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LAW OF THE ITERATED LOGARITHM – CLUSTER POINTS OF DETERMINISTIC AND RANDOM SUBSEQUENCES

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Abstract. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. random variables with mean 0 and finite, positive variance σ^2 and let

$$S_n = \sum_{k=1}^{n} X_k, \quad n \geqslant 1.$$

Further, let

$$\varepsilon^*(\{n_k\}) = \inf\{\varepsilon > 0;\ \sum_{k=3}^{\infty} (\log n_k)^{-\varepsilon^2/2} < \infty\},$$

where $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing subsequence of the positive integers. Then the set of cluster points of $\{S_{n_k}/\sqrt{n_k \log\log n_k}\}_{k=3}^{\infty}$ equals $[-\sigma\sqrt{2}, \sigma\sqrt{2}]$ a.s. if $\liminf_{k\to\infty} n_k/n_{k+1} > 0$, and $[-\sigma\varepsilon^*(\{n_k\}), \sigma\varepsilon^*(\{n_k\})]$ a.s. if $\limsup_{k\to\infty} n_k/n_{k+1} < 1$. These results are then applied to randomly indexed partial sums.

1. Introduction. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. random variables with mean 0 and finite, positive variance σ^2 and let S_n, $n \geqslant 1$, be the sum of the first n terms in this sequence. In [5] we find the first version of the law of the iterated logarithm (LIL) for this case and in [8] a more complete formulation is given, which, in particular, states that the set of cluster points of the sequence $\{S_n/\sqrt{n \log\log n}\}_{n=3}^{\infty}$ coincides with $[-\sigma\sqrt{2}, \sigma\sqrt{2}]$ almost surely.

The first proof of the LIL in this formulation, that is based only on basic probability tools, is given in [1].

Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing subsequence of the positive integers. In [6] it is proved that the cluster set of $\{S_{n_k}/\sqrt{n_k \log\log n_k}\}_{k=3}^{\infty}$ equals $[-\sigma\sqrt{2}, \sigma\sqrt{2}]$ almost surely if n_{k+1}/n_k has a finite limit as $k \to \infty$, and in [4] results for the cases

$$\limsup_{k\to\infty} n_k/n_{k+1} < 1 \quad \text{and} \quad \liminf_{k\to\infty} n_k/n_{k+1} > 0$$

are proved in an elementary way.

Following the lines of [1] and using the results in [4] we will, in Section 4, prove that, with probability one, the set of cluster points of the sequence $\{S_{n_k}/\sqrt{n_k \log\log n_k}\}_{k=3}^{\infty}$ coincides with $[-\sigma\varepsilon^*(\{n_k\}), \sigma\varepsilon^*(\{n_k\})]$ if $\limsup n_k/n_{k+1} < 1$ $(k \to \infty)$ and with $[-\sigma\sqrt{2}, \sigma\sqrt{2}]$ if $\liminf n_k/n_{k+1} > 0$ $(k \to \infty)$. These results will then be used in Section 6 to prove an extension of the following theorem, which can be considered as an Anscombe theorem for the LIL, and which is contained in [6], [3] and [2]:

THEOREM 1.1. *Let $\{X_k\}_{k=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$ be as above and let $\{b_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive reals, increasing to infinity, such that*

$$b_{k+1}/b_k \to B \quad \text{as } k \to \infty,\ 1 \leqslant B < \infty. \tag{1.1}$$

Further, let $\{\nu_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive, integer valued random variables with $\nu_1 \geqslant 3$ and such that $\nu_k/b_k \xrightarrow{a.s.} \theta$ as $k \to \infty$, $0 < \theta < \infty$.

Then, the set of cluster points of the sequence $\{S_{\nu_k}/\sqrt{\nu_k \log\log \nu_k}\}_{k=1}^{\infty}$ coincides with $[-\sigma\sqrt{2}, \sigma\sqrt{2}]$ a.s.

2. Results for deterministic subsequences. Denote by $C(\{x_k\})$ the set of cluster points (the cluster set) of the sequence $\{x_k\}_{k=3}^{\infty}$.

THEOREM 2.1. *Let $\{X_k\}_{k=1}^{\infty}$ be i.i.d. random variables with $\mathrm{E}X_1 = 0$ and $\mathrm{E}X_1^2 = \sigma^2 < \infty$ and let*

$$S_n = \sum_{k=1}^{n} X_k, \quad n \geqslant 1.$$

Further, let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing subsequence of the positive integers and define $\varepsilon^*(\{n_k\})$ by

$$\varepsilon^*(\{n_k\}) = \inf\{\varepsilon > 0;\ \sum_{k=3}^{\infty} (\log n_k)^{-\varepsilon^2/2} < \infty\}. \tag{2.1}$$

Then

$$C\left(\left\{\frac{S_{n_k}}{\sqrt{n_k \log\log n_k}}\right\}\right) = [-\sigma\varepsilon^*(\{n_k\}), \sigma\varepsilon^*(\{n_k\})] \ \text{a.s.} \tag{2.2}$$

if $\limsup_{k\to\infty} n_k/n_{k+1} < 1$, and

$$C\left(\left\{\frac{S_{n_k}}{\sqrt{n_k \log\log n_k}}\right\}\right) = [-\sigma\sqrt{2}, \sigma\sqrt{2}] \ \text{a.s.} \tag{2.3}$$

if $\liminf_{k\to\infty} n_k/n_{k+1} > 0$.

For subsequences such that $\varepsilon^*(\{n_k\}) = 0$, the normalization $\sqrt{n_k \log\log n_k}$ is too strong. If we instead use $\sqrt{n_k \log k}$, we have the following theorem:

THEOREM 2.2. *Let $\{X_k\}_{k=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$ be as in Theorem 2.1 and suppose that* $\limsup_{k\to\infty} n_k/n_{k+1} < 1$. *Then*

$$C\left(\left\{\frac{S_{n_k}}{\sqrt{n_k \log k}}\right\}\right) = [-\sigma\sqrt{2}, \sigma\sqrt{2}] \ \text{a.s.}$$

3. **Preparatory theorems and lemmas.** In this section we state some results that will be used in the proof of Theorem 2.1.

THEOREM 3.1. *Under the assumptions of Theorem 2.1. we have*

$$\limsup_{k\to\infty}(\liminf_{k\to\infty})\frac{S_{n_k}}{\sqrt{n_k \log\log n_k}} = (\pm)\sigma\varepsilon^*(\{n_k\}) \ \text{a.s.}$$

if $\limsup_{k\to\infty} n_k/n_{k+1} < 1$.

This theorem, see [4], provides a closed, finite interval which contains the cluster set of $\{S_{n_k}/\sqrt{n_k \log\log n_k}\}_{k=3}^{\infty}$ with probability one, in the case $\limsup_{k\to\infty} n_k/n_{k+1} < 1$.

To show the opposite inclusion we need the following two lemmas, where the first one is due to de Acosta [1].

LEMMA 3.2. *Let $\{X_k\}$ be i.i.d. random variables.* $\mathrm{E}X_1 = 0$, $\mathrm{E}X_1^2 = \sigma^2 < \infty$. *Let $m_k \in N$, $\alpha_k > 0$, $\alpha_k/m_k \to 0$, $\alpha_k^2/m_k \to \infty$. Then, for every $b \in \mathbf{R}$ and $\varepsilon > 0$,*

$$\liminf_{k\to\infty} \frac{m_k}{\alpha_k^2} \log P\left(\left|\frac{S_{m_k}}{\alpha_k} - b\right| < \varepsilon\right) \geqslant -\frac{1}{2}\left(\frac{b}{\sigma}\right)^2.$$

LEMMA 3.3. *Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers and let $\varepsilon^*(\{n_k\})$ be defined by (2.1). Then, for every fixed integer $\nu \geqslant 1$, $\varepsilon^*(\{n_{\nu k}\}) = \varepsilon^*(\{n_k\})$.*

Proof. By definition it immediatelly follows that $\varepsilon^*(\{n_{\nu k}\}) \leqslant \varepsilon^*(\{n_k\})$. It thus remains to verify the opposite inequality.

Choose ε arbitrarily in the interval $0 < \varepsilon < \varepsilon^*(\{n_k\})$ and consider the identity

$$\sum_{k=1}^{\infty} (\log n_k)^{-\varepsilon^2/2} = \sum_{j=1}^{\nu}\sum_{k=0}^{\infty} (\log n_{\nu k+j})^{-\varepsilon^2/2}.$$

Since the left-hand side is infinite, there must exist j, $1 \leqslant j \leqslant \nu$, such that

$$\sum_{k=0}^{\infty} (\log n_{\nu k+j})^{-\varepsilon^2/2} = \infty.$$

Thus

$$\sum_{k=1}^{\infty} (\log n_{\nu k})^{-\varepsilon^2/2} \geqslant \sum_{k=1}^{\infty} (\log n_{\nu k+j})^{-\varepsilon^2/2} = \infty$$

and, since ε is arbitrary, the proof is finished.

4. Proofs of Theorems 2.1. and 2.2.

Proof of Theorem 2.1. As already mentioned in Section 3, it follows from Theorem 3.1 that, for the case $\limsup_{k\to\infty} n_k/n_{k+1} < 1$, we have

$$C\left(\left\{\frac{S_{n_k}}{\sqrt{n_k \log\log n_k}}\right\}\right) \subseteq [-\sigma\varepsilon^*(\{n_k\}), \sigma\varepsilon^*(\{n_k\})] \text{ a.s.,} \tag{4.1.}$$

so to prove (2.2) it remains to verify the opposite inclusion. Since there is nothing to prove if $\varepsilon^*(\{n_k\}) = 0$, we assume $\varepsilon^*(\{n_k\}) > 0$.

Consider the subsequence $\{n_{\nu k}\}_{k=1}^{\infty}$, where $\nu \geqslant 1$ is an integer to be determined later. Since $\limsup_{k\to\infty} n_k/n_{k+1} < 1$, there exists a $\lambda > 1$ such that $n_k \geqslant \lambda n_{k-1}$ and thus

$$n_{\nu(k-1)} \leqslant \lambda^{-\nu} n_{\nu k}. \tag{4.2.}$$

We first show that, for each $|b| < \sigma\varepsilon^*(\{n_k\})\sqrt{1-\lambda^{-\nu}}$,

$$P\left(\liminf_{k\to\infty}\left|\frac{S_{n_{\nu k}} - S_{n_{\nu(k-1)}}}{\sqrt{n_{\nu k}\log\log n_{\nu k}}} - b\right| = 0\right) = 1. \tag{4.3}$$

Let $\varepsilon > 0$ be given. By Lemma 3.2, applied with $m_k = n_{\nu k} - n_{\nu(k-1)}$ and $\alpha_k = \sqrt{n_{\nu k}\log\log n_{\nu k}}$, there exists, for every $b \in \boldsymbol{R}$ and $\delta > 0$, k_0 such that

$$P\left(\left|\frac{S_{n_{\nu k}} - S_{n_{\nu(k-1)}}}{\sqrt{n_{\nu k}\log\log n_{\nu k}}} - b\right| < \varepsilon\right) \geqslant \exp\left\{-\frac{1}{2}\left(\left(\frac{b}{\sigma}\right)^2 + \delta\right)\frac{n_{\nu k}}{n_{\nu k} - n_{\nu(k-1)}}\log\log n_{\nu k}\right\}$$

$$\geqslant \exp\left\{-\frac{1}{2}\left(\left(\frac{b}{\sigma}\right)^2 + \delta\right)\frac{1}{1-\lambda^{-\nu}}\log\log n_{\nu k}\right\} = (\log n_{\nu k})^{-[(b/\sigma)^2+\delta]/2(1-\lambda^{-\nu})}$$

for $k \geqslant k_0$, where the second inequality comes from (4.2).

Now choose $|b| < \sigma\varepsilon^*(\{n_k\})\sqrt{1-\lambda^{-\nu}}$ and then $\delta > 0$ so that

$$\left(\left(\frac{b}{\sigma}\right)^2 + \delta\right)\frac{1}{1-\lambda^{-\nu}} < (\varepsilon^*(\{n_k\}))^2.$$

It then follows from Lemma 3.3 and the definition of $\varepsilon^*(\{n_{\nu k}\})$ that

$$\sum_{k=3}^{\infty} \mathrm{P}\left(\left|\frac{S_{n_{\nu k}}-S_{n_{\nu(k-1)}}}{\sqrt{n_{\nu k}\log\log n_{\nu k}}}-b\right|<\varepsilon\right)=\infty .$$

From this we obtain (4.3) by applying the Borel-Cantelli lemma. Next we prove that

$$\liminf_{k\to\infty}\left|\frac{S_{n_k}}{\sqrt{n_k\log\log n_k}}-b\right|=0 \text{ a.s. for every } |b|<\sigma\varepsilon^*(\{n_k\}). \tag{4.4.}$$

Choose $\varepsilon>0$ and $|b|<\sigma\varepsilon^*(\{n_k\})$ arbitrarily, and then let ν be so large that $|b|<\sigma\varepsilon^*(\{n_k\})\sqrt{1-\lambda^{-\nu}}$ and $\lambda^{-\nu/2}(\sigma\varepsilon^*(\{n_k\})+\varepsilon)<\varepsilon$. By (4.1) we have

$$\limsup_{k\to\infty}\left|\frac{S_{n_{\nu(k-1)}}}{\sqrt{n_{\nu(k-1)}}\log\log n_{\nu(k-1)}}\right|\leqslant\sigma\varepsilon^*(\{n_k\}) \text{ a.s.,}$$

i.e. for each ω outside a set of measure zero, there exists a $k_0=k_0(\omega)$ such that

$$\left|\frac{S_{n_{\nu(k-1)}}}{\sqrt{n_{\nu(k-1)}\log\log n_{\nu(k-1)}}}\right|\leqslant\sigma\varepsilon^*(\{n_k\})+\varepsilon,\qquad k>k_0. \tag{4.5}$$

Thus we have

$$\left|\frac{S_{n_{\nu k}}}{\sqrt{n_{\nu k}}\log\log n_{\nu k}}-b\right|\leqslant\left|\frac{S_{n_{\nu(k-1)}}}{\sqrt{n_{\nu(k-1)}}\log\log n_{\nu(k-1)}}\right|\times$$

$$\times\sqrt{\frac{n_{\nu(k-1)}\log\log n_{\nu(k-1)}}{n_{\nu k}\log\log n_{\nu k}}}+\left|\frac{S_{n_{\nu k}}-S_{n_{\nu(k-1)}}}{\sqrt{n_{\nu k}\log\log n_{\nu k}}}-b\right|$$

$$<(\sigma\varepsilon^*(\{n_k\})+\varepsilon)\lambda^{-\nu/2}+\left|\frac{S_{n_{\nu k}}-S_{n_{\nu(k-1)}}}{\sqrt{n_{\nu k}\log\log n_{\nu k}}}-b\right|$$

$$<\varepsilon+\left|\frac{S_{n_{\nu k}}-S_{n_{\nu(k-1)}}}{\sqrt{n_{\nu k}\log\log n_{\nu k}}}-b\right|,$$

valid for $k>k_0$. The second inequality above comes from (4.5) and (4.2). This, together with (4.3), now yields

$$\liminf_{k\to\infty}\left|\frac{S_{n_{\nu k}}}{\sqrt{n_{\nu k}\log\log n_{\nu k}}}-b\right|<\varepsilon+\liminf_{k\to\infty}\left|\frac{S_{n_{\nu k}}-S_{n_{\nu(k-1)}}}{\sqrt{n_{\nu k}\log\log n_{\nu k}}}-b\right|=\varepsilon \text{ a.s.}$$

for $|b|<\sigma\varepsilon^*(\{n_k\})\sqrt{1-\lambda^{-\nu}}$. Since

$$\liminf_{k\to\infty}\left|\frac{S_{n_k}}{\sqrt{n_k\log\log n_k}}-b\right|\leqslant\liminf_{k\to\infty}\left|\frac{S_{n_{\nu k}}}{\sqrt{n_{\nu k}\log\log n_{\nu k}}}-b\right|,$$

ε may be chosen arbitrarily small and ν arbitrarily large, (4.4.) follows.

To finish off the proof of (2.2) we finally note that it follows from (4.4) that

$$P\left(D \subseteq C\left(\left\{\frac{S_{n_k}}{\sqrt{n_k \log\log n_k}}\right\}\right)\right) = 1$$

for every countable dense subset $D \subseteq (-\sigma\varepsilon^*(\{n_k\}), \sigma\varepsilon^*(\{n_k\}))$, and, since the set of cluster points is closed, we have

$$[-\sigma\varepsilon^*(\{n_k\}), \sigma\varepsilon^*(\{n_k\})] \subseteq C\left(\left\{\frac{S_{n_k}}{\sqrt{n_k \log\log n_k}}\right\}\right)$$

with probability one, which completes the proof of (2.2).

To prove (2.3) we define a strictly increasing subsequence $\{m_k\}_{k=1}^{\infty}$ of $\{n_k\}_{k=1}^{\infty}$ by

$$m_k = \min\{n_j : n_j > k^k\}, \quad k = 1, 2, \ldots$$

Since the condition $\liminf n_k/n_{k+1} > 0$ provides the existence of $\lambda > 1$ such that $n_k \leqslant \lambda n_{k-1}$, we have $k^k < m_k \leqslant \lambda k^k$ and $m_k/m_{k+1} < \lambda k^k/(k+1)^{k+1}$, $k = 1, 2, \ldots$, which in turn yields $\varepsilon^*(\{m_k\}) = \sqrt{2}$ and $\limsup_{k\to\infty}(m_k/m_{k+1}) < 1$.

Applying (2.2), we now obtain

$$C\left(\left\{\frac{S_{n_k}}{\sqrt{n_k \log\log n_k}}\right\}\right) \supseteq C\left(\left\{\frac{S_{m_k}}{\sqrt{m_k \log\log m_k}}\right\}\right) = [-\sigma\sqrt{2}, \sigma\sqrt{2}] \text{ a.s.}$$

and, since the opposite inclusion is trivial, the proof is finished.

Proof of Theorem 2.2. From Theorem 11.1 in [4] it follows that $C(\{S_{n_k}/\sqrt{n_k \log k}\}) \subseteq [-\sigma\sqrt{2}, \sigma\sqrt{2}]$ a.s. To prove the opposite inclusion we use the proof of (2.2) with obvious modifications.

5. Results for random subsequences. In this section we state corresponding results concerning randomly index partial sums (cf. [6]).

THEOREM 5.1. *Let $\{X_k\}_{k=1}^{\infty}$ be i.i.d. random variables with $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$, and set*

$$S_n = \sum_{i=1}^{n} X_k, \quad n = 1, 2, \ldots$$

Suppose that $\{b_k\}_{k=1}^{\infty}$ is a sequence of positive reals, strictly increasing to infinity and let $\{v_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive, integer valued random variables with $v_1 \geqslant 3$ and such that

(5.1.) $$v_k/b_k \xrightarrow{a.s.} \theta \text{ as } k \to \infty, \quad 0 < \theta < \infty.$$

Finally, let $n_k = [\theta b_k]$ *denote the integer part of* θb_k, $k = 1, 2, \ldots,$ *and let* $\varepsilon^*(\{n_k\})$ *be defined by* (2.1). *Then*

$$C\left(\left\{\frac{S_{v_k}}{\sqrt{v_k \log\log v_k}}\right\}\right) = [-\sigma\varepsilon^*(\{n_k\}), \sigma\varepsilon^*(\{n_k\})] \ a.s. \tag{5.2.}$$

if $\limsup_{k\to\infty} b_k/b_{k+1} < 1$, *and*

$$C\left(\left\{\frac{S_{v_k}}{\sqrt{v_k \log\log v_k}}\right\}\right) = [-\sigma\sqrt{2}, \sigma\sqrt{2}] \ a.s. \tag{5.3}$$

if $\liminf_{k\to\infty} b_k/b_{k+1} > 0$.

Remark. Suppose that $b_k/b_{k+1} \to B$, $k \to \infty$, $0 < B \leqslant 1$. Then we have $\varepsilon^*(\{[\theta b_k]\}) = \sqrt{2}$ and we rediscover Theorem 1.1.

THEOREM 5.2. *Under the conditions of Theorem 5.1 we have*

$$C\left(\left\{\frac{S_{v_k}}{\sqrt{v_k \log k}}\right\}\right) = [-\sigma\sqrt{2}, \sigma\sqrt{2}] \ a.s.$$

if $\limsup_{k\to\infty} b_k/b_{k+1} < 1$.

6. Proofs of Theorems 5.1. and 5.2. The proofs in this section are based on Theorem 2.1, the Lévy inequality and the following result (cf. [4], Lemma 4.1):

LEMMA 6.1. *Let* $\{X_k\}_{k=1}^{\infty}$ *be i.i.d. random variables with* $\mathrm{E}X_1 = 0$ *and* $\mathrm{E}X_1^2 = \sigma^2 < \infty$ *and let* $\{n_k\}_{k=1}^{\infty}$ *be a strictly increasing subsequence of the positive integers such that* $\limsup_{k\to\infty} n_k/n_{k+1} < 1$.

Finally, let $\varepsilon^*(\{n_k\})$ *be defined by* (2.2). *Then*

$$\sum_{k=3}^{\infty} P(|S_{n_k}| > \varepsilon\sqrt{n_k \log\log n_k}) < \infty \qquad \text{for all } \varepsilon > \sigma\varepsilon^*(\{n_k\})$$

and

$$\sum_{k=3}^{\infty} P(|S_{n_k}| > \varepsilon\sqrt{n_k \log k}) < \infty \qquad \text{for all } \varepsilon > \sigma\sqrt{2}.$$

Proof of Theorem 5.1. To prove (5.2) we observe that, if $\limsup_{k\to\infty} b_k/b_{k+1} < 1$, the sequence $\{n_k\}_{k=l}^{\infty}$ is strictly increasing for l large enough and that $\limsup_{k\to\infty} n_k/n_{k+1} < 1$. The conclusion thus follows from (2.2) once we know that

$$C\left(\left\{\frac{S_{\nu_k}}{\sqrt{\nu_k \log\log \nu_k}}\right\}\right) = C\left(\left\{\frac{S_{n_k}}{\sqrt{n_k \log\log n_k}}\right\}\right) \text{ a.s.},$$

but this in turn follows easily if we prove that

$$\frac{S_{\nu_k} - S_{n_k}}{\sqrt{n_k \log\log n_k}} \xrightarrow{a.s.} 0, \quad k \to \infty. \tag{6.2}$$

Let $\gamma > 0$ be given and choose $\delta > 0$ such that $\gamma/2\delta^{1/2} > \sigma\varepsilon^*(\{m_k\}) = \sigma\varepsilon^*(\{m_k'\})$, where $m_k = n_k - [(1-\delta)n_k] - 1$ and $m_k' = [(1+\delta)n_k] - n_k$, $k = 1, 2, \ldots$, and let l be an integer to be determined later. Then

$$\text{(6.3)} \quad \mathrm{P}\left(\bigcup_{k=l}^{\infty}\left\{\left|\frac{S_{\nu_k} - S_{n_k}}{\sqrt{n_k \log\log n_k}}\right| > \gamma\right\}\right)$$

$$\leqslant \sum_{k=l}^{\infty} \mathrm{P}\left(\left\{\left|\frac{S_{\nu_k} - S_{n_k}}{\sqrt{n_k \log\log n_k}}\right| > \gamma\right\} \cap \left\{\left|\frac{\nu_k}{n_k} - 1\right| \leqslant \delta\right\}\right) + \mathrm{P}\left(\bigcup_{k=l}^{\infty}\left\{\left|\frac{\nu_k}{n_k} - 1\right| > \delta\right\}\right)$$

$$\leqslant \sum_{k=l}^{\infty}\left\{\mathrm{P}\left(\max_{n_k - m_k \leqslant j < n_k}\left|\frac{S_j - S_{n_k}}{\sqrt{n_k \log\log n_k}}\right| > \gamma\right) + \right.$$

$$\left. + \mathrm{P}\left(\max_{n_k \leqslant j \leqslant n_k + m_k'}\left|\frac{S_j - S_{n_k}}{\sqrt{n_k \log\log n_k}}\right| > \gamma\right)\right\} + \mathrm{P}\left(\bigcup_{k=l}^{\infty}\left\{\left|\frac{\nu_k}{n_k} - 1\right| > \delta\right\}\right).$$

From the definition of m_k and m_k' it follows that there exist k_0 and k_1 such that

$$\gamma\sqrt{n_k \log\log n_k} - \sigma\sqrt{2m_k} \geqslant (\gamma/\delta^{1/2})\sqrt{m_k \log\log m_k} - \sigma\sqrt{2m_k}$$

$$\geqslant (\gamma/2\delta^{1/2})\sqrt{m_k \log\log m_k} \quad \text{for all } k \geqslant k_0$$

and

$$\gamma\sqrt{n_k \log\log n_k} - \sigma\sqrt{2m_k'} \geqslant (\gamma/2\delta^{1/2})\sqrt{m_k' \log\log m_k'} \quad \text{for all } k \geqslant k_1.$$

Thus, if we choose $l > \max(k_0, k_1)$, the Lévy inequality (see [7], p. 248) yields that the sum in the right most side of (6.3) is bounded by

$$2\sum_{k=l}^{\infty}\left\{\mathrm{P}\left(|S_{m_k}| > \frac{\gamma}{2\delta^{1/2}}\sqrt{m_k \log\log m_k}\right) + \mathrm{P}\left(|S_{m_k'}| > \frac{\gamma}{2\delta^{1/2}}\sqrt{m_k' \log\log m_k'}\right)\right\},$$

which in turn is finite by Lemma 6.1.

Further, condition (5.1) implies that

$$\lim_{l\to\infty} \mathrm{P}\left(\bigcup_{k=l}^{\infty}\left\{\left|\frac{\nu_k}{n_k} - 1\right| > \delta\right\}\right) = 0$$

and hence it follows from (6.3) that

$$\lim_{l\to\infty} P\left(\bigcup_{k=l}^{\infty}\left\{\left|\frac{S_{\nu_k}-S_{n_k}}{\sqrt{n_k \log\log n_k}}\right| > \gamma\right\}\right) = 0.$$

Since γ is arbitrary, (6.2) follows.

To prove (5.3), we define $m_k = \min\{j: [\theta b_j] > k^k\}$, $k = 1, 2, \ldots$ Since

$$\limsup_{k\to\infty} b_{m_k}/b_{m_{k+1}} < 1 \quad \text{and} \quad \varepsilon^*(\{[\theta b_{m_k}]\}) = \sqrt{2},$$

(5.3) follows from (5.2) applied to the sequence $\{S_{\nu_{m_k}}/\sqrt{\nu_{m_k}\log\log \nu_{m_k}}\}_{k=3}^{\infty}$ (cf. the proof of (2.3)).

Proof of Theorem 5.2. Theorem 5.2. follows from Theorem 2.2 once we know that $C(\{S_{\nu_k}/\sqrt{\nu_k \log k}\}) = C(\{S_{n_k}/\sqrt{n_k \log k}\})$ a.s., and to prove this we use proof of (5.2) with obvious modifications.

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