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DOMAINS OF ATTRACTION AND MOMENTS

BY

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Abstract. The limit behaviour of scalar modifications of powers of probability measures under a generalized convolution is considered. In particular, some necessary and sufficient conditions in terms of moments and medians for a probability measure to belong to the domain of attraction of a compact set consisting of non-degenerate at the origin measures are established.

1. Notation and preliminaries. Generalized convolutions were introduced in [3]. We recall some basic definitions. Let P denote the set of all Borel probability measures on the positive half-line $R_+ = [0, \infty)$. The set P is endowed with the topology of weak convergence. For $\mu \in P$ and $a > 0$ we define the map T_a by setting $(T_a \mu)(E) = \mu(a^{-1} E)$ for all Borel subsets E of R_+. By δ_c we denote the probability measure concentrated at the point c.

A continuous in each variable separately commutative and associative P-valued binary operation $\circ$ on P is called a *generalized convolution* if it is distributive with respect to convex combinations and maps T_a $(a > 0)$ with δ_0 as the unit element. Moreover, there exist a sequence $\{c_n\}$ of positive norming constants and a measure $\gamma \in P$ other than δ_0 such that $T_{c_n} \delta_1^{\circ n} \to \gamma$, where $\delta_1^{\circ n}$ is the n-th power of δ_1 under $\circ$. The measure γ is called the *characteristic measure* of $\circ$. By Propositions 4.4 and 4.5 in [4] it is defined *uniquely* up to a scale change T_a $(a > 0)$ and fulfils the equation

$$T_a \gamma \circ T_b \gamma = T_{g_\varkappa(a,b)} \gamma \quad (a, b > 0),$$

where $0 < \varkappa \leqslant \infty$, $g_\varkappa(a, b) = (a^\varkappa + b^\varkappa)^{1/\varkappa}$ if $0 < \varkappa < \infty$ and $g_\infty(a, b) = \max(a, b)$. The constant $\varkappa$ is called the *characteristic exponent* of $\circ$. By Proposition 4.5 and Lemma 2.1 in [4], $\varkappa = \infty$ if and only if $\circ$ is the max-convolution.

Let m_0 be the sum of δ_0 and the Lebesgue measure on R_+. By P_0 we shall denote the subset of P consisting of all absolutely continuous with respect to m_0 measures. It has been proved in [4] (Theorem 4.1 and Corollary 4.4) that each generalized convolution $\circ$ admits a weak characteristic function, i.e. a one-to-one correspondence $\mu \leftrightarrow \hat{\mu}$ between measures μ from P and real-valued functions $\hat{\mu}$ from $L_\infty(m_0)$ such that the functions $\hat{\lambda}$ are continuous for $\lambda \in P_0$, $(c\mu+(1-c)\nu)\hat{} = c\hat{\mu}+(1-c)\hat{\nu}$ $(0 \leqslant c \leqslant 1)$, $[T_a\mu]\hat{}(t) = \hat{\mu}(at)$ $(a > 0)$ and $[\mu \circ \nu]\hat{} = \hat{\mu}\hat{\nu}$ for all $\mu, \nu \in P$. Moreover, the weak convergence $\mu_n \to \mu$ is equivalent to the convergence $\hat{\mu}_n \to \hat{\mu}$ in the $L_1(m_0)$-topology of $L_\infty(m_0)$. The weak characteristic function is uniquely determined up to a scale change and for any $\mu \in P$

$$\hat{\mu}(t) = \int_0^\infty \hat{\delta}_1(tx)\,\mu(dx) \tag{1.1}$$

m_0-almost everywhere.

For our purpose it is convenient to describe the weak convergence of measures in terms of the m_0-almost sure convergence of their weak characteristic functions.

LEMMA 1.1. *Let $\mu_n, \mu \in P$ $(n = 1, 2, \ldots)$. Then $\mu_n \to \mu$ if and only if each subsequence of indices contains a subsequence $n_1 < n_2 < \ldots$ such that $\hat{\mu}_{n_k} \to \hat{\mu}$ m_0-almost everywhere.*

Proof. Suppose that $\mu_n \to \mu$. Then, by Proposition 2.4 in [5], $\mu_n \circ \mu_n \to \mu \circ \mu$ and $\mu_n \circ \mu \to \mu \circ \mu$. Consequently, for every $\lambda \in P_0$,

$$\int_0^\infty \hat{\mu}_n^2(t)\,\lambda(dt) \to \int_0^\infty \hat{\mu}^2(t)\,\lambda(dt)$$

and

$$\int_0^\infty \hat{\mu}_n(t)\,\hat{\mu}(t)\,\lambda(dt) \to \int_0^\infty \hat{\mu}^2(t)\,\lambda(dt)$$

which yields

$$\int_0^\infty \left(\hat{\mu}_n(t)-\hat{\mu}(t)\right)^2 \lambda(dt) \to 0.$$

Taking a measure λ equivalent to m_0 we get the condition in question. Conversely, this condition and the boundedness of weak characteristic functions ([4], Lemma 4.4) imply the convergence

$$\int_0^\infty \hat{\mu}_n(t)\,\lambda(dt) \to \int_0^\infty \hat{\mu}(t)\,\lambda(dt)$$

for every $\lambda \in P_0$. Thus $\hat{\mu}_n \to \hat{\mu}$ in the $L_1(m_0)$-topology of $L_\infty(m_0)$ which yields $\mu_n \to \mu$. This completes the proof.

It has been shown in [5], Chapter 2, that the generalized convolution $\circ$ can be extended to the space $\bar{P}$ of all Borel probability measures on the compactified half-line $\bar{R}_+ = [0, \infty]$. Since the space $\bar{P}$ is compact in the topology of weak convergence, this enables us to use compactness arguments and, therefore, is a useful tool in the study of generalized convolutions. We identify the space P with the subspace of $\bar{P}$ consisting of measures with zero mass at ∞. By Theorem 4.2 and Corollaries 3.2 and 3.5 in [5], for any $\mu \in P$ other than δ_0 we have

$$\mu^{\circ n} \to \delta_c \text{ in } \bar{P}, \tag{1.2}$$

where $0 < c \leqslant \infty$. Moreover, $c = \infty$ whenever $\varkappa < \infty$.

Given $\mu \in P$ and a norming sequence of positive numbers $\{a_n\}$, by $G(\{a_n\}, \mu)$ we shall denote the set of all cluster points in $\bar{P}$ of the sequence $T_{a_n}\mu^{\circ n}$. Of course, the set $G(\{a_n, \mu)$ is compact in $\bar{P}$.

We say that μ *belongs to the domain of attraction of a compact subset of* $P \backslash \{\delta_0\}$ if $G(\{a_n\}, \mu) \subset P \backslash \{\delta_0\}$ for a norming sequence $\{a_n\}$. For the symmetric convolution this compactness property was introduced and studied by W. Feller in [1]. The aim of this paper is to give a necessary and sufficient condition for μ to belong to the domain of attraction of a compact subset of $P \backslash \{\delta_0\}$ in terms of the moments of $\mu^{\circ n}$. Another condition in terms of medians of $\mu^{\circ n}$ is contained in [7].

Given $\lambda \in \bar{P}$, by $m(\lambda)$ and $M(\lambda)$ we shall denote the lowest and the greatest median of λ, respectively. It is clear that the functions $\lambda \to m(\lambda)$ and $\lambda \to M(\lambda)$ are lower and upper semicontinuous respectively and

$$m(T_a \lambda) = am(\lambda), \quad M(T_a \lambda) = aM(\lambda) \quad (a > 0). \tag{1.3}$$

Moreover, by (1.2), $\lim_{n \to \infty} M(\mu^{\circ n}) > 0$ for $\mu \in P$ other than δ_0.

Denoting by r the greatest index for which $M(\mu^{\circ r}) = 0$, we put

$$c_n(\mu) = M(\mu^{\circ n})^{-1} \quad (n > r)$$

and

$$c_n(\mu) = 1 \quad (1 \leqslant n \leqslant r).$$

By (1.2) we have

$$c_n(\mu) \to 0 \quad \text{if } \varkappa < \infty. \tag{1.4}$$

For $p > 0$ we shall also use the notation

$$M_p(\mu) = \int_0^\infty x^p \mu(dx) \quad \text{and} \quad N_p(\mu) = M_p(\mu)^{1/p}.$$

It is evident that

$$M_p(c\mu + (1-c)\nu) = cM_p(\mu) + (1-c)M_p(\nu) \quad (0 \leqslant c \leqslant 1). \tag{1.5}$$

Our next result lies somewhat deeper.

LEMMA 1.2. *Suppose that $0 < p < \varkappa$. If $M_p(\mu^{\circ k}) < \infty$ for a positive integer k, then $M_p(\mu^{\circ n}) < \infty$ for all positive integers n.*

Proof. Let $\lambda, \nu \in P$. By Lemma 4.4 in [4] we have the inequalities $|\hat{\lambda}(t)| \leqslant 1$ and $|\hat{\nu}(t)| \leqslant 1$ m_0-almost everywhere. Consequently, $1-[\lambda \circ \nu]\hat{}(t) + 1-\hat{\nu}(t) \geqslant 1-\hat{\nu}(t)$ m_0-almost everywhere. Since, for $0 < p < \varkappa$,

$$\int_0^\infty \frac{1-\hat{\varrho}(t)}{t^{1+p}}\,dt = d_p M_p(\varrho) \qquad (\varrho \in P),$$

where $0 < d_p < \infty$ ([6], formula (5)), we get the inequality

$$M_p(\lambda) \leqslant M_p(\nu) + M_p(\lambda \circ \nu) \quad \text{for all } \lambda, \nu \in P. \tag{1.6}$$

Suppose now that $M_p(\mu^{\circ r}) < \infty$ and $r > 1$. Since in the case $\mu = \delta_0$ the Lemma is obvious, we may assume that $\mu \neq \delta_0$. There exists then a positive number b such that $0 < \mu([0, b)) < 1$. Setting $c = \mu([0, b))$, $\mu_1(E) = c^{-1}\mu(E \cap [0, b))$, $\mu_2(E) = (1-c)^{-1}\mu(E \cap [b, \infty))$, we have $\mu = c\mu_1 + (1-c)\mu_2$, $M_p(\mu_1) < \infty$ and, by (1.5), $M_p(\mu^{\circ(r-1)} \circ \mu_1) < \infty$. Substituting $\lambda = \mu^{\circ(r-1)}$ and $\nu = \mu_1$ into (1.6) we get the inequality $M_p(\mu^{\circ(r-1)}) < \infty$. An inductive repetition of this argument leads to the inequality $M_p(\mu) < \infty$. Applying Lemma 1 in [6] we obtain the inequality $M_p(\mu^{\circ n}) \leqslant nM_p(\mu)$ for every n, which completes the proof.

Given $0 < p < \varkappa$, we put

$$K_p(\mu) = \begin{cases} \overline{\lim\limits_{n\to\infty}} \dfrac{N_{2p}(\mu^{\circ n})}{N_p(\mu^{\circ n})}, & \text{whenever } 0 < N_p(\mu) < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

Observe that, by Lemma 2.3 in [5], $N_p(\mu^{\circ n}) > 0$ for all n provided $N_p(\mu) > 0$. This fact and Lemma 1.2 show that the above definition makes sense.

2. Norming sequences. In order to discuss properties of norming sequences we have to make a brief digression to describe the behaviour of tails of μ_k under the assumption that $\mu_k^{\circ n_k}$ is convergent for a subsequence $n_1 < n_2 < \ldots$ We begin with auxiliary results on generalized convolutions with finite exponent.

LEMMA 2.1. *Suppose that $\varkappa < \infty$ and $\mu \in P$. If the set $\{t\colon \hat{\mu}(t) = 1\}$ has positive Lebesgue measure, then $\mu = \delta_0$.*

Proof. Taking a probability measure ν with the support contained in $\{t\colon \hat{\mu}(t) = 1\}$ and absolutely continuous with respect to the Lebesgue measure on R_+ we have, by Lemma 3.11, Propositions 3.3 and 3.4, and Theorem 4.1 in [4],

$$\lim_{t\to\infty} \hat{\nu}(t) = 0. \tag{2.1}$$

Further, by Corollary 4.1 in [4],

$$\int_0^\infty \hat{\nu}(t)\,\mu^{\circ n}(dt) = \int_0^\infty \hat{\mu}^n(t)\,\nu(dt) = 1 \quad (n = 1, 2, \ldots). \tag{2.2}$$

Suppose that $\mu \neq \delta_0$. Then, by (1.2), $\mu^{\circ n} \to \delta_\infty$ and, consequently, by the continuity of $\hat{\nu}$ and (2.1),

$$\int_0^\infty \hat{\nu}(t)\,\mu^{\circ n}(dt) \to 0$$

which contradicts (2.2). Thus $\mu = \delta_0$.

LEMMA 2.2. *Suppose that $\varkappa < \infty$, $\nu \in P_0$ and $\nu \neq \delta_0$. Then, for every $a > 0$,*

$$\sup\{\hat{\nu}(t):\ t \geqslant a\} < 1.$$

Proof. Suppose the contrary. Since, by Lemma 3.11, Propositions 3.3 and 3.4, and Theorem 4.1 in [4],

$$\lim_{t\to\infty} \hat{\nu}(t) = \nu(\{0\}) < 1$$

and, by Lemma 4.4 in [4], $|\hat{\nu}(t)| \leqslant 1$, the continuity of $\hat{\nu}$ yields the existence of a number $u \geqslant a$ such that $\hat{\nu}(u) = 1$. Using formula (1.1) we get the equality $\hat{\delta}_u(x) = \hat{\delta}_1(ux) = 1$ for ν-almost all x. But this contradicts Lemma 2.1, which completes the proof.

LEMMA 2.3. *Suppose that $\varkappa < \infty$. If $n_1 < n_2 < \ldots$ and $\mu_k^{\circ n_k}$ $(k = 1, 2, \ldots)$ is convergent in P, then $\mu_k \to \delta_0$.*

Proof. By Corollary 2.3 in [5] the sequences μ_k and $\mu_k^{\circ(n_k - r)}$ $(n_k > r;\ r = 1, 2, \ldots)$ are conditionally compact in P. Passing to a subsequence if necessary we may assume without loss of generality that $\mu_k^{\circ n_k} \to \lambda$, $\mu_k \to \nu$ and $\mu_k^{\circ(n_k - r)} \to \nu_r$, where $\lambda, \nu, \nu_r \in P$ $(r = 1, 2, \ldots)$. Then, of course, $\nu^{\circ r} \circ \nu_r = \lambda$ $(r = 1, 2, \ldots)$ and, by Corollary 2.3 in [5], the sequence $\nu^{\circ r}$ is conditionally compact in P. Comparing this with (1.2) we conclude that $\nu = \delta_0$, which completes the proof.

LEMMA 2.4. *Suppose that $\varkappa < \infty$. If $n_1 < n_2 < \ldots$ and $\mu_k^{\circ n_k} \to \lambda$ in P, then $\hat{\lambda}(t) > 0$ m_0-almost everywhere.*

Proof. Applying Lemma 2.3 we obtain $\mu_k \to \delta_0$. Let s be a positive integer and p_k the integral part of n_k/s. Write $n_k = sp_k + r_k$, where $0 \leqslant r_k < s$, $\nu_k = \delta_0$ if $r_k = 0$ and $\nu_k = \mu_k^{\circ r_k}$ otherwise. Then $\nu_k \to \delta_0$ and $\mu_k^{n_k} = (\mu_k^{\circ p_k})^{\circ s} \circ \nu_k$ $(k = 1, 2, \ldots)$. Hence, by Corollary 2.3 in [5], it follows that the sequence $\mu_k^{\circ p_k}$ is conditionally compact in P. Let λ_s be its cluster point. Then

$$\lambda_s^{\circ s} = \lambda \quad (s = 1, 2, \ldots) \tag{2.3}$$

and, consequently, $\hat{\lambda}_s(t)^s = \hat{\lambda}(t)$ m_0-almost everywhere. This yields the inequality $\hat{\lambda}(t) \geqslant 0$ m_0-almost everywhere. Moreover, for odd indices s, $\hat{\lambda}_s(t) = \hat{\lambda}(t)^{1/s}$ m_0-almost everywhere. Put $E = \{t\colon \hat{\lambda}(t) = 0\}$. Then, for odd s, $\hat{\lambda}_s(t) = 0$ m_0-almost everywhere on E. We have to show that $m_0(E) = 0$. Contrary to this let us assume that $m_0(E) > 0$. Taking a measure ϱ from P_0 with the support contained in E we get

$$\int_0^\infty \hat{\lambda}_s(t)\,\varrho(dt) = 0 \quad (s = 1, 2, \ldots). \tag{2.4}$$

Since, by (2.3) and Lemma 2.3, $\lambda_s \to \delta_0$ and, by Lemma 4.1 in [4], $\hat{\lambda}_0(t) = 1$ $(t \in R_+)$, we have

$$\int_0^\infty \hat{\lambda}_s(t)\,\varrho(dt) \to 1 \quad \text{as } s \to \infty.$$

But this contradicts (2.4), which completes the proof.

LEMMA 2.5. *Suppose that* $n_1 < n_2 < \ldots$ *and* $\mu_k^{\circ n_k} \to \lambda$ *in* P. *Then*

$$\overline{\lim_{k\to\infty}}\, n_k \mu_k[(b, \infty)) < \infty$$

if either $\varkappa = \infty$ *and* $\lambda([0, b)) > 0$ *or* $\varkappa < \infty$ *and* $b > 0$.

Proof. First consider the case $\varkappa = \infty$. Then $\circ$ is the max-convolution and, consequently,

$$\mu_k^{n_k}([0, b)) \to \lambda([0, b))$$

for all continuity points b of λ. Hence, by standard calculations, we get the assertion of the Lemma.

Suppose now that $\varkappa < \infty$. From Lemma 2.3 it follows that $\mu_k \to \delta_0$. Passing to a subsequence if necessary and applying Lemma 1.1 we may assume without loss of generality that

$$\hat{\mu}_k^{n_k} \to \hat{\lambda} \tag{2.5}$$

and

$$\hat{\mu}_k \to 1 \tag{2.6}$$

m_0-almost everywhere. By Egorev Theorem ([2], Section 21, Theorem 1) there exists a Borel subset B of R_+ with $m_0(B) > 1$ such that, in view of Lemma 2.4, $\hat{\lambda}$ is bounded from below by a positive number on B and convergences (2.5) and (2.6) are uniform on B. This yields

$$\lim_{k\to\infty} n_k(1 - \hat{\mu}_k(t)) = -\log \hat{\lambda}(t) \tag{2.7}$$

uniformly on B. Since $m_0(B) > 1$, we can find a measure ϱ from P_0, other than δ_0, with the support contained in B. Then, by (2.7),

$$\lim_{k\to\infty} n_k \int_0^\infty \left(1-\hat{\mu}_k(t)\right)\varrho(dt) < \infty. \tag{2.8}$$

Observe that, by Corollary 4.1 in [4],

$$\int_0^\infty \hat{\mu}_k(t)\,\varrho(dt) = \int_0^\infty \hat{\varrho}(t)\,\mu_k(dt) \qquad (k = 1, 2, \ldots).$$

Given $b > 0$, we have, by Lemma 2.2, the inequality

$$c = \inf\{1-\hat{\varrho}(t)\colon\ t \geqslant b\} > 0,$$

which yields

$$\int_0^\infty \left(1-\hat{\mu}_k(t)\right)\varrho(dt) \geqslant c\mu_k\left([b, \infty)\right) \qquad (k = 1, 2, \ldots).$$

Now the assertion of Lemma 2.5 is an immediate consequence of inequality (2.8).

To state the next result we introduce some new notation.

Given $\mu \in P$, by $A(\mu)$ we denote the set of all norming sequences $\{a_n\}$ for which the inclusion $G(\{a_n\}, \mu) \subset P\setminus\{\delta_0\}$ is true. Two sequences $\{a_n\}$ and $\{b_n\}$ of non-negative numbers are said to be *equivalent*, in symbols $\{a_n\} \sim \{b_n\}$, if

$$0 < \varliminf_{n\to\infty} \frac{a_n}{b_n} \quad \text{and} \quad \varlimsup_{n\to\infty} \frac{a_n}{b_n} < \infty.$$

Let $\{a_n\} \in A(\mu)$. Put

$$b\left(\{a_n\}, \mu\right) = \inf\{b\colon\ \lambda\left([0, b)\right) > 0 \text{ for all } \lambda \in G\left(\{a_n\}, \mu\right)\}.$$

By the compactness of $G(\{a_n\}, \mu)$ the inequality $b(\{a_n\}, \mu) < \infty$ is true.

LEMMA 2.6. *Let* $\{a_n\} \in A(\mu)$ *and* $I(b) = \sup\{n\mu^{\circ r}([ba_{nr}^{-1}, \infty))\colon\ n, r = 1, 2, \ldots\}$.

Then $I(b) < \infty$ *if either* $\varkappa = \infty$ *and* $b > b(\{a_n\}, \mu)$ *or* $\varkappa < \infty$ *and* $b > 0$.

Proof. Suppose that b fulfils the conditions of the Lemma. Let (n_k, r_k) be a sequence of pairs for which

$$I(b) = \lim_{k\to\infty} n_k \mu^{\circ r_k}\left([ba_{n_k r_k}^{-1}, \infty)\right).$$

If the sequence $\{n_k\}$ is bounded, then the inequality $I(b) < \infty$ is obvious. Otherwise we may assume without loss of generality that $n_1 < n_2 < \ldots$ Moreover, we may also assume that the sequence $T_{a_{n_k n_k}} \mu^{\circ n_k r_k}$ is convergent in P. Setting

$$\mu_k = T_{a_{n_k r_k}} \mu^{\circ r_k},$$

we conclude that the sequence $\mu_k^{\circ n_k}$ is convergent in P and

$$\mu_k([b, \infty)) = \mu^{\circ r_k}([ba_{n_k r_k}^{-1}, \infty)) \quad (k = 1, 2, \ldots).$$

Applying Lemma 2.5 we get the inequality $I(b) < \infty$, which completes the proof.

LEMMA 2.7. *Let* $\{a_n\} \in A(\mu)$. *Then*

$$\overline{\lim_{n \to \infty}}\, a_n N_p(\mu^{\circ n}) < \infty$$

for sufficiently small p.

Proof. Passing, by Lemmas 1 and 2 in [7], to an equivalent sequence we may assume without loss of generality that $\{a_n\}$ is monotone non-increasing and $b(\{a_n\}, \mu) < 1$.

First consider the case $\lim_{n \to \infty} a_n > 0$. Then, by Lemma 3 in [7], $\circ$ is the max-convolution and $\mu^{\circ n} \to \delta_c$, where $0 < c < \infty$. It is easy to verify that the support of $\mu^{\circ n}$ is contained in $[0, c]$. Thus $N_p(\mu^{\circ n}) \leqslant c$ for all $p > 0$, which yields the assertion of the Lemma.

Suppose now that $\lim_{n \to \infty} a_n = 0$. Since, by Corollary 1 in [7], $\{a_n\} \sim \{a_{2n}\}$, we can find a positive number q such that

$$a_2^{-1} \leqslant 2^q, \quad \frac{a_n}{a_{2n}} \leqslant 2^q \quad (n = 1, 2, \ldots). \tag{2.9}$$

Put $u(k, n) = a_{2^k n}^{-1}$ $(k = 0, 1, \ldots; n = 1, 2, \ldots)$. Clearly $u(k, n) \leqslant u(k+1, n)$, $\lim_{k \to \infty} u(k, n) = \infty$ and, by (2.9), $u(k, n) \leqslant 2^{kq} a_n^{-1}$. Moreover, taking into account the inequality $b(\{a_n\}, \mu) < 1$ and applying Lemma 2.6, we obtain the inequality

$$d = \sup\{2^k \mu^{\circ n}([u(k, n), \infty)) \colon k = 0, 1, \ldots; n = 1, 2, \ldots\} < \infty.$$

Thus, for every $p > 0$,

$$\int_{u(k,n)}^{u(k+1,n)} x^p \mu^{\circ n}(dx) \leqslant u(k+1, n)^p \mu^{\circ n}([u(k, n), \infty))$$
$$\leqslant 2^{(k+1)qp} a_n^{-p} 2^{-k} d \quad (k = 0, 1, \ldots; n = 1, 2, \ldots)$$

and

$$\int_0^{u(0,n)} x^p \mu^{\circ n}(dx) \leqslant u(0, n)^p = a_n^{-p} \quad (n = 1, 2, \ldots).$$

For p fulfilling the condition $0 < p < q^{-1}$ the above inequalities imply, by a routine computation,

$$M_p(\mu^{\circ n}) \leqslant a_n^{-p}\left(1+\frac{d \cdot 2^{qp}}{1-2^{qp-1}}\right) \quad (n = 1, 2, \ldots),$$

which yields the assertion of the Lemma.

3. Main results. We are now in a position to establish a necessary and sufficient condition for μ to belong to the domain of attraction of a compact subset of $P \backslash \{\delta_0\}$ in terms of the asymptotic behaviour of $N_p(\mu^{\circ n})$ $(n = 1, 2, \ldots)$. We begin with the following result:

THEOREM 3.1. *If μ belongs to the domain of attraction of a compact subset of $P \backslash \{\delta_0\}$, then $\{N_p(\mu^{\circ n})^{-1}\} \in A(\mu)$ and $\{N_p(\mu^{\circ n})\} \sim \{M(\mu^{\circ n})\}$ for sufficiently small p.*

Proof. By Theorem 1 in [7] the sequence $\{c_n(\mu)\}$, defined in Section 1, belongs to $A(\mu)$. Consequently, by Lemma 2.7,

$$\overline{\lim_{n\to\infty}}\, c_n(\mu)\, N_p(\mu^{\circ n}) < \infty$$

whenever p is small enough. Further, the obvious inequality

$$\frac{1}{2} M^p(\mu^{\circ n}) \leqslant M_p(\mu^{\circ n}) \quad (n = 1, 2, \ldots)$$

for all $p > 0$ yields

$$\underline{\lim_{n\to\infty}}\, c_n(\mu)\, N_p(\mu^{\circ n}) \geqslant 2^{-1/p}.$$

Thus $\{N_p(\mu^{\circ n})^{-1}\} \sim \{c_n(\mu)\}$ which, by Lemma 1 in [7], implies the assertion of the Theorem.

PROPOSITION 3.1. *If $K_p(\mu) < \infty$ for an index $p < \varkappa$, then μ belongs to the domain of attraction of a compact subset of $P \backslash \{\delta_0\}$.*

Proof. It follows from the assumption that $0 < N_{2p}(\mu^{\circ n}) < \infty$ $(n = 1, 2, \ldots)$. Put $b_n = N_p(\mu^{\circ n})^{-1}$ $(n = 1, 2, \ldots)$. Then

$$\overline{\lim_{n\to\infty}}\, M_{2p}(T_{b_n}\mu^{\circ n}) = K_p(\mu)^{2p} < \infty.$$

Hence it follows that $G(\{b_n\}, \mu) \subset P$ and, for $q < 2p$, the function x^q is uniformly integrable with respect to all measures λ from $G(\{b_n\}, \mu)$ and $T_{b_n}\mu^{\circ n}$ $(n = 1, 2, \ldots)$. Consequently, the equalities $M_p(T_{b_n}\mu^{\circ n}) = 1$ $(n = 1, 2, \ldots)$ imply $M_p(\lambda) = 1$ for all $\lambda \in G(\{b_n\}, \mu)$, which shows that $G(\{b_n\}, \mu) \subset P \backslash \{\delta_0\}$. This completes the proof.

THEOREM 3.2. *The following conditions are equivalent:*

(i) *μ belongs to the domain of attraction of a compact subset of $R \backslash \{\delta_0\}$;*

(ii) *$\mu \neq \delta_0$ and $\{N_p(\mu^{\circ n})\} \sim \{M(\mu^{\circ n})\}$ for sufficiently small p;*

(iii) *$K_p(\mu) < \infty$ for an index $p < \varkappa$.*

Proof. By Theorem 3.1 the implication (i) $\to$ (ii) is true. Suppose that $0 < q < \varkappa$ and (ii) holds for $p < q$. Taking $2p < q$ we have the equivalence $\{N_{2p}(\mu^{\circ n})\} \sim \{N_p(\mu^{\circ n})\}$, which yields (iii). Finally, Proposition 3.1 yields the implication (iii) $\to$ (i), which completes the proof.

LEMMA 3.1. *Suppose that $\{b_n\}$ is monotone non-increasing and $\{b_{sn}\} \in A\{\mu\}$ for a positive integer s. Then $\{b_n\} \in A(\mu)$.*

Proof. Let p_n be the integral part of n/s. From Lemmas 1 and 4 in [7] it follows that $\{b_{sp_n}\} \sim \{b_{sn}\}$. Since $sp_n \leqslant n \leqslant sn$, we have the inequalities $b_{sp_n} \geqslant b_n \geqslant b_{sn}$, which yield $\{b_n\} \sim \{b_{sn}\}$. Now our assertion is an immediate consequence of Lemma 1 in [7].

A generalized convolution $\circ$ is said to be *regular* if the set P with the operation $\circ$ and the operations of convex combinations admits a non-constant continuous homomorphism into the algebra of real numbers with the operations of multiplication and convex combinations. From Theorem 3 in [3] and Lemma 4.5 in [4] it follows that $\circ$ is regular if and only if for every $\mu \in P$ the weak characteristic function $\hat{\mu}$ is equal to a continuous function $\tilde{\mu}$ m_0-almost everywhere on R_+. This continuous version $\mu \to \tilde{\mu}$ is called a *characteristic function* of $\circ$. The weak convergence $\mu_n \to \mu$ is equivalent to the uniform convergence $\tilde{\mu}_n \to \tilde{\mu}$ on every compact subset of R_+. We note that regular generalized convolutions have always finite exponent.

Let $\circ$ be a regular generalized convolution and $\mu \neq \delta_0$. Put

$$B_n = \{t\colon\ n\big(1-\int_0^1 \tilde{\mu}(tx)\,dx\big) = 1\} \qquad (n = 1, 2, \ldots).$$

Since $\tilde{\mu}(0) = 1$ and $\tilde{\mu}$ is not identically equal to 1, we infer that there exists an index n_0 such that $B_n = \emptyset$ if $n < n_0$ and $B_n \neq \emptyset$ if $n \geqslant n_0$. Put $b_b(\mu) = \min B_n$ if $n \geqslant n_0$ and $b_n(\mu) = b_{n_0}(\mu)$ if $n < n_0$. Of course, $b_n(\mu) > 0$ and

$$b_n(\mu) \leqslant a \ \text{ if } \ n\big(1-\int_0^1 \tilde{\mu}(ax)\,dx\big) \geqslant 1. \tag{3.1}$$

Hence, in particular, it follows that the sequence $\{b_n(\mu)\}$ is monotone non-increasing.

LEMMA 3.2. *Suppose that $\circ$ is regular and $\mu \neq \delta_0$. Then $\delta_0 \notin G\big(\{b_{sn}(\mu), \mu\}\big)$ for any positive integer n.*

Proof. Suppose, on the contrary, that there exist a positive integer s and a subsequence $n_1 < n_2 < \ldots$ such that $T_{d_k}\mu^{\circ n_k} \to \delta_0$, where $d_k = b_{sn_k}(\mu)$ $(k = 1, 2, \ldots)$. Then

$$\tilde{\mu}(d_k t)^{n_k} \to 1 \tag{3.2}$$

uniformly on every compact subset of R_+. By the continuity of $\tilde{\mu}$ there exist real numbers t_k satisfying the conditions $0 \leqslant t_k \leqslant 1$ and

$$\int_0^1 \tilde{\mu}(d_k x)\,dx = \tilde{\mu}(d_k t_k) \quad (k = 1, 2, \ldots).$$

Consequently,

$$\tilde{\mu}(d_k t_k)^{n_k} = \left(1 - \frac{1}{sn_k}\right)^{n_k} \quad (k = 1, 2, \ldots),$$

which contradicts (3.2). The Lemma is thus proved.

LEMMA 3.3. *Suppose that* $\circ$ *is regular and* $\mu \neq \delta_0$. *There exists then a positive integer* s *such that* $b_{sn}(\mu) \leqslant c_n(\mu)$ *for sufficiently large* n.

Proof. First we shall prove the inequality

$$\varliminf_{n \to \infty} n\left(1 - \int_0^1 \tilde{\mu}\left(c_n(\mu)x\right)dx\right) > 0. \tag{3.3}$$

Contrary to this let us suppose that there exists a subsequence $n_1 < n_2 < \ldots$ with the property

$$n_k\left(1 - \int_0^1 \tilde{\mu}\left(c_{n_k}(\mu)x\right)dx\right) \to 0. \tag{3.4}$$

We may assume without loss of generality that the sequence $\nu_k = T_{d_k}\mu^{\circ n_k}$, where $d_k = c_{n_k}(\mu)$, converges in $\bar{P}$, say to ν. From (3.4) it follows that

$$\left(\int_0^1 \tilde{\mu}\left(c_{n_k}(\mu)x\right)dx\right)^{n_k} \to 1$$

which, by virtue of the inequality $0 < \mu(t) \leqslant 1$ for t small enough ([3], Theorem 5), yields

$$\int_0^1 \tilde{\mu}^{n_k}\left(c_{n_k}(\mu)x\right)dx \to 1$$

or, equivalently,

$$\int_0^1 \tilde{\nu}_k(x)\,dx \to 1.$$

Denoting by ω the uniform distribution on the interval [0, 1] and applying Corollary 4.1 in [4] we have

$$\int_0^\infty \tilde{\omega}(x)\,\nu_k(dx) \to 1. \tag{3.5}$$

Since, by Lemma 3.11, Propositions 3.3 and 3.4 and Theorem 4.1 in [4],

$$\lim_{x\to\infty} \tilde{\omega}(x) = 0,$$

we infer, by (3.5), that $\nu \in P$ and

$$\int_0^\infty \tilde{\omega}(x)\,\nu(dx) = 1,$$

which, by Lemma 2.2, yields $\nu = \delta_0$. But, in view of (1.3), $M(\nu_k) = d_k M(\mu^{\circ n_k}) = 1$ for sufficiently large k, which implies $M(\delta_0) \geqslant 1$. Thus we have reached the desired contradiction. Inequality (3.3) is thus proved. As its immediate consequence we get the existence of a positive integer s for which the inequality

$$sn\Big(1 - \int_0^1 \tilde{\mu}(c_n(\mu)\,x)\,dx\Big) \geqslant 1$$

is true for sufficiently large n. Now, applying (3.1), we get the assertion of the Lemma.

THEOREM 3.3. *Let $\circ$ be a regular generalized convolution. Then a measure μ belongs to the domain of attraction of a compact subset of $P\backslash\{\delta_0\}$ if and only if $\mu \neq \delta_0$ and*

$$\varliminf_{n\to\infty} \frac{b_{2n}(\mu)}{b_n(\mu)} > 0. \tag{3.6}$$

If it is the case, then $\{b_n(\mu)\} \in A(\mu)$.

Proof. Necessity. Suppose that μ belongs to the domain of attraction of a compact subset of $P\backslash\{\delta_0\}$. Then, of course, $\mu \neq \delta_0$ and, by Theorem 1 in [7], $\{c_n(\mu)\} \in A(\mu)$. Since, by Lemma 3.3, $b_{sn}(\mu) \leqslant c_n(\mu)$ for a positive integer s and sufficiently large n, we have the inclusion

$$G(\{b_{sn}(\mu)\}, \mu) \subset \{T_a \lambda\colon\ 0 \leqslant a \leqslant 1,\ \lambda \in G(\{c_n(\mu), \mu)\} \subset P,$$

which, together with Lemma 3.2, yields $G(\{b_{sn}(\mu)\}, \mu) \subset P\backslash\{\delta_0\}$. In other words, $\{b_{sn}(\mu)\} \in A(\mu)$. Since the sequence $\{b_n(\mu)\}$ is monotone non-increasing, we have, by Lemma 3.1, $\{b_n(\mu)\} \in A(\mu)$. Further, by Corollary 1 in [7], $\{b_n(\mu)\} \sim \{b_{2n}(\mu)\}$, which implies condition (3.6).

Sufficiency. Suppose that $\mu \neq \delta_0$ and condition (3.6) is fulfilled. By Lemma 3.3 there exists a positive integer s such that $b_{sn}(\mu) \leqslant c_n(\mu)$ for sufficiently large n. Since, by (3.3),

$$\varliminf_{n\to\infty} \frac{b_{2sn}(\mu)}{b_{sn}(\mu)} > 0,$$

we have, by Lemma 6 in [6], the inclusion $G(\{b_{sn}(\mu)\}, \mu) \subset P$, which, together with Lemma 3.2, shows that μ belongs to the domain of attraction of a compact subset of $P \backslash \{\delta_0\}$. The Theorem is thus proved.

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