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OPERATOR SEMI-STABLE PROBABILITY MEASURES ON BANACH SPACES

BY

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Abstract. This paper is concerned with the operator semi-stable probability measures on a real separable space. The aim is to prove the singularity of a Gaussian measure ϱ and a Poisson measure $\tilde{e}(M)$ which are the components in the Levy–Khinchine decomposition of an operator semi-stable measure.

1. First we show a characterization of the spectral radius of an operator. This characterization is not really necessary in the proofs of the remaining lemmas and theorems but the use of it makes reasonings more clear and readible.

Let X be an arbitrary linear space with a norm $\|\cdot\|$, A be a continuous operator on X and $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$ be its spectral radius. By $\{\|\cdot\|_t\}_{t\in T}$ we understand the set of all norms equivalent to the norm $\|\cdot\|$.

LEMMA 1.1. *With the above assumptions the following conditions hold:*

(i) $r(A) = \inf_{t\in T} \|A\|_t$;

(ii) $\lim_{n\to\infty} A^n = 0$ *iff there exists a* $t\in T$ *such that* $\|A\|_t < 1$;

(iii) *if* X *is a finite-dimensional space, then, for all* $x\in X$, $\lim_{n\to\infty} A^n x = 0$ *iff there exists a* $t\in T$ *such that* $\|A\|_t < 1$.

Proof. (i) It suffices to prove that $r(A) \geqslant \inf \|A_t\|$ $(t\in T)$, the opposite inequality being trivial.

We show that for an arbitrary $z > r(A)$ there exists a $t_0\in T$ such that $\|A\|_{t_0} \leqslant z$. Indeed, let $r(A) < z$. Hence there exists a positive integer n_0 such that, for $n > n_0$, $\|(A/z)^n\| < 1$. Then $\|(A/z)^n\| < C$ $(n = 1, 2, \ldots)$ for a constant $C > 0$.

Putting

$$\|x\|_{t_0} = \sup_{n=0,1,2,\ldots} \|(A/z)^n x\| \qquad \big((A/z)^0 = I\big)$$

we get $\|x\| < \|x\|_{t_0} \leqslant C\|x\|$ and

$$\|(A/z)x\|_{t_0} = \sup_{n=0,1,2,\ldots} \|(A/z)(A/z)^n x\| \leqslant \sup_{n=0,1,2,\ldots} \|(A/z)^n x\| = \|x\|_{t_0}.$$

These inequalities imply that $\|A\|_{t_0} \leqslant z$ and $t_0 \in T$.

(ii) Since $\lim A^n = 0$, there exists a positive integer n_0 such that, for $n = 1, 2, \ldots$, we have $\|A^{n_0 n}\| < 1$. Hence $r(A^{n_0}) < 1$ and, since $r(A^{n_0}) = r(A)^{n_0}$, $r(A) < 1$.

From (i) it follows that there exists a $t \in T$ such that $\|A\|_t < 1$.

(iii) easily follows from (ii) because in a finite-dimensional space the strong operator convergence is equivalent to the norm convergence of a sequence of operators.

2. Let X denote a real separable Banach space with the norm $\|\cdot\|$ and with the dual space X^*. By $\langle\cdot,\cdot\rangle$ we denote the dual parting between X and X^*.

A measure μ on X is said to be *full* if its support is not contained in any hyperplane on X. By δ_x we denote the probability measure concentrated at the point $x \in X$. Given a measure μ, we define μ^- by putting $\mu^-(A) = \mu(-A)$, where $-A = \{-y\colon y \in A\}$.

A probability measure μ on X is called *infinitely divisible* if, for every natural n, there exists a probability measure μ_n such that $\mu_n^{*n} = \mu$ (the power $*$ is taken in the sense of convolution).

Tortrat [6], p. 311 (see also [1]), proved the following analogue of the Levy–Khinchine (L-K) representation of infinitely divisible laws: every infinitely divisible measure μ on X has a unique representation $\mu = \varrho * \tilde{e}(M)$, where ϱ is a symmetric Gaussian measure on X and M is a generalized Poisson exponent of $\tilde{e}(M)$.

Let $M(X)$ denote the set of all generalized Poisson exponents of X. The characteristic functionals of these measures are:

(2.1)
$$\hat{\tilde{e}}(M)(x^*) = \exp\{i\langle x^*, x_0\rangle + \int_X [e^{i\langle x^*, x\rangle} - 1 - i\langle x^*, x\rangle 1_{D_\tau}(x)]\, M(dx)\}\ (^1),$$

(2.2)
$$\hat{\varrho}(x^*) = \exp(-\tfrac{1}{2}\langle x^*, Rx^*\rangle),$$

where $x_0 \in X$, D_τ is the ball of radius τ concentrated at 0 (D_τ is the set of continuity of the measure M ([2], Th. 2.3)), R is a covariance operator, and $x^* \in X^*$ ([8], p. 173).

We say that a probability measure μ is *operator semi-stable* if its characteristic functional $\hat{\mu}$ satisfies the functional equation $\hat{\mu}(x^*)^c$

(1) Such a measure $\tilde{e}(M)$ we also denote by $\tilde{e}_\tau(M) * \delta_{x_0}$.

$= \mu(B^* x^*) e^{i\langle x^*, x_0\rangle}$ for all $x^* \in X^*$, where B is a continuous linear invertible operator on X, $x_0 \in X$ $c \in (0, 1)$.

Operator semi-stable measures have been considered by Krakowiak [4]. It is known that every operator semi-stable measure μ on X is infinitely divisible $(\varrho * \tilde{e}(M))$ and that

$$(2.3) \qquad BM = cM \quad \big(BM(A) = M(B^{-1}A)\big),$$

$$(2.4) \qquad cR = BRB^*,$$

where B is a continuous linear invertible operator on X, $c \in (0, 1)$, $M \in M(X)$ and ϱ is a symmetric Gaussian measure on X with the covariance operator R (see Prop. 2.1 and Prop 4.1 in [4]).

The proofs of (2.3) and (2.4) immediately follow from (2.1) and (2.2).

By the triple $(\varrho * \tilde{e}(M), B, c)$ we mean the operator semi-stable measure with an operator B and a parameter c. The representation L-K of this measure is of the form $\varrho * \tilde{e}(M)$, where ϱ is a symmetric Gaussian measure and $M \in M(X)$. The set of such triples we denote by $S(X)$.

By K we denote the unit ball on X, $x^* M$ denotes the measure define by $x^* M(A) = M(x^{*-1} A)$ for $M \in M(X)$ and $x^* \in X^*$.

3. In this section we show the singularity of the Gaussian measure ϱ and the Poisson measure $\tilde{e}(M)$ in the L-K representation for the operator semi-stable measure $\varrho * \tilde{e}(M)$.

THEOREM 3.1. *Let* $(\varrho * \tilde{e}(M), B, c) \in S(X)$ *and* $\lim_{n\to\infty} B^n = 0$. *Then there exists a Borel subset A in X such that*

$$\text{(i)} \qquad \tilde{e}(M)(A^c + x_0) = 0 \quad \textit{for some } x_0 \in X,$$

$$\text{(ii)} \qquad M(A^c) = 0,$$

$$\text{(iii)} \qquad \varrho(A+y) = 0 \quad \textit{for all } y \in X$$

(A^c *denotes the complement of* A).

First we introduce some necessary notation. Put $D = B/\sqrt{c}$. Let $x_0^* \in X^*$ be such that $x_0^* \varrho \neq \delta_0$ and $\{n_j\}_{j=1}^{\infty}$ be such a sequence of positive integers that

$$(3.1) \qquad \int_K \langle x_0^*, D^{n_j} x\rangle^2 M(dx) \leqslant 2^{-j}.$$

The existence of such a sequence follows from the observation that we can assume $\|B\| < 1$ (see Lemma 1.1 (ii)) and (see (2.3))

$$(3.2) \qquad \int_K \langle x_0^*, D^{n_j} x\rangle^2 M(dx) = \int_{B^{n_j}K} \langle x_0^*, x\rangle^2 M(dx).$$

Finally, put

$$A = \{x\colon \lim_{j\to\infty} \langle x_0^*, D^{n_j} x\rangle = 0\}. \tag{3.3}$$

Now recall a well-known fact concerning the Poisson measures:

LEMMA 3.1. *If $M \in M(X)$, M and $\tilde{e}(M)$ are symmetric, then*

$$\int_K \langle x^*, x\rangle^2 M(dx) = \int_K \langle x^*, x\rangle^2 \tilde{e}(M)(dx)$$

for all $x^* \in X^*$ (see Lemma 1.8, p. 10, in [5]).

Proof of theorem 3.1. (i) Notice first that

$$1_{A^c}(x+y) \leqslant 1_{A^c}(x) + 1_{A^c}(y) \quad \text{for all } x, y \in X. \tag{3.4}$$

Step 1. Let us assume that $\tilde{e}(M)$ and M are symmetric measures. Fix $\varepsilon > 0$ and a positive integer n_0 such that, for $K_0 = B^{-n_0} K$,

$$M(K_0^c) < \varepsilon. \tag{3.5}$$

Put $m = M|_{K_0^c}$. Then by the definition of the Poisson measure $\tilde{e}(m)$ we get

$$\begin{aligned}
\tilde{e}(m)(A^c) &= e^{-m(X)} \sum_{k=0}^{\infty} \frac{m^{*k}(A^c)}{k!} \\
&= e^{-m(X)} \sum_{k=1}^{\infty} \frac{1}{k!} \underbrace{\int_X \ldots \int_X}_{k\text{times}} 1_{A^c}(x_1 + \ldots + x_k)\, m(dx_1) \ldots m(dx_k) \\
&\leqslant e^{-m(X)} \sum_{k=1}^{\infty} \frac{1}{k!} k m(A^c) m(X)^{k-1} && \text{(see (3.4))} \\
&= e^{-m(X)} m(A^c) \sum_{k=0}^{\infty} \frac{m(X)^k}{k!} = M(A^c \cap K_0^c) < \varepsilon && \text{(see (3.5))}
\end{aligned}$$

which gives the inequality

$$\tilde{e}(M|_{K_0^c})(A^c) < \varepsilon. \tag{3.6}$$

Now we show that

$$\tilde{e}(M|_{K_0})(A^c) = 0. \tag{3.7}$$

We have

$$A^c = \bigcup_{k=1}^{\infty} \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \left\{ |\langle x_0^*, D^{n_j} x\rangle| \geqslant \frac{1}{k} \right\}. \tag{3.8}$$

Thus, by the Borel–Cantelli lemma, formula (3.7) is justified by the estimations

$$\sum_{j=1}^{\infty} \tilde{e}(M|_{K_0})\left(\left\{x\colon |\langle x_0^*, D^{n_j} x\rangle| \geqslant \frac{1}{k}\right\}\right)$$

$$\leqslant k^2 \sum_{j=1}^{\infty} \int_X \langle x_0^*, D^{n_j} x\rangle^2 \tilde{e}(M|_{K_0})(dx)$$

$$= k^2 \sum_{j=1}^{\infty} \int_X \langle x_0^*, D^{n_j} x\rangle^2 M|_{K_0}(dx) \qquad \text{(see Lemma 3.1)}$$

$$= k^2 \sum_{j=1}^{\infty} \Big(\int_K \langle x_0^*, D^{n_j} x\rangle^2 M(dx) + \sum_{i=1}^{n_0} \int_{B^{-i}P} \langle x_0^*, D^{n_j} x\rangle^2 M(dx)\Big)$$

$$(P = K \setminus BK;\ \text{see also (2.3)})$$

$$\leqslant k^2 \Big(\sum_{j=1}^{\infty} 2^{-j} + \sum_{i=1}^{n_0} \int_{B^{-i}K} \langle x_0^*, x\rangle^2 M(dx)\Big) < +\infty \qquad \text{(see (3.2))}.$$

Finally, from (3.6) and (3.7), we obtain

$$\tilde{e}(M)(A^c) = \tilde{e}(M|_{K_0} + M|_{K_0^c})(A^c) = \tilde{e}(M|_{K_0}) * \tilde{e}(M|_{K_0^c})(A^c)$$

$$\leqslant \int_X \int_X \big(1_{A^c}(x) + 1_{A^c}(y)\big) \tilde{e}(M|_{K_0})(dx)\, \tilde{e}(M|_{K_0^c})(dy) \qquad \text{(see 3.4))}$$

$$= \tilde{e}\,(M|_{K_0})(A^c) + \tilde{e}(|M|_{K_0^c})(A^c) < \varepsilon.$$

Since $\varepsilon > 0$ is an arbitrary number, we obtain (i).

Step 2. Let M and $\tilde{e}(M) = \tilde{e}_\tau(M) * \delta_{y_0}$ be not necessarily symmetric. Let $M + M^-$ be the symmetrization of the measure M and $\tilde{e}(M + M^-)$ be a symmetric measure. Notice that $(\tilde{e}(M + M^-),\ B,\ c) \in S(X)$. Hence

$$\tilde{e}(M + M^-)(A^c) = \int_X \tilde{e}_\tau(M)(A^c - x)\, \tilde{e}_\tau(M^-)(dx) = 0,$$

i.e. $\tilde{e}_\tau(M)(A^c - x) = 0$ a.e. with respect to $\tilde{e}_\tau(M^-)$.

Putting $x_0 = x + y_0$, we get $\tilde{e}(M)(A^c + x_0) = 0$.

(ii) It suffices to prove that, for all integers m,

$$0 = M\big(A^c \cap B^m P\big) \qquad (P = K \setminus BK). \tag{3.9}$$

By a similar argument as in the proof of (i), condition (3.8) results from the following estimations:

$$\sum_{j=1}^{\infty} M\big(\{x\colon |\langle x_0^*, D^{n_j} x\rangle| \geqslant 1/k\} \cap B^m P\big)$$

$$\leqslant k^2 \sum_{n=0}^{\infty} \int_{B^m P} \langle x_0^*, D^n x\rangle^2 M(dx) = k^2 \sum_{n=0}^{\infty} \int_{B^{m+n}P} \langle x_0^*, x\rangle^2 M(dx)$$

$$= k^2 \int_{B^m K} \langle x_0^*, x\rangle^2 M(dx) < +\infty \qquad \text{(see (2.3))}.$$

(iii) First notice that

$$D\varrho = \varrho. \tag{3.10}$$

This equality can be proved by counting the characteristic functionals of both measures applying (2.2) and (2.4).

Let $y \in X$. Then, by (3.10) and since $x_0^* \varrho \neq \delta_0$, we obtain

$$\varrho(A+y) = \varrho\Big(\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} \{x\colon |\langle x_0^*, D^{n_j} x\rangle| < 1/k\} + y\Big)$$

$$\leqslant \overline{\lim_k}\, \overline{\lim_i}\, \varrho(\{x\colon |\langle x_0^*, D^{n_i} x\rangle < 1/k\} + y)$$

$$= \overline{\lim_k}\, \overline{\lim_i}\, D^{n_i} \varrho(\{x\colon |\langle x_0^*, x\rangle - \langle x_0^*, D^{n_i} y\rangle| < 1/k\})$$

$$= \overline{\lim_k}\, \overline{\lim_i}\, x_0^* \varrho(\{t \in \boldsymbol{R}\colon |t - \langle x_0^*, D^{n_i} y\rangle| \langle 1/k\})$$

$$\leqslant \lim_k x_0^* \varrho(\{t \in \boldsymbol{R}\colon |t| < 1/k\}) = 0.$$

The last inequality follows from the observation that a Gaussian measure of an interval of a fixed length is the greatest possible when the interval is a symmetric neighbourhood of zero. This completes the proof.

COROLLARY 3.1. *If* $(\tilde{e}(M) * \varrho, B, c) \in S(X)$ $\lim_{n\to\infty} B^n = 0$, *then*

(i) $\tilde{e}(M)$ *and* ϱ *are singular,*

(ii) M *and* ϱ *are singular.*

COROLLARY 3.2 (cf. [3], p. 31). *Let* X *be a finite-dimensional space and* $(\tilde{e}(M) * \varrho, B, c) \in S(X)$ *be a full measure. Then there exist two* B*-invariant subspaces* X_1 *and* X_2 *such that:*

(i) $\varrho(X_1) = 1$;

(ii) $M(X_2^c) = 0$ *and* $\tilde{e}(M)(X_2 + y) = 0$ *for some* $y \in X$;

(iii) $r(B|_{X_2}) < \sqrt{c}$;

(iv) $r(B|_{X_1}) = \sqrt{c}$;

(v) $X = X_1 \oplus X_2$.

Proof. We can assume that $\lim_{n\to\infty} B^n = 0$ ($B^n(\mu * \mu^-) = (\mu * \mu^-)^{c^n} \Rightarrow \delta_0$ and see (ii) in [7], p. 120).

Let $X_1 = RX^*$ (RX^* is the image of the space X^* by the covariance operator R corresponding to the Gaussian measure ϱ).

(i) is obvious for every finite-dimensional space.

(ii) Evidently, $X_2 \subset A$ (see (3.3)), which together with Theorem 3.1 (i), (ii), implies (ii).

(iii) obviously follows from Lemma 1.1 (i), (iii).

(iv) Let $\ker R = \{x^*: Rx^* = 0\}$. Since the space RX^* is isomorphic to $X^*/\ker R$, we can consider all elements of RX^* as $x^* + \ker R = [x^*]$, and the operator D as $D[x^*] = [D^* x^*]$.

Now we can define a norm on $X^*/\ker R$ by putting

$$\|[x^*]\|_0 = \sqrt{\langle x^*, Rx^* \rangle}.$$

Condition (2.4) gives at once that D is isometric on $X^*/\ker R$ and thus we obtain that $r(D|_{X_2}) = 1$.

(v) is obvious in view of (iii) and (iv), for the considered measure is full.

We shall show a one more fact concerning the considered measures:

THEOREM 3.2. *Suppose* $(\varrho * \tilde{e}(M),\ B,\ c) \in S(X)$ *and* $r(B)^p/c < 1$. *Then* $\int\limits_K \|x\|^p M(dx) < +\infty$.

Proof. By Lemma 1.1 (ii) we can assume that $\|B\|^p/c < 1$. Since

$$K = \bigcup_{n=0}^{\infty} B^n P \quad (n = 1, 2, \ldots)$$

and (2.3) holds, we obtain

$$\int\limits_K \|x\|^p M(dx) = \sum_{n=0}^{\infty} \int\limits_P \frac{\|B^n x\|^p}{c^n} M(dx) \leqslant M(P) \sum_{n=0}^{\infty} \frac{\|B\|^{pn}}{c^n} < +\infty,$$

which concludes the proof.

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