \delta < 1/3. \tag{3.7}$$

Proof. Following the lines of [8] we first note that $|ES'_n| = o(\sqrt{n \log\log n})$ as $n \to \infty$. Next, by using the exponential bound as formulated in [8], Lemma 2.2, with $t = 2\delta b_n^{-1}$, we obtain, for large n,

$$\begin{aligned}P(|S'_n| > \varepsilon\sqrt{n \log\log n}) &\leqslant P(|S'_n - ES'_n| > \varepsilon(1-\delta)\sqrt{n \log\log n}) \\ &\leqslant 2\exp\left\{-2\delta b_n^{-1} \varepsilon(1-\delta)\sqrt{n \log\log n} + \frac{1}{2} n 4\delta^2 b_n^{-2} \sigma^2 (1+\delta)\right\} \\ &= 2\exp\left\{-\frac{\varepsilon^2}{2\sigma^2}(1-3\delta) \log\log n\right\},\end{aligned}$$

which proves (3.6).

To prove (3.7) we use the lower exponential bound (see [16], p. 262). However, first $\operatorname{Var}(S'_n)$ is estimated. Trivially,

$$\operatorname{Var}(X'_{k,n}) \leqslant E(X'_{k,n})^2 \leqslant \sigma^2.$$

Also,

$$\operatorname{Var}(X'_{k,n}) = EX_k^2 - EX_k^2 I\{|X_k| > \frac{1}{2}b_n\} - (E(X'_{k,n}))^2$$

and it follows that

$$s_n^2 = \operatorname{Var}(S'_n)\begin{cases} \leqslant n\sigma^2 & \text{for all } n, \\ \geqslant n\sigma^2(1-\delta) & \text{for large } n. \end{cases} \tag{3.8}$$

The lower exponential bound thus yields

$$P(S'_n > \varepsilon\sqrt{n\log\log n}) \geqslant P\left(\frac{S'_n}{s_n} > \frac{\varepsilon}{\sigma\sqrt{1-\delta}}\sqrt{\log\log n}\right)$$
$$\geqslant \exp\left\{-\frac{\varepsilon^2}{2\sigma^2(1-\delta)}\log\log n\cdot(1+\gamma)\right\},$$

which is the same as (3.7).

LEMMA 3.3. *Let $\{X_n\}_{n=1}^{\infty}$ be as in Lemma 3.2 and let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing subsequence of the positive integers satisfying* (3.2). *Then, for all* $\eta > 0$,

$$\sum_{k=3}^{\infty} P(|S''_{n_k}| > \eta\sqrt{n_k\log\log n_k}) < \infty, \tag{3.9}$$

$$\sum_{k=3}^{\infty} P(|S'''_{n_k}| > \eta\sqrt{n_k\log\log n_k}) < \infty. \tag{3.10}$$

Proof. Since $P(|S''_{n_k}| > \eta\sqrt{n_k\log\log n_k}) \leqslant n_k P(|X_1| > \sqrt{n_k})$, an application of (3.3) yields (3.9).

To prove (3.10) we argue like in [3], p. 635. Since

$$|EX'''_{1,n}| \leqslant \int_{b_n/2}^{\infty} |x|\, dF(x) = o(\sqrt{n^{-1}\log\log n}),$$

it follows that $|ES'''_n| = o(\sqrt{n\log\log n})$ as $n \to \infty$ and hence that, for large k,

$$P(|S'''_{n_k}| > \eta\sqrt{n_k\log\log n_k}) \leqslant P\left(|S'''_{n_k} - ES'''_{n_k}| > \frac{\eta}{2}\sqrt{n_k\log\log n_k}\right)$$
$$\leqslant \frac{4\operatorname{Var}(S'''_{n_k})}{\eta^2 n_k\log\log n_k} \leqslant \frac{4E(X'''_{1,n_k})^2}{\eta^2\log\log n_k}.$$

Thus, by changing the order of integration and summation,

$$\sum_{k=k_0}^{\infty} P(|S_{n_k}'''| > \eta\sqrt{n_k \log\log n_k}) \leqslant 4\eta^{-2} \sum_{k=k_0}^{\infty} \frac{1}{\log\log n_k} EX^2 I\left\{\frac{1}{2}b_{n_k} \leqslant |X_1| \leqslant \sqrt{n_k}\right\}$$

$$= 4\eta^{-2} \int \Big(\sum_{A(k,x)} \frac{1}{\log\log n_k} \Big) x^2\, dF(x),$$

where $A(k, x) = \{k;\ \frac{1}{2}b_{n_k} \leqslant |x| \leqslant \sqrt{n_k}\}$.

By inverting these inequalities we find that, for large k (and $|x|$),

$$A(k, x) \subset A^*(k, x) = \left\{k;\ x^2 \leqslant n_k \leqslant 2\left(\frac{\varepsilon}{\delta\sigma^2}\right)^2 x^2 \log\log|x|\right\}, \tag{3.11}$$

which, keeping (3.4) in mind, yields

$$\sum_{A(k,x)} \frac{1}{\log\log n_k} \leqslant \frac{1}{\log\log(x^2)} \operatorname{Card}\{A^*(k, x)\} = O\left(\frac{\log\log\log|x|}{\log\log|x|}\right) \quad \text{as } |x| \to \infty$$

and so

$$\sum_{k=k_0}^{\infty} P\left(|S_{n_k}''| > \eta\sqrt{n_k \log\log n_k}\right) \leqslant \text{Const}\cdot EX_1^2 < \infty \tag{3.12}$$

and the proof is complete.

4. Proof of the upper class results. Since

$$\limsup_{k\to\infty} \frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} \leqslant \limsup_{n\to\infty} \frac{S_n}{\sqrt{n \log\log n}} = \sigma\sqrt{2} \text{ a.s.},$$

the upper class result for Theorem 2.1 is immediate. For Theorem 2.2 this estimate is too crude and we need the following

LEMMA 4.1. *Assume that* (2.3) *holds and let* ε^* *be defined by* (2.4). *Then, for all* $\varepsilon > \sigma\varepsilon^*$,

$$\sum_{k=3}^{\infty} P(|S_{n_k}| > \varepsilon\sqrt{n_k \log\log n_k}) < \infty, \tag{4.1}$$

$$P(S_{n_k} > \varepsilon\sqrt{n_k \log\log n_k} \text{ i.o.}) = 0, \tag{4.2}$$

$$P(|S_{n_k}| > \varepsilon\sqrt{n_k \log\log n_k} \text{ i.o.}) = 0, \tag{4.3}$$

$$\limsup_{k\to\infty} \frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} \leqslant \sigma\varepsilon^* \ \textit{a.s.} \tag{4.4}$$

$$\limsup_{k\to\infty} \frac{|S_{n_k}|}{\sqrt{n_k \log\log n_k}} \leqslant \sigma\varepsilon^* \ \textit{a.s.} \tag{4.5}$$

Remark 4.1. If $\varepsilon^* = 0$, then,

$$(4.6) \qquad \frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} \to 0 \text{ a.s.} \quad \text{as } k \to \infty$$

and the sufficiency has been proved for that case. Recall that e.g. the example preceding Theorem 2.2 was such a case.

Proof. It is clear that once (4.1) is proved, the other conclusions are immediate.

By (3.6) it follows that

$$(4.7) \qquad \sum_{k=k_0}^{\infty} P(|S'_{n_k}| > \varepsilon\sqrt{n_k \log\log n_k}) \leqslant 2 \sum_{k=k_0}^{\infty} (\log n_k)^{-\varepsilon^2(1-3\delta)/2\sigma^2} < \infty$$

if $\varepsilon > \sigma\varepsilon^*(1-3\delta)^{-1/2}$. Thus, given $\varepsilon > \sigma\varepsilon^*$, let $\eta > 0$ and $\delta > 0$ be so small that $(\varepsilon-\eta)^2(1-3\delta) > \sigma^2(\varepsilon^*)^2$. The fact that

$$\{|S_n| > \varepsilon\sqrt{n\log\log n}\} \subset \{|S'_n| > (\varepsilon-\eta)\sqrt{n\log\log n}\} \cup$$
$$\cup \left\{|S''_n| > \frac{\eta}{2}\sqrt{n\log\log n}\right\} \cup \left\{|S'''_n| > \frac{\eta}{2}\sqrt{n\log\log n}\right\}$$

together with (4.7) and Lemma 3.3 now implies that

$$\sum_{k=k_0}^{\infty} P(|S_{n_k}| > \varepsilon\sqrt{n_k \log\log n_k}) < \infty$$

and, since η and δ may be chosen arbitrarily small, (4.1) follows.

5. Proof of the lower class results. Since the lower class result in Theorem 2.1 can be deduced from the corresponding result in Theorem 2.2, we begin by considering the latter. As mentioned above, there is nothing to prove when $\varepsilon^* = 0$.

LEMMA 5.1. *Suppose that* (2.3) *holds. Let* ε^* *be defined by* (2.4) *and suppose also that* $\varepsilon^* > 0$. *Then*

$$(5.1) \qquad \sum_{k=3}^{\infty} P(S_{n_k} > \varepsilon\sqrt{n_k \log\log n_k}) = +\infty \quad \textit{for all } \varepsilon < \sigma\varepsilon^*.$$

Remark 5.1. Just like in the classical proof, the events contained in (5.1) are not independent, i.e. there are no immediate further conclusions to be made at this point.

Proof. By (3.7) it follows, for k_0 large, that

$$(5.2) \quad \sum_{k=k_0}^{\infty} P(S'_{n_k} > \varepsilon\sqrt{n_k \log\log n_k}) \geqslant \sum_{k=k_0}^{\infty} (\log n_k)^{-\varepsilon^2(1+\gamma)/2\sigma^2(1-\delta)} = +\infty$$

if $\varepsilon < \sigma\varepsilon^*\sqrt{(1-\delta)/(1+\gamma)}$.

Now, let $\varepsilon < \sigma\varepsilon^*$ be given and choose η, γ and δ so small that $(\varepsilon+\eta)^2(1+\gamma)(1-\delta)^{-1} < \sigma^2(\varepsilon^*)^2$. Since

$$\{S_n' > \varepsilon\sqrt{n\log\log n}\} \subset \{S_n > (\varepsilon-\eta)\sqrt{n\log\log n}\} \cup$$
$$\cup \left\{|S_n''| > \frac{\eta}{2}\sqrt{n\log\log n}\right\} \cup \left\{|S_n'''| > \frac{\eta}{2}\sqrt{n\log\log n}\right\},$$

it follows from (5.2) and Lemma 3.3 that

$$\sum_{k=k_0}^{\infty} P(S_{n_k} > \varepsilon\sqrt{n_k\log\log n_k}) = +\infty,$$

which proves the lemma.

Like in the classical proof (see e.g. [16], p. 271), we now pass to increments in order to apply the converse of the Borel-Cantelli lemma and we shall consider the subsequence $\{n_{\nu k}\}_{k=1}^{\infty}$, where ν is an integer to be chosen later.

The first step is to show that (5.1) holds for this subsequence. Let ε_ν^* be the ε^* corresponding to this subsequence, that is,

$$\varepsilon_\nu^* = \inf\{\varepsilon > 0;\ \sum_k (\log n_{\nu k})^{-\varepsilon^2/2} < \infty\}. \tag{5.3}$$

Now, let $0 < \varepsilon < \varepsilon^*$. Then, since $\sum(\log n_k)^{-\varepsilon^2/2} = \infty$, at least one of the series

$$\sum_k (\log n_{\nu k+j})^{-\varepsilon^2/2}, \quad j = 0, 1, 2, \ldots, \nu-1,$$

must diverge. Since $\{n_k\}$ is strictly increasing we must, in particular, have

$$\sum_k (\log n_{\nu k})^{-\varepsilon^2/2} = +\infty \quad \text{for } 0 < \varepsilon < \varepsilon^*, \tag{5.4}$$

that is, $\varepsilon_\nu^* \geqslant \varepsilon^*$. However, since trivially $\varepsilon_\nu^* \leqslant \varepsilon^*$, it follows that $\varepsilon_\nu^* = \varepsilon^*$. Finally, since, by (3.4), $n_{\nu k}/n_{\nu(k+1)} \leqslant \lambda^{-\nu} < 1$, it follows from Lemma 5.1, applied to the subsequence $\{n_{\nu k}\}_{k=1}^{\infty}$, that

$$\sum_{k=3}^{\infty} P(S_{n_{\nu k}} > \varepsilon\sqrt{n_{\nu k}\log\log n_{\nu k}}) = +\infty \quad \text{for all } \varepsilon < \sigma\varepsilon^*. \tag{5.5}$$

Next we note that

$$\begin{aligned} P(S_{n_{\nu k}} &> \varepsilon\sqrt{n_{\nu k}\log\log n_{\nu k}}) \\ &\leqslant P(S_{n_{\nu k}} - S_{n_{\nu(k-1)}} > \varepsilon(1-\delta_1)\sqrt{n_{\nu k}\log\log n_{\nu k}}) + \\ &\quad + P(S_{n_{\nu(k-1)}} > \varepsilon\delta_1\sqrt{n_{\nu k}\log\log n_{\nu k}}), \end{aligned} \tag{5.6}$$

where $0 < \delta_1 < 1/3$, but otherwise δ_1 is arbitrary.

Moreover, recalling (3.4) and (4.1), we have

$$(5.7)\qquad \sum_{k=4}^{\infty} P(|S_{n_{\nu(k-1)}}| > \varepsilon\delta_1 \sqrt{n_{\nu k} \log\log n_{\nu k}})$$

$$\leqslant \sum_{k=4}^{\infty} P(|S_{n_{\nu(k-1)}}| > \varepsilon\delta_1 \lambda^{\nu/2} \sqrt{n_{\nu(k-1)} \log\log n_{\nu(k-1)}}) < \infty$$

if ν is chosen so large that $\varepsilon\delta_1 \lambda^{\nu/2} > \sigma\varepsilon^*$, which, in view of (5.6) and (5.5), yields, for $\varepsilon < \sigma\varepsilon^* < \varepsilon\delta_1 \lambda^{\nu/2}$,

$$(5.8)\qquad \sum_{k=4}^{\infty} P(S_{n_{\nu k}} - S_{n_{\nu(k-1)}} > \varepsilon(1-\delta_1)\sqrt{n_{\nu k} \log\log n_{\nu k}}) = +\infty$$

and thus, by independence and Borel-Cantelli, that

$$(5.9)\qquad P(S_{n_{\nu k}} - S_{n_{\nu(k-1)}} > \varepsilon(1-\delta_1)\sqrt{n_{\nu k} \log\log n_{\nu k}} \text{ i.o.}) = 1$$

for $\varepsilon < \sigma\varepsilon^* < \varepsilon\delta_1 \lambda^{\nu/2}$.

By (5.7) we know, in particular, that

$$(5.10)\qquad P(|S_{n_{\nu(k-1)}}| > 2\varepsilon\delta_1 \sqrt{n_{\nu k} \log\log n_{\nu k}} \text{ i.o.}) = 0,$$

which, together with (5.9), yields

$$(5.11)\qquad P(S_{n_{\nu k}} > \varepsilon(1-3\delta_1)\sqrt{n_{\nu k} \log\log n_{\nu k}} \text{ i.o.}) = 1$$

for $\varepsilon < \sigma\varepsilon^* < \varepsilon\delta_1 \lambda^{\nu/2}$.

Thus,

$$\limsup_{k\to\infty} \frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} \geqslant \limsup_{k\to\infty} \frac{S_{n_{\nu k}}}{\sqrt{n_{\nu k} \log\log n_{\nu k}}} \geqslant \sigma\varepsilon^*(1-3\delta_1),$$

which, in view of the arbitrariness of δ_1, proves the lower class result for Theorem 2.2.

As for Theorem 2.1 we define, like in the proof of the classical case (see [16], p. 271, also [12], Lemma 1),

$$(5.12)\qquad m_j = \min\{k;\ n_k > M^j\},$$

where $j = 1, 2, \ldots$ and M is an integer $\geqslant 2$.

Now, (2.1) implies that

$$\inf_k \frac{n_k}{n_{k+1}} > 0;$$

in particular, there exists an integer $L > 1$ such that

$$(5.13)\qquad n_{k+1} \leqslant L n_k, \quad k = 1, 2, \ldots,$$

which, together with (5.12), implies that

(5.14) $$M^j \leqslant n_{m_j} \leqslant LM^j \text{ and } (LM)^{-1} \leqslant \frac{n_{m_{j-1}}}{n_{m_j}} \leqslant LM^{-1} \qquad (j = 1, 2, \ldots).$$

The sequence $\{n_{m_j}\}_{j=1}^{\infty}$ thus satisfies formula (2.3). Further, it follows that $\varepsilon^* = \sqrt{2}$ and, therefore, by what has already been shown, we conclude that

$$\limsup_{k\to\infty} \frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} \geqslant \limsup_{j\to\infty} \frac{S_{n_{m_j}}}{\sqrt{n_{m_j} \log\log n_{m_j}}} \geqslant \sigma\sqrt{2}$$

and we are done.

Remark 5.2. The subsequences satisfying (2.1) are "at most geometrically increasing" and the sequence considered in (5.12) is, in view of (5.14), "approximately geometrically increasing". The subsequences satisfying (2.3) are "at least geometrically increasing".

6. The necessity. We thus suppose that

(6.1) $$P\left(\limsup_{k\to\infty} \frac{|S_{n_k}|}{\sqrt{n_k \log\log n_k}} < \infty\right) > 0.$$

First we assume that (2.3) holds and that $\varepsilon^* > 0$. By using Feller's proof for the classical case (see e.g. [16], p. 297) applied to symmetric random variables, with ε^* playing the rôle of $\sqrt{2}$ in the classical case, it follows that $EX_1^2 < \infty$, after which we desymmetrize and conclude that $EX_1 = 0$ by the Kolmogorov strong law of large numbers. This concludes the proof of Theorem 2.2 (with $\varepsilon^* > 0$).

Concerning Theorem 2.1 we observe that (6.1) in particular implies that

(6.2) $$P\left(\limsup_{j\to\infty} \frac{|S_{n_{m_j}}|}{\sqrt{n_{m_j} \log\log n_{m_j}}} < \infty\right) > 0,$$

which, together with the fact that $\{n_{m_j}\}_{j=1}^{\infty}$ satisfies (2.3), proves the conclusion. Alternatively, one may proceed like in [9], Section 3, and first conclude that

(6.3) $$\sum_{j=1}^{\infty} P(|S_{n_{m_j}} - S_{n_{m_{j-1}}}| > \varepsilon\sqrt{n_{m_j} \log\log n_{m_j}}) < \infty \quad \text{for some } \varepsilon > 0$$

and then, recalling (5.14), that

$$\sum_{j=1}^{\infty} P(\sup_{k_{j-1} \leqslant n \leqslant k_j} |S_n/\sqrt{n \log\log n}| > \varepsilon_1) < \infty \qquad \text{for some } \varepsilon_1 > 0,$$

where ε_1 may be chosen as $2\varepsilon M\sqrt{L/(M-L)}$ and where $k_j = M^j - LM^{j-1}$,

$j = 1, 2, \ldots$ (and $M > L$). This implies that

$$(6.4) \qquad P(|S_n| > \varepsilon_1 \sqrt{n \log\log n} \text{ i.o.}) = 0 \quad \text{for some } \varepsilon_1 > 0,$$

from which it follows that $EX_1^2 < \infty$ and $EX_1 = 0$ by the converse of the classical law of the iterated logarithm.

This leaves the case $\varepsilon^* = 0$ without a converse at present. The next section treats this case in more detail.

7. The case $\varepsilon^* = 0$. Since Feller's proof for the necessity does not work when $\varepsilon^* = 0$, one may be tempted to guess that for such sequences one can obtain positive results also when the variance does not exist. We begin this section by finding necessary moment conditions and then proceed to prove that the above guess is correct.

Suppose that $\varepsilon^* = 0$ and that

$$P\left(\limsup_{k\to\infty} \frac{|S_{n_k}|}{\sqrt{n_k \log\log n_k}} < \infty\right) > 0.$$

Like in Section 6, we obtain (cf. (6.3)) that

$$(7.1) \qquad \sum_{k=3}^{\infty} P(|S_{n_k} - S_{n_{k-1}}| > \varepsilon\sqrt{n_k \log\log n_k}) < \infty \quad \text{for some } \varepsilon > 0.$$

Now assume that the variables have a symmetric distribution. Since

$$|X_n| \leqslant |S_n| + |S_{n-1}| \leqslant 2 \max_{1 \leqslant k \leqslant n} |S_k|,$$

it follows from the i.i.d. assumption, Lévy's inequality and (3.4) that

$$\begin{aligned}
2P(|S_{n_k} - S_{n_{k-1}}| > \varepsilon\sqrt{n_k \log\log n_k}) &= 2P(|S_{n_k - n_{k-1}}| > \varepsilon\sqrt{n_k \log\log n_k}) \\
&\geqslant P(\max_{i \leqslant n_k - n_{k-1}} |S_i| > \varepsilon\sqrt{n_k \log\log n_k}) \\
&\geqslant P(\max_{i \leqslant n_k - n_{k-1}} |X_i| > 2\varepsilon\sqrt{n_k \log\log n_k}) \\
&= 1 - (1 - P(|X_1| > 2\varepsilon\sqrt{n_k \log\log n_k}))^{n_k - n_{k-1}} \\
&\geqslant 1 - (1 - P(|X_1| > 2\varepsilon\sqrt{n_k \log\log n_k}))^{(1-\lambda^{-1})n_k}.
\end{aligned}$$

In view of (7.1) it now follows that

$$(1 - P(|X_1| > 2\varepsilon\sqrt{n_k \log\log n_k}))^{(1-\lambda^{-1})n_k} \to 1 \quad \text{as } k \to \infty$$

and hence that

$$n_k P(|X_1| > 2\varepsilon\sqrt{n_k \log\log n_k}) \to 0 \quad \text{as } k \to \infty.$$

By Taylor expansion this implies that, for large k,

$$(7.2)\qquad 2P(|S_{n_k}-S_{n_{k-1}}|>\varepsilon\sqrt{n_k\log\log n_k})$$
$$\geqslant \frac{1}{2}(1-\lambda^{-1})\,n_k\,P(|X_1|>2\varepsilon\sqrt{n_k\log\log n_k})$$

from which we conclude that

$$(7.3)\qquad \sum_{k=3}^{\infty} n_k\,P(|X_1|>2\varepsilon\sqrt{n_k\log\log n_k})<\infty \quad \text{for some } \varepsilon>0,$$

and, hence, by (3.1), that

$$(7.4)\qquad EM\big(\psi(CX_1^2/\log^+\log^+|X_1|)\big)<\infty \quad \text{for some } C<\infty.$$

If, in addition, $M(\psi(x))\geqslant C_1x$ as $x\to\infty$ for some C_1, $0<C_1<\infty$, then $EX_1^2/\log^+\log^+|X_1|<\infty$ (recall Remark 3.1).

If the random variables are non-symmetric, one symmetrizes and concludes that the moment condition must hold for the symmetrized variables after which one desymmetrizes and, for the case $M(\psi(x))\geqslant C_1x$, uses the law of large numbers to conclude that $EX_1=0$ [5, 8].

We have thus proved that (7.4) is necessary and that if, in addition, $M(\psi(x))\geqslant C_1x$ as $x\to\infty$ for some C_1 $(0<C_1<\infty)$, then

$$(7.5)\qquad EX_1^2/\log^+\log^+|X_1|<\infty \quad\text{and}\quad EX_1=0$$

is a necessary condition.

Now, suppose that $EX_1^2/(\log^+\log^+|X_1|)^{1-\delta}<\infty$ for some (small) $\delta>0$. Truncation and Chebyshev's inequality yield

$$P(|S_n|>\varepsilon\sqrt{n\log\log n})\leqslant nP(|X_1|>\sqrt{n\log\log n})+\frac{EX_1^2I\{|X_1|\leqslant\sqrt{n\log\log n}\}}{\varepsilon^2\log\log n}$$
$$\leqslant nP(|X_1|>\sqrt{n\log\log n})+\frac{EX_1^2/(\log^+\log^+|X_1|)^{1-\delta}}{\varepsilon^2(\log\log n)^{\delta}}\to 0 \quad\text{as } n\to\infty,$$

i.e. $S_n/\sqrt{n\log\log n}\to 0$ in probability as $n\to\infty$ and thus *there exists* a subsequence converging to 0 a.s.

This indicates that positive results are possible even if there is no finite variance. In fact, the following result can be obtained:

THEOREM 7.1. *Let* $\{n_k\}_{k=1}^{\infty}$ *be a strictly increasing subsequence of the positive integers such that*

$$\limsup_{k\to\infty} n_k/n_{k+1}<1$$

and $\varepsilon^* = 0$. *Suppose, in addition, that*

$$\frac{\log k}{\log\log n_k} \downarrow 0 \quad \text{as } k \to \infty. \tag{7.6}$$

Let $\{X_n\}_{n=1}^{\infty}$ *be i.i.d. random variables with* $EX_1 = 0$ *and suppose that*

$$EX_1^2 \frac{\log^+ \psi(X_1^2 \log^+ \log^+ |X_1|)}{\log^+ \log^+ |X_1|} < \infty. \tag{7.7}$$

Then

$$\frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} \to 0 \text{ a.s.} \quad \text{as } k \to \infty. \tag{7.8}$$

Conversely, if

$$P\left(\limsup_{k\to\infty} \frac{|S_{n_k}|}{\sqrt{n_k \log\log n_k}} < \infty\right) > 0,$$

then $EM(\psi(CX_1^2/\log^+ \log^+ |X_1|)) < \infty$ *for some* $C < \infty$. *If, in addition,* $M(\psi(x)) \geq C_1 x$ $(0 < C_1 < \infty)$ *for large* x, *then* $EX_1^2/\log^+ \log^+ |X_1| < \infty$ *and* $EX_1 = 0$.

An elementary computation shows that

$$\varepsilon^* = 0 \Leftrightarrow \frac{\log k}{\log\log n_k} \to 0 \text{ as } k \to \infty \Leftrightarrow \frac{\log \psi(k)}{\log\log k} \to 0 \text{ as } k \to \infty, \tag{7.9}$$

i.e. the theorem captures those subsequences with $\varepsilon^* = 0$ where the convergences in (7.9) are monotone. Further, (7.9) implies that $\log \psi(x^2 \log\log x) = o(\log\log x)$ as $x \to \infty$, i.e. requirement (7.7) is always strictly weaker than the assumption that $EX_1^2 < \infty$. Note also that for all "reasonably well-behaved" sequences $\{n_k\}$, (7.7) is equivalent to

$$EX_1^2 \frac{\log^+ \psi(|X_1|)}{\log^+ \log^+ |X_1|} < \infty. \tag{7.10}$$

Remark 7.1. If, e.g. $\{n_k\}$ is such that $\log \psi(x^2)/\log \psi(x) \leq C_1$ for all large x or if $\psi(x^2)/\psi(x) \leq C_2$ for all large x, then (7.7) and (7.10) are equivalent.

If $n_k = 2^{2^k}$ $(k = 1, 2, \ldots)$, then $\psi(x) \sim \log\log x$ as $x \to \infty$ and conditions (7.7) and (7.10) amount to requiring

$$EX_1^2 \frac{\log^+ \log^+ \log^+ |X_1|}{\log^+ \log^+ |X_1|} < \infty.$$

Furthermore, if $n_k = 2^{2^{\cdot^{\cdot^{2^k}}}}$ with m 2:s, the inverse behaves like

$\log_m(x)$, the m times iterated logarithm, and (7.7) and (7.10) become

$$EX_1^2 \frac{\log_m^+ |X_1|}{\log_2^+ |X_1|} < \infty.$$

By choosing m large, it follows that one can reach arbitrarily close to $EX_1^2/\log^+ \log^+ |X_1| < \infty$ by requiring sufficiently rapidly increasing sequences. It is appropriate at this point to mention that we have not been able to provide a condition which is both necessary and sufficient for the case $\varepsilon^* = 0$.

The following is a kind of boundary case when $M(\psi(x)) \geqslant Cx$. Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. random variables with $EX_1 = 0$ and $EX_1^2/\log^+ \log^+ |X_1| < \infty$. Then there exists a strictly increasing function $G(x) \nearrow \infty$ as $x \to \infty$, such that

$$EX_1^2 \frac{G(|X_1|)}{\log^+ \log^+ |X_1|} < \infty$$

(see e.g. [13], p. 38). Thus, by choosing $\{n_k\}$ in such a way that its inverse ψ satisfies $\log \psi(x^2 \log\log x) \leqslant G(x)$ as $x \to \infty$, it follows from Theorem 7.1 that the conclusion of the theorem holds for this very distribution and this choice of subsequence.

Finally, we mention [4], where a law of the iterated logarithm is proved for variables without finite variance. For a complement to Theorem 7.1 we also refer to Section 11.5 below.

8. Proof of Theorem 7.1. The proof of the sufficiency consists of a suitable modification of the proof of Theorem 2.2 (cf. also [8]). Since $\varepsilon^* = 0$, however, only the upper class result is needed. Define, for $n = 1, 2, \ldots$ and $0 < \delta < 1/3$,

$$b_n = \frac{2\delta^2}{\varepsilon} \frac{\sqrt{n \log^+ \log^+ n}}{\log^+ \psi(n)} \quad \text{and} \quad c_n = \sqrt{\frac{n \log^+ \log^+ n}{\log^+ \psi(n)}} \tag{8.1}$$

and set, for $k = 1, 2, \ldots, n$, $X'_{k,n} = X_k I\{|X_k| < \frac{1}{2} b_n\}$, $X''_{k,n} = X_k I\{|X_k| > c_n\}$, $X'''_{k,n} = X_k - X'_{k,n} - X''_{k,n}$ and let $S'_n = \sum_{k=1}^{n} X'_{k,n}$ etc. Also, set

$$g(x) = x^2 \frac{\log^+ \psi(x^2 \log^+ \log^+ x)}{\log^+ \log^+ x} \quad \text{and} \quad h(x) = x^2 \frac{\log^+ \psi(x^2)}{\log^+ \log^+ x}, \qquad x > 0.$$

Thus, $Eg(|X_1|) < \infty$ and $Eh(|X_1|) < \infty$.

First, we shall give estimates for EX'_{1,n_k} and $\operatorname{Var}(X'_{1,n_k})$. It follows from (7.9) that $b_n \geqslant 2\delta^2 \varepsilon^{-1} \sqrt{n(\log\log n)^{-1}}$ for large n and thus, by a repeated application of (7.9), that

$$b_n^2 \log\log b_n \geqslant n \qquad \text{for large } n.$$

Furthermore, $\log b_n \leqslant \log n$ for large n. Since $EX_1 = 0$ and $g(x)/x$ is monotone for large x, we obtain

$$|EX'_{1,n_k}| \leqslant \int\limits_{|x|>\frac{1}{2}b_{n_k}} |x|\,dF(x) \leqslant \frac{\log\log b_{n_k}}{\frac{1}{2}b_{n_k}\log\psi(b_{n_k}^2\log\log b_{n_k})} \int\limits_{|x|>\frac{1}{2}b_{n_k}} g(|x|)\,dF(x)$$

$$\leqslant \frac{\log\log n_k}{b_{n_k}\log\psi(n_k)}o(1) = o\left(\sqrt{\frac{\log\log n_k}{n_k}}\right) \quad \text{as } k\to\infty,$$

i.e.

(8.2) $$|ES'_{n_k}| = o(\sqrt{n_k\log\log n_k}) \quad \text{as } k\to\infty.$$

Since

$$\sup_{\frac{1}{2}b_{n_{i-1}}\leqslant|x|\leqslant\frac{1}{2}b_{n_i}} \frac{\log\log x}{\log\psi(x^2\log\log x)} \leqslant 2\frac{\log\log n_i}{\log i} \quad \text{for large } i,$$

a similar computation, together with (7.6), yields

$$\operatorname{Var}(X'_{1,n_k}) \leqslant E(X'_{1,n_k})^2 = \int\limits_{|x|\leqslant A} x^2\,dF(x) + \int\limits_{A<|x|\leqslant\frac{1}{2}b_{n_k}} x^2\,dF(x)$$

$$\leqslant A^2 + 2\frac{\log\log n_k}{\log k}\int\limits_{|x|\geqslant A} g(|x|)\,dF(x).$$

By choosing A fixed, large, it thus follows (since $Eg(|X_1|)<\infty$) that

(8.3) $$\operatorname{Var}(X'_{1,n_k}) = o\left(\frac{\log\log n_k}{\log k}\right) \quad \text{as } k\to\infty.$$

In particular, for k large,

(8.4) $$|ES'_{n_k}| \leqslant \varepsilon\delta\sqrt{n_k\log\log n_k} \quad \text{and} \quad \operatorname{Var}(X'_{1,n_k}) \leqslant \delta\frac{\log\log n_k}{\log k}.$$

Remark 8.1. If we compare with b_n as chosen in Section 3, it would have been natural to choose b_n here comparable to

$$2\delta\varepsilon^{-1}\operatorname{Var}(X'_{1,n})\sqrt{n(\log\log n)^{-1}}.$$

However, by (8.3) it follows that this quantity is close to the same as b_n as chosen in (8.1).

By proceeding exactly like in the proof of (3.6) with $t = 2\delta b_n^{-1}$, we obtain, recalling (8.4), that, for large k,

$$P(|S'_{n_k}| > \varepsilon\sqrt{n_k\log\log n_k}) \leqslant 2\exp\left\{-\frac{\varepsilon^2}{2\delta}(1-3\delta)\log k\right\} = 2k^{\varepsilon^2(1-3\delta)/2\delta},$$

i.e.

$$(8.5)\qquad \sum_{k=k_0}^{\infty} P(|S'_{n_k}| > \varepsilon\sqrt{n_k \log\log n_k})$$
$$\leqslant 2\sum_{k=k_0}^{\infty} k^{-\varepsilon^2(1-3\delta)/2\delta} < \infty \qquad \text{for } \varepsilon > \sqrt{\frac{2\delta}{1-3\delta}}.$$

Turning over to S''_{n_k} we find, in view of (7.9), that

$$\{|X_1| > c_{n_k}\} \subset \{\log^+ \psi(X_1^2) > \log k\} \qquad \text{for large } k.$$

Thus,

$$P(|S''_{n_k}| > \eta\sqrt{n_k \log\log n_k}) \leqslant n_k P(|X_1| > c_{n_k})$$
$$\leqslant n_k P\left(|X_1|\sqrt{\log^+ \psi(X_1^2)} > \sqrt{n_k \log\log n_k}\right) \qquad \text{for large } k$$

and, consequently, by (3.3), we obtain

$$\sum_{k=k_0}^{\infty} P(|S''_{n_k}| > \eta\sqrt{n_k \log\log n_k}) \leqslant \sum_{k=k_0}^{\infty} n_k P\left(|X_1|\sqrt{\log^+ \psi(X_1^2)} > \sqrt{n_k \log\log n_k}\right)$$
$$\leqslant EX_1^2 \log^+ \psi(X_1^2)/\log^+ \log^+ \left(|X_1|\sqrt{\log^+ \psi(X_1^2)}\right) \leqslant \text{const} + Eh(|X_1|),$$

which proves that

$$(8.6)\qquad \sum_{k=3}^{\infty} P(|S''_{n_k}| > \eta\sqrt{n_k \log\log n_k}) < \infty \qquad \text{for all } \eta > 0.$$

As for the third sum, we first note, recalling (8.2), that

$$|ES'''_{n_k}| \leqslant n_k \int_{\frac{1}{2}b_{n_k} \leqslant |x| \leqslant c_{n_k}} |x|\,dF(x)$$
$$\leqslant n_k \int_{|x| \geqslant \frac{1}{2}b_{n_k}} |x|\,dF(x) = o(\sqrt{n_k \log\log n_k}) \qquad \text{as } k \to \infty$$

and that, by (7.9), $c_n^2 \geqslant n$ for large n, from which it follows that

$$(8.7)\qquad E(X'''_{1,n_k})^2 \leqslant \frac{\log\log n_k}{\log k} E\left(h(|X_1|)\cdot I\left\{\frac{1}{2}b_{n_k} \leqslant |X_1| \leqslant c_{n_k}\right\}\right)$$

for large k.

By proceeding like in the proof of (3.10) we now obtain

$$\sum_{k=k_0}^{\infty} P(|S'''_{n_k}| > \eta\sqrt{n_k \log\log n_k}) \leqslant 4\eta^{-2} \sum_{k=k_0}^{\infty} \frac{1}{\log\log n_k} E(X'''_{1,n_k})^2$$
$$\leqslant 4\eta^{-2} \sum_{k=k_0}^{\infty} \frac{1}{\log \psi(n_k)} Eh(|X_1|) I\left\{\frac{1}{2}b_{n_k} \leqslant |X_1| \leqslant c_{n_k}\right\}$$
$$= 4\eta^{-2} \int \sum_{B(k,x)} \frac{1}{\log \psi(n_k)} h(|x|)\,dF(x),$$

where $B(k, x) = \{k;\ \frac{1}{2}b_{n_k} \leqslant |x| \leqslant c_{n_k}\}$.

By inverting the inequalities we find that, for large k (and $|x|$),

$$(8.8)\quad B(k, x) \subset B^*(k, x) = \left\{k;\ \frac{1}{2}\frac{x^2 \log\psi(|x|)}{\log\log|x|} \leqslant n_k \leqslant 16\frac{x^2(\log\psi(|x|))^2}{\log\log|x|}\right\},$$

which, together with (7.9), yields

$$\sum_{B(k,x)} (\log\psi(n_k))^{-1} \leqslant \left(\log\psi\left(\frac{1}{2}\frac{x^2\log\psi(|x|)}{\log\log|x|}\right)\right)^{-1} \operatorname{Card}\{B^*(k, x)\}$$

$$\leqslant (\log\psi(|x|))^{-1}\cdot O(\log\log\psi(|x|)) \quad \text{as } x\to\infty$$

and, finally,

$$(8.9)\quad \sum_{k=3}^{\infty} P(|S'''_{n_k}| > \eta\sqrt{n_k\log\log n_k}) \leqslant \text{Const}\cdot Eh(|X_1|) < \infty \ \text{ for all } \eta > 0.$$

By combining (8.5), (8.6) and (8.9) (cf. Section 4) we conclude that

$$(8.10)\quad \sum_{k=3}^{\infty} P(|S_{n_k}| > (\varepsilon+\eta)\sqrt{n_k\log\log n_k}) < \infty \ \text{ if } \varepsilon > \sqrt{\frac{2\delta}{1-3\delta}} \ \text{ and } \eta > 0,$$

and, since δ and η may be arbitrarily small, it follows that

$$(8.11)\qquad \sum_{k=3}^{\infty} P(|S_{n_k}| > \varepsilon\sqrt{n_k\log\log n_k}) < \infty \qquad \text{for all } \varepsilon > 0.$$

The conclusion now follows by the Borel-Cantelli lemma.

For the necessity we refer to (7.4) and (7.5).

9. A dominated ergodic theorem. In [7] a dominated ergodic theorem related to the law of large numbers for subsequences was proved. In this section, a corresponding result related to the LIL will be given.

THEOREM 9.1. *Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing subsequence of the positive integers, let ψ be the inverse and suppose that $\{X_n\}_{n=1}^{\infty}$ are i.i.d. random variables with $EX_1 = 0$.*

(a) *Suppose that*

$$\liminf_{k\to\infty} n_k/n_{k+1} > 0.$$

Then

$$(9.1)\qquad E\sup_k \frac{S_{n_k}^2}{n_k\log\log n_k} < \infty \Leftrightarrow EX_1^2\frac{\log^+|X_1|}{\log^+\log^+|X_1|} < \infty.$$

(b) *Suppose that*

$$\limsup_{k\to\infty} n_k/n_{k+1} < 1$$

and define

$$(9.2)\qquad H(x) = \sum_{k=1}^{[x]} (\log^+\log^+ n_k)^{-1}, \qquad x > 0.$$

If $H(\infty) < \infty$, *then*

(9.3) $$E \sup_k \frac{S_{n_k}^2}{n_k \log\log n_k} < \infty \Leftrightarrow EX_1^2 < \infty .$$

If $H(\infty) = \infty$, *then*

(9.4) $$E \sup_k \frac{S_{n_k}^2}{n_k \log\log n_k} < \infty \Leftrightarrow EX_1^2 (\psi(\ldots))(\psi(X_1^2/\log^+ \log^+ |X_1|)) < \infty .$$

Just like in Section 7, we have, for all reasonably well-behaved sequences $\{n_k\}$, simpler moment conditions (cf. Remark 7.1). For example, if $H(\infty) = \infty$ and $(\psi(x))^{-1} \cdot \psi(x^2) \leqslant C$ as $x \to \infty$, then

(9.5) $$E \sup_k \frac{S_{n_k}^2}{n_k \log\log n_k} < \infty \Leftrightarrow EX_1^2 H(\psi(|X_1|)) < \infty .$$

Also, if $H(\infty) = \infty$ and $k^{-1} \log\log n_k \to 0$ as $k \to \infty$, then

(9.6) $$EX_1^2 H(\psi(|X_1|)) < \infty \Leftrightarrow EX_1^2 \frac{\psi(|X_1|)}{\log^+ \log^+ |X_1|} < \infty .$$

For very rapidly increasing sequences $\psi(x) = o(\log\log x)$ as $x \to \infty$ and for "slowly" increasing sequences (such that $\limsup_{k\to\infty} n_k/n_{k+1} < 1$), $\psi(x)/\log\log x \to \infty$ as $x \to \infty$. The boundary point is where (9.6) begins/ceases to hold, i.e. typically when $n_k = [2^{2^{k^\alpha}}]$, $\alpha = 1$.

When $0 < \alpha < 1$, the relevant assumption is $EX_1^2 (\log^+ \log^+ |X_1|)^{1/\alpha - 1} < \infty$, for $\alpha > 1$ it is $EX_1^2 < \infty$, but when $\alpha = 1$ it is $EX_1^2 \log^+ \log^+ \log^+ |X_1| < \infty$.

Another boundary case is $n_k = [2^{2^{k \log k}}]$, which requires

$$EX_1^2 \log^+ \log^+ \log^+ \log^+ |X_1| < \infty .$$

We also remark that $n_k = 2^k$ yields

$$EX_1^2 \frac{\log^+ |X_1|}{\log^+ \log^+ |X_1|} < \infty$$

(in both (a) and (b)) and that the condition in (a) is the same as that of Siegmund [14], where the case $n_k = k$ is treated.

By using examples like those of Section 2 and of Gut [7], Section 2, different cases with

$$\limsup_{k\to\infty} n_k/n_{k+1} = 1 \quad \text{and} \quad \liminf_{k\to\infty} n_k/n_{k+1} = 0$$

can be constructed.

The proof is a mixture of the proofs in [14], [6] and [7] and will only be hinted at whenever the resemblance is very strong.

10. Proof of Theorem 9.1. For $n = 1, 2, \ldots$ set $b_n = \sqrt{n/\log^+ \log^+ n}$ and $c_n = \sqrt{n \log^+ \log^+ n}$ $(b_0 = c_0 = 0)$ and

$$\mu_n' = EX_n I\left\{|X_n| < \frac{1}{2} b_n\right\}, \qquad \mu_n'' = EX_n I\left\{|X_n| > \frac{1}{2} c_n\right\},$$

$$\mu_n''' = EX_n I\left\{\frac{1}{2} b_n \leqslant |X_n| \leqslant \frac{1}{2} c_n\right\} \qquad (\text{i.e. } \mu_n' + \mu_n'' + \mu_n''' = EX_n = 0).$$

For $n = 1, 2, \ldots$ we now define

$$X_n' = X_n I\left\{|X_n| < \frac{1}{2} b_n\right\} - \mu_n', \qquad X_n'' = X_n I\left\{|X_n| > \frac{1}{2} c_n\right\} - \mu_n'',$$

$$X_n''' = X_n I\left\{\frac{1}{2} b_n \leqslant |X_n| \leqslant \frac{1}{2} c_n\right\} - \mu_n''' \qquad (\text{i.e. } X_n = X_n' + X_n'' + X_n''').$$

Further,

$$S_n' = \sum_{k=1}^{n} X_k', \qquad Y_k' = S_{n_k}' - S_{n_{k-1}}', \qquad W' = \sup_k |Y_k'/\sqrt{n_k \log^+ \log^+ n_k}|,$$

$$V' = \sup_k |S_{n_k}'/\sqrt{n_k \log^+ \log^+ n_k}|$$

and similarly for S_n'', $Y_k'', \ldots$ and S_n''', $Y_k''', \ldots$ Finally,

$$W = \sup_k |Y_k/\sqrt{n_k \log^+ \log^+ n_k}| \quad \text{and} \quad V = \sup_k |S_{n_k}/\sqrt{n_k \log^+ \log^+ n_k}|.$$

Proof of the sufficiencies. Since

$$V \leqslant \sup_n |S_n/\sqrt{n \log \log n}|,$$

the sufficiency in (a) follows from [14].

To prove the sufficiency in (b) we first note that

$$EX_1^2 < \infty \Rightarrow E \sup_n (S_n'/\sqrt{n \log \log n})^2 < \infty$$

by [14]; in particular it follows that

(10.1) $$EX_1^2 < \infty \Rightarrow E(W')^2 \leqslant 4E(V')^2 < \infty.$$

Next, by proceeding like in [3] (cf. also Lemma 3.3 above) it follows that

$$\sum_{n=3}^{\infty} \frac{1}{n \log \log n} E(X_n''')^2 < \infty$$

and thus, by Kronecker's lemma, that

$$\frac{S_n'''}{\sqrt{n \log \log n}} \to 0 \text{ a.s.} \quad \text{as } n \to \infty,$$

which, together with the fact that $|X_n'''/\sqrt{n \log^+ \log^+ n}| \leqslant 1$ and Corollary 3.4 of Hoffmann-Jørgensen [11], proves that

$$EX_1^2 < \infty \Rightarrow E \sup_n (S_n'''/\sqrt{n \log\log n})^2 < \infty;$$

in particular it follows that

$$EX_1^2 < \infty \Rightarrow E(W''')^2 \leqslant 4E(V''')^2 < \infty. \tag{10.2}$$

As for $E(W'')^2$ we proceed like in [7] to obtain

$$\begin{aligned} E(W'')^2 &\leqslant \sum_{k=1}^{\infty} \frac{E(Y_k'')^2}{n_k \log^+ \log^+ n_k} \leqslant \sum_{k=1}^{\infty} \frac{n_k - n_{k-1}}{n_k \log^+ \log^+ n_k} EX_1^2 I\{|X_1| > c_{n_{k-1}}\} \\ &\leqslant \sum_{k=1}^{\infty} (\log^+ \log^+ n_k)^{-1} \sum_{j=k}^{\infty} EX_1^2 I\{c_{j-1} < |X_1| \leqslant c_j\} \\ &\leqslant \sum_{j=1}^{\infty} \Big(\sum_{k=1}^{j} (\log^+ \log^+ n_k)^{-1}\Big) EX_1^2 I\{c_{j-1} < |X_1| \leqslant c_j\} \\ &= \sum_{j=1}^{\infty} H(j) EX_1^2 I\{c_{j-1} < |X_1| \leqslant c_j\}. \end{aligned} \tag{10.3}$$

If $H(\infty) < \infty$, the sum in (10.3) is dominated by

$$\sum_{j=1}^{\infty} H(\infty) EX_1^2 I\{c_{j-1} < |X_1| \leqslant c_j\} = H(\infty) EX_1^2$$

and, if $H(\infty) = \infty$, the sum is dominated by $EX_1^2 H(\psi(X_1^2/\log^+ \log^+ |X_1|))$. We thus conclude that $E(W'')^2 < \infty$, which, together with (10.1) and (10.2), proves that $EW^2 < \infty$.

Finally, by Theorem 2.2 we know that $P(V < \infty) = 1$. An application of [11], Corollary 3.4, therefore yields $EV^2 < \infty$.

Proof of the necessities. Like in [6] and [7] we assume without restriction that $0 < P(|X_1| < 1) < 1$. Since $EX_1^2 < \infty$ is trivially necessary, it follows from the LIL and (7.1) that

$$\sum_{k=3}^{\infty} P(|Y_k| > \varepsilon\sqrt{n_k \log\log n_k}) < \infty \quad \text{for some } \varepsilon > 0 \tag{10.4}$$

(in fact, the sum is convergent at least for all $\varepsilon > 2\sigma\sqrt{2}$).

Set

$$A(x) = \prod_{k=1}^{\infty} P(|Y_k| \leqslant x\sqrt{n_k \log^+ \log^+ n_k}).$$

In view of (10.4) we have (cf. [7], formula (4.1))

$$EX_1^2 < \infty \Rightarrow A(x) > 0 \quad \text{for } x > \text{some } x_0. \tag{10.5}$$

To prove the necessity in (a) one now proceeds exactly like in [7], (2.3)$\Rightarrow$(2.4a). We thus begin by assuming that the distribution is symmetric. First, we observe that

$$EV^2 < \infty \Rightarrow EW^2 < \infty.$$

Next,

$$(10.6)\qquad E \sup_{n>n_0} \frac{S_n^2}{n \log\log n} \leqslant 2 \sum_{k=k_0}^{\infty} E \sup_{n_{k-1} \leqslant n \leqslant n_k} \frac{(S_n - S_{n_{k-1}})^2}{n_{k-1} \log\log n_{k-1}} + 2EV^2$$

after which one needs to show that

$$(10.7)\qquad \sum_{k=k_0}^{\infty} \sum_{m=m_0}^{\infty} P\big(\sup_{n_{k-1} \leqslant n \leqslant n_k} (S_n - S_{n_{k-1}})^2 > mn_{k-1} \log\log n_{k-1}\big) < \infty.$$

By (10.5)

$$P(W^2 > m) = P(W > \sqrt{m}) \geqslant A(\sqrt{m}) \sum_{k=k_0}^{\infty} P(|Y_k| > \sqrt{mn_k \log\log n_k}) > 0$$

if $\sqrt{m} > x_0$ and, finally, (10.7) follows, i.e.

$$EV^2 < \infty \Rightarrow E \sup_n \frac{S_n^2}{n \log\log n} < \infty,$$

from which we conclude that

$$EX_1^2 \frac{\log^+ |X_1|}{\log^+ \log^+ |X_1|} < \infty$$

by [14].

A desymmetrization concludes the proof. We omit further details.

To prove the necessity in (b) we proceed like in [7], (2.3)$\Rightarrow$(2.4.b). Thus, omitting details,

$$
\begin{aligned}
(10.8)\qquad \infty > 4EV^2 \geqslant EW^2 &\geqslant \sum_{m=m_0}^{\infty} P(W > \sqrt{m}) \\
&\geqslant A(\sqrt{m_0}) \sum_{k=k_0}^{\infty} \sum_{m=m_0}^{\infty} P(|Y_k| > \sqrt{mn_k \log\log n_k}) \\
&\geqslant c_1 A(\sqrt{m_0}) \sum_{k=k_0}^{\infty} \sum_{m=m_0}^{\infty} n_k P(X_1^2 > c_2 mn_k \log\log n_k) \\
&\geqslant c_1 A(\sqrt{m_0}) \sum_{i=1}^{\infty} \Big(\sum_{mn_k \log\log n_k \leqslant i} n_k \Big) P\big(c_2 i < X_1^2 \leqslant c_2(i+1)\big) - c_3,
\end{aligned}
$$

where now $\sqrt{m_0} > x_0$ (cf. (10.5)) and c_1, c_2 and c_3 are numerical constants. By rescaling $\{X_n\}$ we can and do assume that $c_2 = 1$.

By inversion we find that, for large i,

$$\Big(\sum_{mn_k \log^+ \log^+ n_k \leqslant i} n_k\Big) \sim \sum_{n_k \leqslant i/\log\log i} \Big(\sum_{m \leqslant i/n_k \log^+ \log^+ n_k} 1\Big) n_k$$

$$= \sum_{n_k \leqslant i/\log\log i} n_k \left[\frac{i}{n_k \log^+ \log^+ n_k}\right] \sim i \sum_{n_k \leqslant i/\log\log i} (\log^+ \log^+ n_k)^{-1}$$

$$= i \sum_{k \leqslant \psi(i/\log\log i)} (\log^+ \log^+ n_k)^{-1} = iH\big(\psi(i/\log\log i)\big).$$

The sum in (10.8) (with $c_2 = 1$) thus majorizes

$$(10.9)\qquad \sum_{i=1}^{\infty} iH\big(\psi(i/\log^+ \log^+ i)\big) P(i < X_1^2 \leqslant i+1)$$
$$\sim EX_1^2 H\big(\psi(X_1^2/\log^+ \log^+ |X_1|)\big),$$

which completes the proof.

11. Miscellania. In this final section we collect some additional results and remarks.

11.1. In the process of proving Theorem 2.2 we found that

$$\sum_k P(|S_{n_k}| > \varepsilon\sqrt{n_k \log\log n_k}) < \infty \; (= \infty)$$

for $\varepsilon > \sigma\varepsilon^*$ ($\varepsilon < \sigma\varepsilon^*$) when

$$\limsup_{k\to\infty} n_k/n_{k+1} < 1$$

(see (4.1) and (5.1)). By choosing the particular sequence $n_k = [c^k]$, where $c > 1$, and by performing computations like those in [5] and [8] we obtain

COROLLARY 11.1. *Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. random variables with mean* 0 *and finite variance σ^2. Then*

$$(11.1)\qquad \sum_{n=3}^{\infty} \frac{1}{n} P(|S_n| > \varepsilon\sqrt{n\log\log n}) < \infty \quad \text{for } \varepsilon > \sigma\sqrt{2},$$

$$(11.2)\qquad \sum_{n=3}^{\infty} P(|S_{[c^n]}| > \varepsilon\sqrt{c^n \log^+ \log^+ c^n}) < \infty \quad \text{for } \varepsilon > \sigma\sqrt{2},$$

$$(11.3)\qquad \sum_{n=3}^{\infty} \frac{1}{n} P(|S_n| > \varepsilon\sqrt{n\log\log n}) = \infty \quad \text{for } \varepsilon < \sigma\sqrt{2},$$

$$(11.4)\qquad \sum_{n=3}^{\infty} P(|S_{[c^n]}| > \varepsilon\sqrt{c^n \log^+ \log^+ c^n}) = \infty \quad \text{for } \varepsilon < \sigma\sqrt{2}.$$

Conversely, if one of the sums is finite for some ε, then so are the others, $EX_1^2 < \infty$ and $EX_1 = 0$.

As for the sufficiency, (11.1) and (11.3) were proved in [2], Theorem 4, by normal approximation. For (11.1) and (11.2), see [8], [9], and for the converse, see [8], Theorem 6.2.

11.2 Closely connected with the sums studied above is the number of boundary crossings, i.e.

$$N(\varepsilon) = \sum_{k=1}^{\infty} \mathrm{I}\{|S_{n_k}| > \varepsilon\sqrt{n_k \log^+ \log^+ n_k}\}, \qquad \varepsilon > 0.$$

In particular,

$$EN(\varepsilon) = \sum_{k=1}^{\infty} P(|S_{n_k}| > \varepsilon\sqrt{n_k \log^+ \log^+ n_k}). \tag{11.5}$$

For $n_k = k$ it was shown in [15] that $EN^r(\varepsilon) = +\infty$ for all r and $\varepsilon > 0$ and, in [8], Corollary 8.3, that $E \log^+ N(\varepsilon) < \infty$ for $\varepsilon > 2\sigma$ provided $EX_1^2 (\log^+ \log^+ |X_1|)^{-1} \log^+ |X_1| < \infty$.

By combining the results obtained earlier we get

COROLLARY 11.2. *Let $\{n_k\}$ be as in Theorem 2.2, i.e. such that*

$$\limsup_{k\to\infty} n_k/n_{k+1} < 1$$

and suppose that $\{X_n\}_{n=1}^{\infty}$ are i.i.d. random variables with $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$. Then

$$EN(\varepsilon) \begin{cases} < \infty & \textit{for } \varepsilon > \sigma\varepsilon^*, \\ = \infty & \textit{for } \varepsilon < \sigma\varepsilon^*. \end{cases} \tag{11.6}$$

Remark 11.1. For the case $\varepsilon^* = 0$ it follows, in view of Theorem 7.1, that $EN(\varepsilon) < \infty$ for all $\varepsilon > 0$ provided (7.6) and (7.7) hold.

11.3. For the case

$$\liminf_{k\to\infty} n_k/n_{k+1} > 0$$

we did not consider

$$\sum_k P(|S_{n_k}| > \varepsilon\sqrt{n_k \log\log n_k}),$$

but

$$\sum_j P(|S_{n_{m_j}}| > \varepsilon\sqrt{n_{m_j} \log\log n_{m_j}}),$$

with $\{n_{m_j}\}_{j=1}^{\infty}$ as defined in (5.12). Nevertheless, one can ask whether the former sum converges. Clearly, $EX_1^2 < \infty$ (and $EX_1 = 0$) is a necessary moment condition. Moreover, by using computations like those leading to (7.3), we find that *if*

$$\sum_{k=3}^{\infty} P(|S_{n_k}| > \varepsilon\sqrt{n_k \log\log n_k}) < \infty \qquad \text{for some } \varepsilon > 0,$$

then necessarily we must have

(11.7) $$\sum_{k=3}^{\infty} n_k P(|X_1| > \varepsilon_1 \sqrt{n_k \log\log n_k}) < \infty \quad \text{for some } \varepsilon_1 > 0.$$

By adding further points to the subsequence we increase the sum in (11.7), i.e. its "largest" value is

$$\sum_{n=3}^{\infty} nP(|X_1| > \varepsilon_1 \sqrt{n \log\log n}),$$

the finiteness of which is equivalent to $EX_1^4/(\log^+ \log^+ |X_1|)^2 < \infty$. Thus, if we only assume finite variance (and zero mean) the sum of interest may be divergent for *all* $\varepsilon > 0$.

But, even more is true; namely, even if the necessary moment conditions are satisfied we need not have convergence for any $\varepsilon > 0$. In fact, suppose that the summands are uniformly bounded and that the sum converges for some ε. It then follows from (3.7) that we must have

$$\sum_k (\log n_k)^{-\varepsilon^2/2\sigma^2} < \infty.$$

However, if for example $n_k = k^d$, where d is a positive integer, then this sum equals $+\infty$ for all $\varepsilon > 0$ and thus

$$EN(\varepsilon) = \sum_{k=3}^{\infty} P(|S_{n_k}| > \varepsilon\sqrt{n_k \log\log n_k}) = +\infty \quad \text{for all } \varepsilon > 0.$$

(For $d = 1$ recall the comments following (11.5)).

As for positive results we shall confine ourselves to the following

Example 11.1. Suppose that $n_k = [e^{\sqrt{k}}]$. Then $M(x) \sim 2\sqrt{x} e^{\sqrt{x}}$, $\psi(x) \sim (\log x)^2$ and hence

$$M(\psi(x)) \sim 2x \log x (\geqslant x) \quad \text{as } x \to \infty.$$

It follows from Lemma 3.1 (and Remark 3.1) that (11.7) holds iff $EX_1^2 \log^+ |X_1| (\log^+ \log^+ |X_1|)^{-1} < \infty$, which hence is necessary for the expressions in (11.5) to be finite. In particular, finite variance is not enough.

Now, by truncating at $\frac{1}{2}b_n$, where

$$b_n = \frac{2\delta\sigma^2}{\varepsilon} \sqrt{n/\log^+ \log^+ n}$$

(cf. Section 3), and by using Lemma 3.2 and the fact that

$$\sum_{k=3}^{\infty} n_k P(|X_1| > \sqrt{n_k/\log\log n_k}) < \infty \Leftrightarrow EX_1^2 \log^+ |X_1| \log^+ \log^+ |X_1| < \infty$$

(Lemma 3.1 and Remark 3.1) it follows that

(11.8) $$\sum_{k=3}^{\infty} P(|S_{n_k}| > \varepsilon\sqrt{n_k \log\log n_k}) \begin{cases} < \infty & \text{for } \varepsilon > 2\sigma, \\ = \infty & \text{for } \varepsilon < 2\sigma \end{cases}$$

and

$$(11.9)\qquad EN(\varepsilon)\begin{cases} < \infty & \text{for } \varepsilon > 2\sigma, \\ = \infty & \text{for } \varepsilon < 2\sigma, \end{cases}$$

provided

$$(11.10)\qquad EX_1^2 \log^+ |X_1| \log^+ \log^+ |X_1| < \infty \quad \text{and} \quad EX_1 = 0.$$

We also recall from Theorem 2.1 that

$$\limsup_{k\to\infty} (n_k \log\log n_k)^{-1/2} |S_{n_k}| = \sigma\sqrt{2} \text{ a.s.}$$

or, equivalently, that $N(\varepsilon) < \infty$ a.s. when $\varepsilon > \sigma\sqrt{2}$. In this particular case we thus are in the position that, if (11.10) holds and $\sigma\sqrt{2} < \varepsilon < 2\sigma$, then

$$\limsup_{k\to\infty} (n_k \log\log n_k)^{-1/2} |S_{n_k}| = \sigma\sqrt{2} < \varepsilon,$$

but

$$(11.11)\qquad \sum_{k=3}^{\infty} P(|S_{n_k}| > \varepsilon\sqrt{\log\log n_k}) = +\infty.$$

We thus have a situation intermediate to the cases $n_k = k^d$ and $\limsup_{k\to\infty} n_k/n_{k+1} < 1$, respectively.

For results related to the above, but dealing with the strong law of large numbers, we refer to [1] and [10].

11.4. A useful tool for proving that sums of tail probabilities converge is formula (3.3), p. 164, of [11] (see for example [5]). Here we shall indicate how a weaker version of Theorem 7.1 can be proved this way.

Suppose that $EX_1^2/(\log^+ \log^+ |X_1|)^{1-\delta} < \infty$ for some δ $(0 < \delta < 1)$ and truncate $X_1, \ldots, X_n$ at $\sqrt{n \log\log n}$. Suppose also that the variables have a symmetric distribution.

From the computations preceding Theorem 7.1 we know that

$$EX_1^2 I\{|X_1| \leqslant \sqrt{n\log\log n}\} \leqslant (\log\log n)^{1-\delta} EX_1^2/(\log^+ \log^+ |X_1|)^{1-\delta},$$

which together with an iteration of the inequality of Hoffmann-Jørgensen (see also [5], Lemma 2.4), applied to S_n' and Chebyshev's inequality yields

$$\sum_k P(|S_{n_k}| > 2\cdot 3^j \varepsilon\sqrt{n_k \log\log n_k})$$
$$\leqslant C_j \sum_k n_k P(|X_1| > \sqrt{n_k \log\log n_k}) + D_j \sum_k \left(\frac{EX_1^2/(\log^+ \log^+ |X_1|)^{1-\delta}}{(\log\log n_k)^{\delta}}\right)^{2^j},$$

where C_j and D_j are numerical constants, which only depend upon j.

By (3.3), the first sum in the RHS converges if $EX_1^2/\log^+ \log^+ |X_1| < \infty$.

The second sum converges if

$$\sum_k (\log\log n_k)^{-\delta\cdot 2^j} < \infty.$$

For example, if $n_k = 2^{2^k}$ this occurs as soon as $\delta \cdot 2^j > 1$. Thus, given a sequence $\{n_k\}$ such that

$$\sum_k (\log\log n_k)^{-a} < \infty \quad \text{for some } a > 0$$

(which, by (7.9), is a stronger assumption than $\varepsilon^* = 0$ only), then if j is so large that $\delta \cdot 2^j > a$ we can conclude that

$$\sum_k P(|S_{n_k}| > 2\cdot 3^j \varepsilon \sqrt{n_k \log\log n_k}) < \infty.$$

Since ε may be chosen arbitrarily small (j remains fixed, depending only on δ), the conclusion follows. Note, however, again that this simpler proof yields a result which is weaker than Theorem 7.1 in that more integrability is required (this can be somewhat weakened) and in that fewer sequences for which $\varepsilon^* = 0$ are included.

11.5. The special feature of the LIL is that it describes the asymptotic fluctuations of the random walk $\{S_n\}$. Theorem 2.1 tells us that the fluctuations are of the same order of magnitude for subsequences which do not increase too rapidly. Theorem 2.2 tells us that for rapidly increasing subsequences the asymptotic fluctuations are smaller. In particular, when $\varepsilon^* = 0$, they seem to be of a smaller order of magnitude – the normalization $\sqrt{n_k \log\log n_k}$ is too strong. The following result describes the fluctuations for the latter cases:

THEOREM 11.1. *Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing subsequence of the positive integers such that*

$$\limsup_{k\to\infty} \frac{n_k}{n_{k+1}} < 1$$

and suppose that $\{X_n\}_{n=1}^{\infty}$ are i.i.d. random variables with $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$. Then

$$\limsup_{k\to\infty}\,(\liminf_{k\to\infty}) \frac{S_{n_k}}{\sqrt{n_k \log k}} = {}_{(-)}^{+}\sigma\sqrt{2} \text{ a.s.} \tag{11.12}$$

Conversely, if

$$P\left(\limsup_{k\to\infty} \frac{|S_{n_k}|}{\sqrt{n_k \log k}} < \infty\right) > 0,$$

then $EX_1^2 < \infty$ and $EX_1 = 0$.

The proof of the sufficiency is the same as that of Theorem 2.2 except that b_n now equals $\varepsilon^{-1} 2\delta\sigma^2 \sqrt{n}(\log^+ \psi(n))^{-1}$ and truncation is performed at

$\frac{1}{2}b_n$ and at $\sqrt{n}$. Further, $\log\log n$ in the previous proof is now replaced by $\log\psi(n)$ at relevant places (note then that $\log\log n_k \leftrightarrow \log\psi(n_k) = \log k$). The proof of the necessity follows like in Section 6 (i.e. like in [16], p. 297), again with $\log\log n_k$ replaced by $\log k$ and with the sufficiency above playing the role of the Hartman-Wintner law. We leave the details to the reader.

When $0 < \varepsilon^* \leqslant \sigma\sqrt{2}$, we know that $\{n_k\}$ increases at least geometrically, i.e. ψ increases at most as the logarithmic function, which means that $\log k = \log\psi(n_k)$ increases at most as $\log\log n_k$. We thus normalize with a sequence which is *smaller* than the previous one, but, in view of (7.9), not of a smaller order of magnitude. For example, the case $n_k = [2^{k^\beta}]$, $\beta > 1$ (cf. Example 4 in Section 2), yields $\psi(x) \sim (\log k)^{1/\beta}$, $\varepsilon^* = \sqrt{2/\beta}$, $\log\log n_k = \beta\log k$. Thus, the limit superiors in Theorems 2.2 and 11.1 only differ by a constant scaling factor β, and nothing essentially new has been obtained. For the limiting case $n_k = 2^k$ Theorem 11.1 is, of course, the same as Theorem 2.2.

When $\varepsilon^* = 0$, however, we know, by (7.9), that $\log k = o(\log\log n_k)$ as $k \to \infty$, i.e. the normalization here is of a *smaller order of magnitude* and the fluctuations of size $o(\sqrt{n_k \log\log n_k})$ have been magnified into a readable size – $O(\sqrt{n_k \log k})$. If, for example, $n_k = 2^{2^k}$, $\log\log n_k \sim k$ as compared to $\log k$ here.

Another observation is that, since the subsequence increases very rapidly when $\varepsilon^* = 0$, the influence of $S_{n_{k-1}}$ on S_{n_k} should be small or, more precisely, $S_{n_{k-1}}$ and S_{n_k} should be fairly uncorrelated (in fact the coefficient of correlation is $\sqrt{n_{k-1}/n_k}$ (which converges to 0 in typical cases like $n_k = 2^{2^k}$ etc.)). Furthermore, $S_{n_k}/\sqrt{n_k}$ is asymptotically $N(0, \sigma^2)$ as $k \to \infty$. Thus, the sequence $\{S_{n_k}/\sqrt{n_k \log k}\}_{k=1}^{\infty}$ can be expected to behave asymptotically like the sequence $\{Z_k/\sqrt{\log k}\}_{k=1}^{\infty}$, where $\{Z_k\}_{k=1}^{\infty}$ are i.i.d. $N(0, \sigma^2)$-distributed random variables. Indeed, by well-known estimates for the normal distribution (cf. e.g. [16], p. 256)

$$P(|Z_1| > \varepsilon\sqrt{\log k}) \sim \frac{\sigma}{\varepsilon\sqrt{\log k}} \exp\left\{-\frac{\varepsilon^2}{2\sigma^2}\log k\right\} \quad \text{as } k \to \infty$$

and thus

$$\sum_{k=3}^{\infty} P(|Z_k| > \varepsilon\sqrt{\log k}) < \infty \Leftrightarrow \varepsilon > \sigma\sqrt{2}. \tag{11.13}$$

Now, since $\{Z_k\}$ are independent, the Borel-Cantelli lemma applies in both directions and we conclude that

$$\limsup_{k\to\infty} \frac{Z_k}{\sqrt{\log k}} = \sigma\sqrt{2} \text{ a.s.} \tag{11.14}$$

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REFERENCES

[1] S. Asmussen and T. G. Kurtz, *Necessary and sufficient conditions for complete convergence in the law of large numbers*, Ann. Probab. 8 (1980), p. 176–182.

[2] J. A. Davis, *Convergence rates for the law of the iterated logarithm*, Ann. Math. Statist. 39 (1968), p. 1479–1485.

[3] W. Feller, *The law of the iterated logarithm for identically distributed random variables*, Ann. Math. 47 (1946), p. 631–638.

[4] – *An extension of the law of the iterated logarithm to variables without variance*, J. Math. Mech. 18 (1968), p. 343–355.

[5] A. Gut, *Marcinkiewicz laws and convergence rates in the law of large numbers for random variables with multidimensional indices*, Ann. Probab. 6 (1978), p. 469–482.

[6] – *Moments of the maximum of normed partial sums of random variables with multidimensional indices*, Z. Wahrsch. verw. Gebiete 46 (1979), p. 205–220.

[7] – *On the integrability of* $\sup|S_n/n|$ *for subsequences*, Ann. Probab. 7 (1979), p. 1059–1065.

[8] – *Convergence rates for probabilities of moderate deviations for sums of random variables with multidimensional indices*, ibidem 8 (1980), p. 298–313.

[9] – *Strong laws for independent identically distributed random variables indexed by a sector*, ibidem 11 (1983), p. 569–577.

[10] – *On complete convergence in the law of large numbers for subsequences*, ibidem 13 (1985), p. 1286–1291.

[11] J. Hoffmann-Jørgensen, *Sums of independent Banach space valued random variables*, Studia Math. 52 (1974), p. 159–186.

[12] R. M. Huggins, *Laws of the iterated logarithm for time changed Brownian motion and martingales with an application to branching processes*, Ann. Probab. 13 (1985), p. 1148–1156.

[13] P. A. Meyer, *Probabilités et potentiel*, Hermann, Paris 1966.

[14] D. Siegmund, *On moments of the maximum of normed partial sums*, Ann. Math. Statist. 40 (1969), p. 527–531.

[15] J. Slivka, *On the law of the iterated logarithm*, Proc. Nat. Acad. Sci. U.S.A. 63 (1969), p. 289–291.

[16] W. F. Stout, *Almost sure convergence*, Academic Press, 1974.

[17] M. J. Wichura, *Some Strassen-type laws of the iterated logarithm for multiparameter stochastic processes with independent increments*, Ann. Probab. 1 (1973), p. 272–296.

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