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APPROXIMATION SCHEMES FOR TWO-PARAMETER STOCHASTIC EQUATIONS

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Abstract. In this paper we introduce several approximation schemes for Itô equations with two parameters which are suggested by the Lie–Trotter product formula from the theory of nonlinear semi-groups.

By using the splitting up method the equation is decomposed into two simpler equations. The convergence and speed of convergence of schemes are discussed.

1. Introduction and notation. Approximation schemes for one-parameter Itô equations have been considered by Glorenecc [3], Milstein [4], Pardoux and Talay [5], Platen [6], Rao et al. [7], Rumelin [9]. For the two-parameter case Ermoliev and Tsarenco [2] have proved the convergence of finite differences, and in [10] some approximation schemes are considered for the infinite dimensional case. Recently in [8] several approximation schemes suggested by the Lie–Trotter formula are proposed (see also [1] for the case of parabolic stochastic equations). The method consists in a separation of the diffusion and the drift terms and obtaining in this way two simpler equations, one of them is deterministic and the other one is stochastic.

In the present paper we give similar schemes for two-parameter Itô equations. Next T is a positive number, m and n are positive integers, and λ is the Lebesgue measure on $\boldsymbol{R}^2$. We introduce the following notation:

$$I = [0, T]^2, \quad h_1 = T/m, \quad h_2 = T/n, \quad h = (h_1, h_2),$$

$$s_i = ih_1, \ i = 0, 1, \ldots, m, \quad t_j = jh_2, \ j = 0, 1, \ldots, n,$$

$$z_{i,j} = (s_i, t_j), \quad I_{i,j} = [s_i, s_{i+1}) \times [0, t_j],$$

$$J_{i,j} = [0, s_i] \times [t_j, t_{j+1}), \quad R_{s,t} = [0, s) \times [0, t).$$

For a rectangle $D = [s, t) \times [u, v)$ and a two-parameter process $(f_{s,t})$ we define the increment of f on D by

$$f(D) = f_{t,v} - f_{t,u} - f_{s,v} + f_{s,u}.$$

Let $a(p, q, x)\colon I\times \mathbf{R}^d\to\mathbf{R}^d$ and $b(p, q, x)\colon I\times\mathbf{R}^d\to\mathbf{R}^d\otimes\mathbf{R}^m$ be measurable mappings. We consider the following hypotheses on a, b:

(K) $$|a(p, q, x)|^2 \leqslant K_1(1+|x|^2), \quad |b(p, q, x)|^2 \leqslant K_2(1+|x|^2)$$

for all $(p, q)\in I$, $x\in\mathbf{R}^d$;

(L) $$|a(p, q, x)-a(p, q, y)|^2 \leqslant L_1|x-y|^2,$$
$$|b(p, q, x)-b(p, q, y)|^2 \leqslant L_2|x-y|^2$$

for all $(p, q)\in I$, $x, y\in\mathbf{R}^d$.

Let $(w_{s,t})_{(s,t)\in I}$ be an $\mathbf{R}^m$-valued two-parameter Wiener process, i.e., $(w_{s,t})$ is continuous, w vanishes on $\{0\}\times[0, T]\cup[0, T]\times\{0\}$ for every rectangle D, $w(D)$ has Gaussian distribution with mean 0 and covariance $\lambda(D)I_m$, and for all disjoint rectangles $D_1, \ldots, D_k$ the increments $w(D_1), \ldots, w(D_k)$ are independent. Let $\mathscr{F}_{s,t} = \mathscr{B}(w_{u,v};\ u\leqslant s,\ v\leqslant t)$ be the canonical filtration associated with w. We consider the two-parameter Itô equation

(1) $$x_{s,t} = x+\int_0^s\int_0^t a(p, q, x_{p,q})dp\,dq+\int_0^s\int_0^t b(p, q, x_{p,q})dw_{p,q},$$

where $x\in\mathbf{R}^d$ and $\int_0^s\int_0^t b\,dw$ is the Itô integral as defined for example in [11].

Remark 1. Under (K) and (L) the equation (1) has a pathwise unique continuous solution $(x_{s,t})_{(s,t)\in I}$ (see [11]). The initial condition x can be replaced by a process $(\eta_{s,t})_{(s,t)\in I}$ which is $\mathscr{F}_{s,t}$-adapted and continuous.

2. Main results. First we introduce two approximation schemes for (1) with adapted and continuous approximating processes. We define recursively the approximating processes u^h, x^h, $\tilde{u}^h$, $\tilde{x}^h$ for $(s, t)\in R_{z_{1,1}}$ by

(2) $$u^h_{s,t} = x+\int_0^s\int_0^t a(p, q, u^h_{p,q})dp\,dq, \quad x^h_{s,t} = u^h_{s,t}+\int_0^s\int_0^t b(p, q, x^h_{p,q})dw_{p,q},$$

(3) $$\tilde{u}^h_{s,t} = x+\int_0^s\int_0^t b(p, q, \tilde{u}^h_{p,q})dw_{p,q}, \quad \tilde{x}^h_{s,t} = \tilde{u}^h_{s,t}+\int_0^s\int_0^t a(p, q, \tilde{x}^h_{s,t})dp\,dq.$$

The processes u^h, x^h, $\tilde{u}^h$, $\tilde{x}^h$ with the time parameter $R_{z_{1,1}}$ are well defined, adapted and continuous (in fact, u^h is deterministic). Suppose that for some (i, j) we defined on $R_{z_{i,j}}$ the above processes which are continuous and adapted and, moreover, $u^h_{s,t}$ is $\mathscr{F}_{s_{i-1},t}$-measurable if $(s, t)\in I_{i-1,j}$ and $u^h_{s,t}$ is $\mathscr{F}_{s,t_{j-1}}$-measurable if $(s, t)\in J_{i,j-1}$.

Now, if $(s, t)\in I_{i,j}$, we define

(4) $$u^h_{s,t} = x^h_{s_i-,t-}+\int_{s_i}^s\int_0^t a(p, q, u^h_{p,q})dp\,dq, \quad x^h_{s,t} = u^h_{s,t}+\int_{s_i}^s\int_0^t b(p, q, x^h_{p,q})dw_{p,q},$$

(5) $$\tilde{u}^h_{s,t} = \tilde{x}^h_{s_i-,t-}+\int_{s_i}^s\int_0^t a(p, q, \tilde{u}^h_{p,q})dw_{p,q}, \quad \tilde{x}^h_{s,t} = \tilde{u}^h_{s,t}+\int_{s_i}^s\int_0^t a(p, q, \tilde{x}^h_{p,q})dp\,dq,$$

where

$$f_{s-,t-} = \lim_{p \nearrow s, q \nearrow t} f_{p,q}.$$

If $(s, t) \in J_{i,j}$, we define

$$(6)\quad u^h_{s,t} = x^h_{s-,t_j-} + \int_0^s \int_{t_j}^t a(p, q, u^h_{p,q})dp\,dq, \qquad x^h_{s,t} = u^h_{s,t} + \int_0^s \int_{t_j}^t b(p, q, x^h_{p,q})dw_{p,q},$$

$$(7)\quad \tilde{u}^h_{s,t} = \tilde{x}^h_{s_i-,t_j-} + \int_0^s \int_{t_j}^t b(p, q, \tilde{u}^h_{p,q})dw_{p,q}, \qquad \tilde{x}^h_{s,t} = \tilde{u}^h_{s,t} + \int_0^s \int_{t_j}^t a(p, q, \tilde{x}^h_{p,q})dp\,dq.$$

If $s = T$ or $t = T$, then we define

$$(8)\qquad u^h_{s,t} = u^h_{s-,t-}, \qquad x^h_{s,t} = x^h_{s-,t-}, \qquad \tilde{u}^h_{s,t} = \tilde{u}^h_{s-,t-}, \qquad \tilde{x}^h_{s,t} = \tilde{x}^h_{s-,t-}.$$

The approximating processes u^h, x^h, $\tilde{u}^h$, $\tilde{x}^h$ are defined for all $(s, t) \in I$ as follows: by (2), (3) if $(s, t) \in R_{z_{1,1}}$; by (4), (5) if $(s, t) \in I_{1,1}$; by (6), (7) if $(s, t) \in J_{2,1}$; by (4), (5) if $(s, t) \in I_{2,2}$; by (6), (7) if $(s, t) \in J_{3,2}$; by (4), (5) if $(s, t) \in I_{3,3}, \ldots$; and by (8) if $s = T$ or $t = T$.

Remark 2. The processes u^h, x^h, $\tilde{u}^h$, $\tilde{x}^h$ are continuous and adapted and, moreover, $u^h_{s,t}$ is $\mathscr{F}_{s_i,t}$-measurable if $(s, t) \in I_{i,j}$ and $u^h_{s,t}$ is $\mathscr{F}_{s,t_j}$-measurable if $(s, t) \in J_{i,j}$.

LEMMA 1. *The following equations hold*:

$$(9)\qquad u^h_{s,t} = x + \int_0^s \int_0^t a(p, q, u^h_{p,q})dp\,dq + \int_0^{[s/h_1]h_1} \int_0^t b(p, q, x^h_{p,q})dw_{p,q}$$

if $(s, t) \in R_{z_{i,j}}$, $i+j$ *is odd*;

$$(10)\qquad u^h_{s,t} = x + \int_0^s \int_0^t a(p, q, u^h_{p,q})dp\,dq + \int_0^s \int_0^{[t/h_2]h_2} b(p, q, x^h_{p,q})dw_{p,q}$$

if $(s, t) \in R_{z_{i,j}}$, $i+j$ *is even*;

$$(11)\qquad x^h_{s,t} = x + \int_0^s \int_0^t a(p, q, u^h_{p,q})dp\,dq + \int_0^s \int_0^t b(p, q, x^h_{p,q})dw_{p,q},$$

$$(12)\qquad \tilde{u}^h_{s,t} = x + \int_0^{[s/h_1]h_1} \int_0^t a(p, q, \tilde{x}^h_{p,q})dp\,dq + \int_0^s \int_0^t b(p, q, \tilde{u}^h_{p,q})dw_{p,q}$$

if $(s, t) \in R_{z_{i,j}}$, $i+j$ *is odd*;

$$(13)\qquad \tilde{u}^h_{s,t} = x + \int_0^s \int_0^{[t/h_2]h_2} a(p, q, \tilde{x}^h_{p,q})dp\,dq + \int_0^s \int_0^t b(p, q, \tilde{u}^h_{p,q})dw_{p,q}$$

if $(s, t) \in R_{z_{i,j}}$, $i+j$ *is even*;

$$(14)\qquad \tilde{x}^h_{s,t} = x + \int_0^s \int_0^t a(p, q, \tilde{x}^h_{p,q})dp\,dq + \int_0^s \int_0^t b(p, q, \tilde{u}^h_{p,q})dw_{p,q}.$$

Proof. On $R_{z_{1,1}}$ the equations are obvious. Assume that they hold on $R_{z_{i,j}}$ and let us prove their validity on $I_{i,j}$ and $J_{i,j}$. By hypothesis, for $(s,t)\in I_{i-1,j}$ we have

$$x^h_{s,t} = x+\int_0^s\int_0^t a(p,q,u^h_{p,q})dpdq+\int_0^s\int_0^t b(p,q,x^h_{p,q})dw_{p,q}. \tag{15}$$

Then, using (15) and the induction hypothesis, we have

$$\begin{aligned} u^h_{s,t} &= x^h_{s_i-,t-}+\int_{s_i}^s\int_0^t a(p,q,u^h_{p,q})dpdq \\ &= x+\int_0^{s_i}\int_0^t a(p,q,u^h_{p,q})dpdq+\int_0^{s_i}\int_0^t b(p,q,x^h_{p,q})dw_{p,q}+\int_{s_i}^s\int_0^t a(p,q,u^h_{p,q})dpdq \\ &= x+\int_0^s\int_0^t a(p,q,u^h_{p,q})dpdq+\int_0^{[s/h_1]h_1}\int_0^t b(p,q,x^h_{p,q})dw_{p,q}, \end{aligned}$$

$$\begin{aligned} x^h_{s,t} &= u^h_{s,t}+\int_{[s/h_1]h_1}^s\int_0^t b(p,q,x^h_{p,q})dw_{p,q} \\ &= x+\int_0^s\int_0^t a(p,q,u^h_{p,q})dpdq+\int_0^{[s/h_1]h_1}\int_0^t b(p,q,x^h_{p,q})dw_{p,q}+\int_{[s/h_1]h_1}^s\int_0^t b(p,q,x^h_{p,q})dw_{p,q} \\ &= x+\int_0^s\int_0^t a(p,q,u^h_{p,q})dpdq+\int_0^s\int_0^t b(p,q,x^h_{p,q})dw_{p,q}. \end{aligned}$$

Similarly one obtains the equations for $(s,t)\in J_{i,j}$ and for $\tilde{u}^h$, $\tilde{x}^h$.

LEMMA 2. *The following estimates hold*:

$$\sup_{(s,t)\in I} \mathrm{E}(|z_{s,t}|^2)\leqslant C_1 := 6(|x|^2+T^4K_1+T^2K_2)\exp\{6T^2(T^2K_1+K_2)\} \tag{16}$$

for $z = u^h, x^h, \tilde{u}^h, \tilde{x}^h$;

$$\sup_{(s,t)\in I} \mathrm{E}(|x^h_{s,t}-u^h_{s,t}|^2)\leqslant C_2(h_1+h_2), \qquad C_2 = TK_2(1+C_1); \tag{17}$$

$$\sup_{(s,t)\in I} \mathrm{E}(|\tilde{x}^h_{s,t}-\tilde{u}^h_{s,t}|^2)\leqslant C_3(h_1^2+h_2^2), \qquad C_3 = K_1T^2(1+C_1). \tag{18}$$

Proof. Define $K_3 = 3(|x|^2+T^4K_1+T^2K_2)$ and $K_4 = 3(T^2K_1+K_2)$. Using Lemma 1 and (K), for $(s,t)\in I$ we obtain

$$\begin{aligned} \mathrm{E}(|u^h_{s,t}|^2) &\leqslant 3|x|^2+3T^2\int_0^s\int_0^t \mathrm{E}(|a(p,q,u^h_{p,q})|^2)dpdq \\ &\quad +3\int_0^s\int_0^t \mathrm{E}(|b(p,q,x^h_{p,q})|^2)dpdq \\ &\leqslant K_3+K_4\int_0^s\int_0^t [\mathrm{E}(|u^h_{p,q}|^2)+\mathrm{E}(|x^h_{p,q}|^2)]dpdq. \end{aligned} \tag{19}$$

Similarly we obtain

(20) $$E(|x^h_{s,t}|^2) \leqslant K_3 + K_4 \int_0^s \int_0^t [E(|u^h_{p,q}|^2) + E(|x^h_{p,q}|^2)]\,dp\,dq.$$

Summing (19), (20) and using Gronwall's lemma we obtain

$$E(|u^h_{s,t}|^2) + E(|x^h_{s,t}|^2) \leqslant 2K_3 \exp(2T^2 K_4).$$

Similarly we deduce (16) for $\tilde{u}^h$, $\tilde{x}^h$.

Next, if $(s, t) \in R_{z_{i,j}}$ and $i+j$ is odd, we have

$$E(|x^h_{s,t} - u^h_{s,t}|^2) = \int_{[s/h_1]h_1}^{s} \int_0^t E(|b(p, q, x^h_{p,q})|^2)\,dp\,dq$$
$$\leqslant \int_{[s/h_1]h_1}^{s} \int_0^t K_2[1 + E(|x^h_{p,q}|^2)]\,dp\,dq \leqslant K_2(1+C_1)Th_1.$$

Similarly, if $i+j$ is even, we have

$$E(|x^h_{s,t} - u^h_{s,t}|^2) \leqslant K_2(1+C_1)Th_2.$$

An analogous argument works for $\tilde{x}^h - \tilde{u}^h$.

THEOREM 1. *Assume* (K) *and* (L) *are satisfied. Then*

(21) $$\sup_{(s,t)\in I} E(|x^h_{s,t} - x_{s,t}|^2) \leqslant C_4(h_1 + h_2),$$

where $C_4 = 3T^2 L_1 C_2 \exp\{3T^2(T^2 L_1 + L_2)\}$;

(22) $$\sup_{(s,t)\in I} E(|\tilde{x}^h_{s,t} - x_{s,t}|^2) \leqslant \tilde{C}_4(h_1^2 + h_2^2),$$

where $\tilde{C}_4 = 3L_2 C_3 \exp\{3T^2(T^2 L_1 + L_2)\}$.

Proof. We justify only (21) (similarly for (22)). We have

$$x^h_{s,t} - x_{s,t} = \int_0^s \int_0^t [a(p, q, x^h_{p,q}) - a(p, q, x_{p,q})]\,dp\,dq$$
$$+ \int_0^s \int_0^t [b(p, q, x^h_{p,q}) - b(p, q, x_{p,q})]\,dw_{p,q}$$
$$+ \int_0^s \int_0^t [a(p, q, u^h_{p,q}) - a(p, q, x^h_{p,q})]\,dp\,dq.$$

Then, using (L) and Lemma 2 (the second estimate), we obtain

$$E(|x^h_{s,t} - x_{s,t}|^2) \leqslant 3(T^2 L_1 + L_2) \int_0^s \int_0^t E(|x^h_{p,q} - x_{p,q}|^2)\,dp\,dq$$
$$+ 3T^2 L_1 \sup_{(p,q)\in I} E(|u^h_{p,q} - x^h_{p,q}|^2)$$
$$\leqslant 3T^2 L_1 C_2(h_1 + h_2) + 3(T^2 L_1 + L_2) \int_0^s \int_0^t E(|x^h_{p,q} - x_{p,q}|^2)\,dp\,dq$$

and with Gronwall's lemma we get

$$E(|x^h_{s,t}-x_{s,t}|^2) \leqslant C_4(h_1+h_2),$$

where $C_4 = 3T^2L_1C_2\exp\{3T^2(T^2L_1+L_2)\}$. In the same manner we estimate $E(|\tilde{x}^h_{s,t}-x_{s,t}|^2)$. Thus the proof is complete.

Next we introduce other approximating processes v^h, y^h, $\tilde{v}^h$, $\tilde{y}^h$ which are more appropriate for the numerical treatment. For $(s,t)\in R_{z_{1,1}}$ we define

$$(23)\quad v^h_{s,t} = x+\int_0^s\int_0^t a(p,q,v^h_{p,q})dp\,dq, \qquad y^h_{s,t} = v^h_{h_1-,h_2-}+\int_0^s\int_0^t b(p,q,y^h_{p,q})dw_{p,q};$$

$$(24)\quad \tilde{v}^h_{s,t} = x+\int_0^s\int_0^t b(p,q,\tilde{v}^h_{p,q})dw_{p,q}, \qquad \tilde{y}^h_{s,t} = \tilde{v}^h_{h_1-,h_2-}+\int_0^s\int_0^t a(p,q,\tilde{y}^h_{p,q})dp\,dq.$$

For some (i,j) we defined the processes v^h, y^h, $\tilde{v}^h$, $\tilde{y}^h$ on $R_{z_{i,j}}$ such that: $y^h_{s,t}$, $\tilde{v}^h_{s,t}$ are $\mathscr{F}_{s,t}$-measurable, $v^h_{s,t}$ is $\mathscr{F}_{s_{i-1},t}$-measurable if $(s,t)\in I_{i-1,j}$, $v^h_{s,t}$ is $\mathscr{F}_{s,t_{j-1}}$-measurable if $(s,t)\in J_{i,j-1}$, and $\tilde{y}^h_{s,t}$ is $\mathscr{F}_{s,t_j}$-measurable if $(s,t)\in I_{i-1,j}\cup J_{i,j-1}$. Now, if $(s,t)\in I_{i,j}$, we define (with the convention $y^h_{0-,t} = y^h_{s,0-} = x$)

$$(25)\quad \begin{cases} v^h_{s,t} = y^h_{s_i-,t-}+\int_{s_i}^s\int_0^t a(p,q,v^h_{p,q})dp\,dq, \\ y^h_{s,t} = v^h_{s_{i+1}-,t_j-}+\int_{s_i}^s\int_0^t b(p,q,y^h_{p,q})dw_{p,q}; \end{cases}$$

$$(26)\quad \begin{cases} \tilde{v}^h_{s,t} = \tilde{y}^h_{s_i-,t-}+\int_{s_i}^s\int_0^t b(p,q,\tilde{v}^h_{p,q})dw_{p,q}, \\ \tilde{y}^h_{s,t} = \tilde{v}^h_{s_{i+1}-,t_j-}+\int_{s_i}^s\int_0^t a(p,q,\tilde{y}^h_{p,q})dp\,dq; \end{cases}$$

and if $(s,t)\in J_{i,j}$, we define

$$(27)\quad \begin{cases} v^h_{s,t} = y^h_{s-,t_j-}+\int_0^s\int_{t_j}^t a(p,q,v^h_{p,q})dp\,dq, \\ y^h_{s,t} = v^h_{s_i-,t_{j+1}-}+\int_0^s\int_{t_j}^t b(p,q,y^h_{p,q})dw_{p,q}; \end{cases}$$

$$(28)\quad \begin{cases} \tilde{v}^h_{s,t} = \tilde{y}^h_{s-,t_j-}+\int_0^s\int_{t_j}^t b(p,q,\tilde{v}^h_{p,q})dw_{p,q}, \\ \tilde{y}^h_{s,t} = \tilde{v}^h_{s_i-,t_{j+1}-}+\int_0^s\int_{t_j}^t a(p,q,\tilde{y}^h_{p,q})dp\,dq. \end{cases}$$

Also, if $s = T$ or $t = T$, we set $v^h_{s,t} = v^h_{s-,t-}$, $y^h_{s,t} = y^h_{s-,t-}$, $\tilde{v}^h_{s,t} = \tilde{v}^h_{s-,t-}$, $\tilde{y}^h_{s,t} = \tilde{y}^h_{s-,t-}$.

The definition of v^h, y^h, $\tilde{v}^h$, $\tilde{y}^h$ on the whole I is obtained as follows: we start with $z \in R_{z_{1,1}}$ and define the processes by (23), (24), and then alternatively on $I_{1,1}$ by (25), (26), on $J_{2,1}$ by (27), (28), on $I_{2,2}$ by (25), (26), etc.

Remark 3. The processes y^h and $\tilde{v}^h$ are $\mathcal{F}_{s,t}$-adapted; $v^h_{s,t}$ is $\mathcal{F}_{s_i,t}$-measurable if $(s, t) \in I_{i,j}$; $v^h_{s,t}$ is $\mathcal{F}_{s,t_j}$-measurable if $(s, t) \in J_{i,j}$; $\tilde{y}^h_{s,t}$ is $\mathcal{F}_{s_{i+1},t_j}$-measurable if $(s, t) \in I_{i,j}$; and $\tilde{y}^h_{s,t}$ is $\mathcal{F}_{s_i,t_{j+1}}$-measurable if $(s, t) \in J_{i,j}$.

LEMMA 3. *The following estimates hold:*

$$\sup_{(s,t)\in I} \mathrm{E}(|z_{s,t}|^2) \leqslant D_1 \quad \textit{for } z = v^h, y^h, \tilde{v}^h, \tilde{y}^h, \tag{29}$$

where $D_1 = (4+|x|^2)\exp\{5T^2(1+K_1+K_2)\}$;

$$\sup_{(s,t)\in I} \mathrm{E}(|y^h_{s,t} - v^h_{s,t}|^2) \leqslant D_2(h_1+h_2+h_1h_2+h_1^2+h_2^2), \tag{30}$$

where $D_2 = (3K_2+6T^2K_1+TK_2)(1+D_1)$;

$$\sup_{(s,t)\in I} \mathrm{E}(|\tilde{y}^h_{s,t} - \tilde{v}^h_{s,t}|^2) \leqslant \tilde{D}_2(h_1+h_2+h_1^2+h_2^2+h_1^2h_2^2), \tag{31}$$

where $\tilde{D}_2 = (6K_1+12TK_2+2T^2K_1)(1+D_1)$.

Proof. By Itô's formula for $\{|v^h_{s,t}|^2\}_{s_i \leqslant s < s_{i+1}}$, $0 \leqslant t < t_j$ is fixed, we have

$$|v^h_{s,t}|^2 = |y^h_{s_i-,t-}|^2 + 2\int_{s_i}^{s}\int_0^t \langle a(p, q, v^h_{p,q}), v^h_{p,t}\rangle dp\,dq$$

$$\leqslant |y^h_{s_i-,t-}|^2 + \int_{s_i}^{s}\int_0^t [|a(p, q, v^h_{p,q})|^2 + |v^h_{p,t}|^2]\,dp\,dq$$

$$\leqslant |y^h_{s_i-,t-}|^2 + K_1Th_1 + \int_{s_i}^{s}\int_0^t (K_1|v^h_{p,q}|^2 + |v^h_{p,t}|^2)\,dp\,dq;$$

$$\mathrm{E}(|v^h_{s,t}|^2) \leqslant \mathrm{E}(|y^h_{s_i-,t-}|^2) + TK_1h_1 + \int_{s_i}^{s}\int_0^t [K_1\mathrm{E}(|v^h_{p,q}|^2) + \mathrm{E}(|v^h_{p,t}|^2)]\,dp\,dq.$$

Then we obtain

$$\sup_{0\leqslant t\leqslant t_j} \mathrm{E}(|v^h_{s,t}|^2) \leqslant \sup_{0\leqslant t\leqslant t_j} \mathrm{E}(|y^h_{s_i-,t-}|^2) + TK_1h_1 + T(1+K_1)\int_{s_i}^{s} \sup_{q<t_j} \mathrm{E}(|v^h_{p,q}|^2)\,dp, \tag{32}$$

so that, by Gronwall's lemma,

$$\sup_{t<t_j} \mathrm{E}(|v^h_{s,t}|^2) \leqslant [\sup_{t\leqslant t_j} \mathrm{E}(|y^h_{s_i-,t-}|^2) + TK_1h_1]\exp\{T(1+K_1)h_1\}. \tag{33}$$

Since $v^h_{s_{i+1}-,t_j-}$ is $\mathcal{F}_{s_i,T}$-measurable and if $s \leqslant s'$, $t \leqslant t'$, and

$$\mathrm{E}\Big[\int_s^{s'}\int_t^{t'} h(p, q)\,dw_{p,q}/\mathcal{B}(\mathcal{F}_{s,T}\cup\mathcal{F}_{T,t})\Big] = 0,$$

we obtain, for $s_i \leqslant s < s_{i+1}$, $0 \leqslant t < t_j$,

$$\mathrm{E}(|y^h_{s,t}|^2) = \mathrm{E}(|v^h_{s_{i+1}-,t_j-}|^2) + \int_{s_i}^{s}\int_0^t \mathrm{E}(|b(p,q,y^h_{p,q})|^2)\,dp\,dq$$

$$\leqslant \mathrm{E}(|v^h_{s_{i+1}-,t_j-}|^2) + K_2\int_{s_i}^{s}\int_0^t [1+\mathrm{E}(|y^h_{p,q}|^2)]\,dp\,dq,$$

so that

(34) $$\mathrm{E}(|y^h_{s,t}|^2) \leqslant \mathrm{E}(|v^h_{s_{i+1}-,t_j-}|^2) + TK_2h_1 + K_2\int_{s_i}^{s}\int_0^t \mathrm{E}(|y^h_{p,q}|^2)\,dp\,dq$$

and, by Gronwall's lemma,

(35) $$\mathrm{E}(|y^h_{s,t}|^2) \leqslant [\mathrm{E}(|v^h_{s_{i+1}-,t_j-}|^2) + TK_2h_1]\exp(TK_2h_1).$$

If we take $s \nearrow s_{i+1}$ in (33) and we use Fatou's lemma, we deduce

(36) $$\sup_{t<t_j}\mathrm{E}(|v^h_{s_{i+1}-,t-}|^2) \leqslant [\sup_{t<t_j}\mathrm{E}(|y^h_{s_i-,t-}|^2) + TK_1h_1]\exp\{T(1+K_1)h_1\},$$

and using (36) in (35) we get

(37) $$\sup_{t<t_j}\mathrm{E}(|y^h_{s,t}|^2)$$
$$\leqslant \{TK_2h_1 + [TK_1h_1 + \sup_{t\leqslant t_j}\mathrm{E}(|y^h_{s_i-,t-}|^2)]\exp\{T(1+K_1)h_1\}\}\exp(TK_2h_1).$$

Taking $s \nearrow s_{i+1}$ in (37) and applying Fatou's lemma we obtain

(38) $$\sup_{t\leqslant t_j}\mathrm{E}(|y^h_{s_{i+1}-,t-}|^2)$$
$$\leqslant [T(K_1+K_2)h_1 + \sup_{t<t_j}\mathrm{E}(|y^h_{s_i-,t-}|^2)]\exp\{T(1+K_1+K_2)h_1\}$$

and inductively we get

(39) $$\sup_{t<t_j}\mathrm{E}(|y^h_{s_{i+1}-,t-}|^2) \leqslant (1+|x|^2)\exp\{2T^2(1+K_1+K_2)\}.$$

Using (39) in (36) we obtain

$$\sup_{t<t_j}\mathrm{E}(|v^h_{s_{i+1}-,t}|^2) \leqslant [TK_1h_1 + (1+|x|^2)\exp\{2T^2(1+K_1+K_2)\}]\exp\{T(1+K_1)h_1\},$$

so that

(40) $$\sup_{t<t_j}\mathrm{E}(|v^h_{s_{i+1}-,t}|^2) \leqslant (2+|x|^2)\exp\{3T^2(1+K_1+K_2)\}.$$

Replacing (40) in (35) we obtain

(41) $$\mathrm{E}(|y^h_{s,t}|^2) \leqslant (3+|x|^2)\exp\{4T^2(1+K_1+K_2)\},$$

which together with (33) implies

(42) $$E(|v^h_{s,t}|^2) \leqslant (4+|x|^2)\exp\{5T^2(1+K_1+K_2)\}.$$

The same estimates, (41) and (42), follow if $0 \leqslant s < s_i$, $t_j \leqslant t < t_{j+1}$. A similar computation works for $\tilde{v}^h$, $\tilde{y}^h$.

Next, if $0 \leqslant t < t_j$, $s_i \leqslant s < s_{i+1}$, we have

(43) $$E(|y^h_{s,t_j-} - y^h_{s,t}|^2) = \int_{s_i}^{s}\int_{t}^{t_j} E(|b(p, q, y^h_{p,q})|^2)dp\,dq \leqslant K_2(1+D_1)h_1h_2,$$

(44) $$E(|v^h_{s_{i+1}-,t_j-} - v^h_{s,t}|^2)$$
$$\leqslant 3E(|y^h_{s_i-,t_j-} - y^h_{s_i-,t}|^2) + 3E[|\int_{s_i}^{s_{i+1}}\int_{0}^{t_j} a(p, q, v^h_{p,q})dp\,dq|^2]$$
$$+3E[|\int_{s_i}^{s}\int_{0}^{t} a(p, q, v^h_{p,q})dp\,dq|^2] \leqslant 3K_2(1+D_1)h_1h_2 + 6T^2K_1(1+D_1)h_1^2 = D_1'.$$

Since $v^h_{s_{i+1}-,t_j-} - v^h_{s,t}$ is $\mathscr{F}_{s_i,T}$-measurable for $t < t_j$, $s_i \leqslant s < s_{i+1}$, we have

$$E(|y^h_{s,t} - v^h_{s,t}|^2) = E(|v^h_{s_{i+1}-,t_j-} - v^h_{s,t}|^2) + \int_{s_i}^{s}\int_{0}^{t} E(|b(p, q, y^h_{p,q})|^2)dp\,dq$$
$$\leqslant D_1' + K_2T(1+D_1)h_1.$$

An analogous computation works for $0 \leqslant s < s_i$, $t_j \leqslant t < t_{j+1}$. Therefore

$$E(|y^h_{s,t} - v^h_{s,t}|^2) \leqslant 3K_2(1+D_1)h_1h_2 + 6T^2K_1(1+D_1)(h_1^2+h_2^2) + TK_2(1+D_1)(h_1+h_2)$$
$$\leqslant D_2(h_1+h_2+h_1h_2+h_1^2+h_2^2)$$

with D_2 defined as above. We proceed in the same manner for $E(|\tilde{y}^h_{s,t} - \tilde{v}^h_{s,t}|^2)$.

THEOREM 2. *Assume that* (K) *and* (L) *are satisfied. Then the following estimates hold*:

(45) $$\sup_{(s,t)\in I} E(|y^h_{s,t} - x_{s,t}|^2) \leqslant D_3(h_1+h_2+h_1h_2+h_1^2+h_2^2),$$

(46) $$\sup_{(s,t)\in I} E(|\tilde{y}^h_{s,t} - x_{s,t}|^2) \leqslant \tilde{D}_3(h_1+h_2+h_1^2+h_2^2),$$

where D_3 and $\tilde{D}_3$ are given explicitly in the proof.

Proof. Let $(s, t)\in I_{i,j}$. From the equality

$$u^h_{s,t} - v^h_{s,t} = x^h_{s-,t-} - y^h_{s_i-,t-} + \int_{s_i}^{s}\int_{0}^{t} [a(p, q, u^h_{p,q}) - a(p, q, v^h_{p,q})]\,dp\,dq$$

we obtain, by Itô's formula, along $t=\text{constant}$,

$$|u^h_{s,t}-v^h_{s,t}|^2 = |x^h_{s_i-,t-}-y^h_{s_i-,t-}|^2 + 2\int_{s_i}^{s}\int_{0}^{t}\langle[a(p,q,u^h_{p,q})-a(p,q,v^h_{p,q})],\, u^h_{p,t}-v^h_{p,t}\rangle\, dp\, dq$$
$$\leqslant |x^h_{s_i-,t-}-y^h_{s_i-,t-}|^2 + 2L_1\int_{s_i}^{s}\int_{0}^{t}|u^h_{p,q}-v^h_{p,q}|\,|u^h_{p,t}-v^h_{p,t}|\, dp\, dq.$$

Then

$$\sup_{t<t_j}\mathrm{E}(|u^h_{s,t}-v^h_{s,t}|^2) \leqslant \sup_{t<t_j}\mathrm{E}(|x^h_{s_i-,t-}-y^h_{s_i-,t-}|^2) + 2TL_1\int_{s_i}^{s}\sup_{q<t_j}\mathrm{E}(|u^h_{p,q}-v^h_{p,q}|^2)\,dp,$$

and thus, by Gronwall's lemma,

$$\sup_{t<t_j}\mathrm{E}(|u^h_{s,t}-v^h_{s,t}|^2) \leqslant \sup_{t<t_j}\mathrm{E}(|x^h_{s_i-,t-}-y^h_{s_i-,t-}|^2)\exp(2TL_1h_1). \tag{47}$$

Next from (44) and (47) we obtain

$$\mathrm{E}(|u^h_{s,t}-v^h_{s_{i+1}-,t_j-}|^2) \leqslant 2\mathrm{E}(|u^h_{s,t}-v^h_{s,t}|^2)+2\mathrm{E}(|v^h_{s,t}-v^h_{s_{i+1}-,t_j-}|^2)$$
$$\leqslant 2D_1'+2\sup_{t<t_j}\mathrm{E}(|x^h_{s_i-,t-}-y^h_{s_i-,t-}|^2)\exp(2TL_1h_1)=\tilde{d}_1. \tag{48}$$

Utilizing the $\mathscr{F}_{s_i,T}$-measurability of $u^h_{s,t}$, $v^h_{s_{i+1}-,t_j-}$ we deduce

$$\mathrm{E}(|x^h_{s,t}-y^h_{s,t}|^2) = \mathrm{E}(|u^h_{s,t}-v^h_{s_{i+1}-,t_j-}|^2) + \int_{s_i}^{s}\int_{0}^{t}\mathrm{E}(|b(p,q,x^h_{p,q})-b(p,q,y^h_{p,q})|^2)\,dp\,dq$$
$$\leqslant \mathrm{E}(|u^h_{s,t}-v^h_{s_{i+1}-,t_j-}|^2) + L_2\int_{s_i}^{s}\int_{0}^{t}\mathrm{E}(|x^h_{p,q}-y^h_{p,q}|^2)\,dp\,dq$$
$$\leqslant \sup_{t<t_j}\mathrm{E}(|u^h_{s,t}-v^h_{s_{i+1}-,t_j-}|^2) + L_2\int_{s_i}^{s}\sup_{q<t_j}\mathrm{E}(|x^h_{p,q}-y^h_{p,q}|^2)\,dp.$$

Hence

$$\sup_{t<t_j}\mathrm{E}(|x^h_{s,t}-y^h_{s,t}|^2) \leqslant \sup_{t<t_j}\mathrm{E}(|u^h_{s,t}-v^h_{s_{i+1}-,t_j-}|^2) + L_2T\int_{s_i}^{s}\sup_{q<t_j}\mathrm{E}(|x^h_{p,q}-y^h_{p,q}|^2)\,dp,$$

so that, by (48) and Gronwall's lemma,

$$\sup_{t<t_j}\mathrm{E}(|x^h_{s,t}-y^h_{s,t}|^2) \leqslant \tilde{d}_1\exp(TL_2h_1). \tag{49}$$

If we take $s\nearrow s_{i+1}$ and $t\nearrow t'$ in (49), we obtain the recursive inequality

$$\alpha_{i+1} := \sup_{t<t_j}\mathrm{E}(|x^h_{s_{i+1}-,t-}-y^h_{s_{i+1}-,t-}|^2) \leqslant [2D_1'+2\alpha_i\exp(2TL_1h_1)]\exp(TL_2h_1)$$

or

$$\alpha_{i+1} \leqslant [6K_2(1+D_1)h_1h_2+12T^2K_1(1+D_1)h_1^2+2\alpha_i]\exp\{T(2L_1+L_2)h_1\}. \tag{50}$$

By induction we deduce from (50) the inequality

(51) $$\sup_{t\leqslant t_j} \mathrm{E}(|x^h_{s_{i+1}-,t-} - y^h_{s_{i+1}-,t-}|^2) \leqslant \frac{6K_2(1+D_1)h_2+12T^2K_1(1+D_1)h_1}{T(2L_1+L_2)} \exp\{T^2(2L_1+L_2)\}.$$

Utilizing (51) in (49) we obtain

(52) $$\mathrm{E}(|x^h_{s,t}-y^h_{s,t}|^2) \leqslant d'_1(h_1+h_2+h_1h_2+h_1^2),$$

where

(53) $$d'_1 = 6(1+D_1)\left[K_2+2T^2K_1+\frac{2(K_2+2T^2K_1)}{T(2L_1+L_2)}\exp\{T^2(2L_1+L_2)\}\right]\exp(T^2L_2).$$

Similarly for $(s, t)\in J_{i,j}$ we obtain

(54) $$\mathrm{E}(|x^h_{s,t}-y^h_{s,t}|^2) \leqslant d'_1(h_1+h_2+h_1h_2+h_2^2).$$

Now (52), (54), and (21) imply (45) with

(55) $$D_3 = 2d'_1+2C_4.$$

Next, for $(s, t)\in I_{i,j}$, utilizing the equality

$$\tilde{u}^h_{s,t}-\tilde{v}^h_{s,t} = \tilde{x}^h_{s_i-,t-}-\tilde{y}^h_{s_i-,t-}+\int_{s_i}^{s}\int_{0}^{t}[b(p, q, \tilde{u}^h_{p,q})-b(p, q, \tilde{v}^h_{p,q})]\,dw_{p,q}$$

and $\mathscr{F}_{s_i,T}$-measurability of $\tilde{x}^h_{s_i-,t-}-\tilde{y}^h_{s_i-,t-}$ we deduce

$$\mathrm{E}(|\tilde{u}^h_{s,t}-\tilde{v}^h_{s,t}|^2) = \mathrm{E}(|\tilde{x}^h_{s_i-,t-}-\tilde{y}^h_{s_i-,t-}|^2)+L_2\int_{s_i}^{s}\int_{0}^{t}\mathrm{E}(|\tilde{u}^h_{p,q}-\tilde{v}^h_{p,q}|^2)\,dp\,dq$$

and, by Gronwall's lemma,

(56) $$\sup_{t<t_j}\mathrm{E}(|\tilde{u}^h_{s,t}-\tilde{v}^h_{s,t}|^2) \leqslant \sup_{t\leqslant t_j}\mathrm{E}(|\tilde{x}^h_{s_i-,t-}-\tilde{y}^h_{s_i-,t-}|^2)\exp(TL_2h_1) = \beta_i\exp(TL_2h_1).$$

Also we have

$$\sup_{t<t_j}\mathrm{E}(|\tilde{x}^h_{s,t}-\tilde{y}^h_{s,t}|^2) \leqslant (1+h_1)\sup_{t<t_j}\mathrm{E}(|\tilde{u}^h_{s,t}-\tilde{v}^h_{s_{i+1}-,t_j-}|^2) +(1+1/h_1)L_1Th_1\int_{s_i}^{s}\sup_{q<t_j}\mathrm{E}(|\tilde{x}^h_{p,q}-\tilde{y}^h_{p,q}|^2)\,dp,$$

and hence

(57) $$\sup_{t<t_j}\mathrm{E}(|\tilde{x}^h_{s,t}-\tilde{y}^h_{s,t}|^2) \leqslant (1+h_1)\sup_{t<t_j}\mathrm{E}(|\tilde{u}^h_{s,t}-\tilde{v}^h_{s_{i+1}-,t_j-}|^2)\exp\{2L_1Th_1(1+h_1)\}.$$

On the other hand (by using (K) and (56)), we can write

$$(58)\qquad \mathrm{E}(|\tilde{u}^h_{s,t}-\tilde{v}^h_{s_{i+1}-,t_j-}|^2) = \mathrm{E}(|\tilde{u}^h_{s,t}-\tilde{v}^h_{s,t}+\tilde{v}^h_{s,t}-\tilde{v}^h_{s_{i+1},t_j-}|^2)$$

$$= \mathrm{E}(|\tilde{u}^h_{s,t}-\tilde{v}^h_{s,t}|^2)+\int_{s_i}^{s_{i+1}}\int_{t_j}^{t}\mathrm{E}\big(|b(p,q,\tilde{v}^h_{p,q})|^2\big)dp\,dq$$

$$\leqslant \beta_i \exp(TL_2h_1)+K_1(1+D_1)h_1h_2.$$

Next, taking $s\nearrow s_{i+1}$ in (57) and using (58), we obtain

$$\beta_{i+1} \leqslant \exp\{1+2L_1T(1+h_1)\}h_1[\beta_i\exp(L_2Th_1)+K_1(1+D_1)h_1h_2],$$

$$\beta_{i+1} \leqslant [\beta_i+K_1(1+D_1)h_1h_2]\exp\{h_1[1+2L_1T(1+h_1)]+2L_2T\}.$$

Hence, by induction and putting

$$(59)\qquad d_2' = \frac{K_1(1+D_1)}{1+2L_1T+2L_2T}\exp\{T[1+2L_1T(1+h_1)+2L_2T]\},$$

we get

$$(60)\qquad \beta_i \leqslant d_2'h_2.$$

Then (58) becomes

$$\mathrm{E}(|\tilde{u}^h_{s,t}-\tilde{v}^h_{s_{i+1}-,t_j-}|^2) \leqslant d_2'\exp(TL_2h_1)h_2+K_1(1+D_1)h_1h_2,$$

which used in (57) implies

$$(61)\qquad \mathrm{E}(|\tilde{x}^h_{s,t}-\tilde{y}^h_{s,t}|^2) \leqslant (1+h_1)[d_2'\exp(TL_2h_1)h_2$$

$$+K_1(1+D_1)h_1h_2]\exp\{2TL_1h_1(1+h_1)\}$$

$$\leqslant (T+1)[d_2'+K_1(1+D_1)](h_1+h_2+h_1h_2)\exp\{T(1+T)(L_2+2TL_1)\}.$$

A similar inequality follows if $(s,t)\in J_{i,j}$. Then from (22), (61) we get

$$(62)\qquad \mathrm{E}(|\tilde{y}^h_{s,t}-x_{s,t}|^2) \leqslant \tilde{D}_3(h_1+h_2+h_1h_2+h_1^2+h_2^2),$$

where

$$(63)\qquad \tilde{D}_3 = 2\tilde{C}_4+[d_2'+K_1(1+D_1)]\exp\{T(1+T)(L_2+2TL_1)\}.$$

The proof is complete.

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