PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 8 (1987), p. 77–80

LINEAR KERNELS OF p-STABLE MEASURES

BY

ZDZISŁAW SUCHANECKI (WROCŁAW)

Abstract. The aim of the paper is to give a description of linear kernels of symmetric p-stable measures in terms of Pettis-type-integral of some operators.

1. Introduction. Let E be a locally convex space and μ a cylindrical measure on E. The *linear kernel* of μ is defined as the topological dual to the space E' endowed with the topology of the convergence in μ. One of the problems concerning the linear kernel of an arbitrary cylindrical measure is its description. Partial results in this direction can be found in [5] and [7]. The first paper concerns p-stable Radon measures with discrete spectrum. The second one – a special class of p-stable processes. We give a description of linear kernels of symmetric p-stable measures in terms of Pettis-type-integral of some operators.

2. Preliminaries. Let E denote a locally convex space (l.c.s.) with topological dual E' and let $(\Omega, \mathscr{A}, P)$ be a probability space.

L^p, $0 < p < \infty$, denotes the space $L^p(\Omega, \mathscr{A}, P)$. If μ is a cylindrical measure on the algebra of cylindrical subsets of E, $\hat{\mu}$ is the characteristic function, i.e.

$$\hat{\mu}(x') = \int_E e^{i\langle x,x'\rangle} d\mu(x'), \qquad x' \in E'.$$

In this paper we consider only cylindrical measures on E which characteristic function has the form

$$(*) \qquad \hat{\mu}(x') = \exp\{-\|Tx'\|_{L^p}^p\},$$

where T is a continuous linear operator from E'_τ (τ denotes the Mackey topology) into L^p.

Note that every symmetric p-stable Radon measure has the form $(*)$ (cf. [3]).

PROPOSITION 1 ([4], p. 270). *Let E be an l.c.s. and let $T: E'_\tau \to L^1(\Omega, \mathscr{A}, P)$ be a continuous linear operator. Then T is Pettis integrable, i.e. for every A from $\mathscr{A}$ there is an element x_A of E such that*

$$\langle x_A, x' \rangle = \int_A Tx'\, dP \quad \text{for every } x' \in E'.$$

We write $x_A = \int_A T\,dP$.

If $p \geqslant 1$ and $T: E'_\tau \to L^p$ is a continuous linear operator, then for every function $\psi \in L^q$, $1/p + 1/q = 1$, the linear operator $\psi T: x' \to \psi Tx'$ is continuous as a map from E'_τ into L^1. Therefore, by Proposition 1, the integral $\int_\Omega \psi T\,dP$ exists and is an element of E.

If we endow the dual space E' with the topology s_μ of the convergence in μ, then the space $H_\mu = (E', s_\mu)'$ is called the *linear kernel* of μ.

3. Main result.

THEOREM 1. *Let E be an l.c.s. and μ a cylindrical measure on E with characteristic function of the form $(*)$. If $1 \leqslant p < \infty$, then*

$$H_\mu = \{\int_\Omega \psi T\,dP \colon \psi \in L^q\},$$

where $1/p + 1/q = 1$ (if $p = 1$, then $q = \infty$).

Proof. We define the sets:

$$V_n = \left\{x' \in E' \colon \|Tx'\|_{L^p} \leqslant \frac{1}{n}\right\}, \qquad A_n = \{\int_\Omega \psi T\,dP \colon \|\psi\|_{L^q} \leqslant n\}.$$

Let V_n^0 denote the *polar* of V_n, i.e. $V_n^0 = \{x \in E \colon |\langle x, x'\rangle| \leqslant 1$ for every $x' \in V^n\}$. First, we show that $A_n = V_n^0$.

Let a be an element of A_n. Then $a = \int_\Omega \psi T\,dP$, and we get

$$|\langle a, x'\rangle| = \left|\langle \int_\Omega \psi T\,dP, x'\rangle\right| = \left|\int_\Omega \psi Tx'\,dP\right|$$
$$\leqslant \|\psi\|_{L^q}\|Tx'\|_{L^p} \leqslant n\frac{1}{n} = 1$$

for every $x' \in V_n$. Therefore, $A_n \subset V_n^0$.

If $x' \in A_n^0$, then, for every $\psi \in L^q$, $\|\psi\|_{L^q} \leqslant 1$, and $\left|\int_\Omega \psi Tx'\,dP\right| \leqslant 1/n$, which implies $\|Tx'\|_{L^p} \leqslant 1/n$. Therefore $x' \in V_n$ and, consequently, $A_n^0 \subset V_n$, which implies $A_n \subset V_n^0 \subset A_n^{00}$.

It is easy to check that A_n is convex and balanced. We show that A_n is weakly closed in E. Let $\{a_\alpha\} = \{\int_\Omega \psi_\alpha T\,dP\}$ be a net in A_n which converges to an element $a \in E$. Since $\{a_\alpha\} \subset A_n$, $\|\psi_\alpha\| \leqslant n$ for each α. Since a ball in L^q (in L^∞) is *-weakly closed, there exists a sub-net $\{\psi_\beta\}$ of $\{\psi_\alpha\}$ which weakly converges in L^q (in L^∞) to some ψ_0, $\|\psi_0\| \leqslant n$. Therefore

$$\langle a_\beta, x' \rangle = \int_\Omega \psi_\beta \, Tx' \, dP \to \int_\Omega \psi_0 \, Tx' \, dP \quad \text{for every } x' \in E'.$$

Since $\{a_\beta\}$ also weakly converges to a, we get $a = \int_\Omega \psi_0 \, T dP$, which shows that A_n is weakly closed.

By the Bipolar Theorem (cf. [2]), $A_n^{00} = A_n$, which gives $A_n = V_n^0$.

Now, we define

$$U_n = \{x' \in E' \colon \mu\{x \colon |\langle x, x' \rangle| \geqslant 1/n\} \leqslant 1/n\}, \qquad n = 1, 2, \ldots$$

The sets U_n form a neighbourhood base of 0 in E for s_μ-topology.

Denoting by U_n° the polar of U_n with respect to the dual pair $\langle E', E'^* \rangle$ (* denotes the algebraical dual), we get

$$H_\mu = \bigcup_{n=1}^{\infty} U_n^\circ.$$

Since $\hat{\mu}$ is of the form (*), $\hat{\mu} \colon E'_\tau \to R$ is continuous. Therefore, the measure μ is scalarly concentrated on the family of all absolutely convex weakly compact subsets of E. Consequently, the canonical embedding of E'_τ into (E', s_μ) is continuous (cf. [1], pp. 26 and 30). So, for every n there is an absolutely convex compact set $K_n \subset E$ such that $K_n^0 \subset U_n$.

Note that $(K_n^0)^\circ = K_n$. Indeed, if $x' \in E'^*$ is an element of $(K_n^0)^\circ$, then $|\langle x, x' \rangle| \leqslant 1$ for every $x' \in K_n^0$. But this means that x is continuous on E. Hence x belongs to E and it follows from Bipolar Theorem that $(K_n^0)^\circ = (K_n^0)^0 = K_n$.

Because of $U_n^\circ = (K_n^0)^\circ = K_n$, we get

$$H_\mu = \bigcup_{n=1}^{\infty} U_n^\circ \subset E.$$

Using (*) once again we conclude that if $\{x_n'\}$ converges to x_0' in s_μ-topology, then $\|Tx_n' - Tx_0'\|_{L^p} \to 0$. So, for any n there is a $k(n)$ such that $U_{k(n)} \subset V_n$.

Conversely, it is easy to check that $Tx_n' \to Tx_0'$ in L^p implies $x_n' \to x_0'$ in μ. So, for every n there is an $l(n)$ such that $V_{l(n)} \subset U_n$.

Finally, we obtain

$$H_\mu = \bigcup_{n=1}^{\infty} V_n^0 = \bigcup_{n=1}^{\infty} A_n = \Big\{\int_\Omega \psi T dP \colon \psi \in L^q\Big\},$$

which completes the proof of our theorem.

Remark 1. The fact that H_μ is a subspace of E was known for Radon measures (cf. [3]).

Remark 2. From Theorem 1 we infer that H_μ *is a Banach space*. Indeed, we can take the Minkowski functional of the set A_1 as a norm on H_μ. This norm is equivalent to the norm in L^q (cf. also [5]).

REFERENCES

[1] A. Badrikian, *Séminaire sur les fonctions aléatoires linéaires et les mesures cylindriques*, Lecture Notes in Math. 139 (1970).

[2] H. H. Schaefer, *Topological Vector Spaces*, New York 1980.

[3] W. Smoleński, *Pre-supports and kernels of probability measures in Fréchet spaces*, Demonstratio Math. 10 (1977), p. 751–762.

[4] Z. Suchanecki, *Remarks on Pettis integrability of cylindrical processes*, Lecture Notes in Math. 828 (1980), p. 269–273.

[5] D. H. Thang, *Weakly admissible translates of probability measures on locally convex spaces*. Preprint (1981).

[6] A. Tortrat, *Lois $e(\lambda)$ dans les espaces vectoriels et lois stables*, Z. Wahrschein. verw. Gebiete 37 (1976), p. 175–182.

[7] J. Zinn, *Admissible translates of stable measures*, Studia Math. 54 (1976), p. 245–257.

Institute of Mathematics
Technical University
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland

Received on 29. 10. 1985