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ERGODICITY AND WEAK MIXING OF SEMISTABLE PROCESSES

BY

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Abstract. We prove that ergodicity and weak mixing coincide for the class of stationary symmetric semistable processes and give several characterizations of ergodic semistable processes.

1. Preliminaries. Before we state and prove our results concerning ergodicity and weak mixing of semistable processes we recall basic definitions and properties of semistable laws which will be exploited in this note.

Let us first recall (cf. [6]) that a real random variable X has a *symmetric r-semistable distribution* with index α, where $r \in (0, 1)$ and $\alpha \in (0, 2\rangle$, if its characteristic function takes the form

$$\phi_x(t) = \exp\Big(-\int_{\Delta} |ts|^{\alpha} k_{\alpha}(ts) \Gamma(ds)\Big),$$

where $\Delta = \{x \in \boldsymbol{R}: r^{1/\alpha} < |x| \leqslant 1\}$, Γ is a symmetric measure defined on the σ-field $\mathscr{B}_{\Delta}$ of Borel subsets of Δ such that $\int_{\Delta} |s|^{\alpha} \Gamma(ds) < \infty$ and a function k_{α} is defined for $t \in \boldsymbol{R}$ by

$$k_{\alpha}(t) = |t|^{\alpha} \sum_{n=-\infty}^{\infty} r^{-n}\big(1-\cos(r^{n/\alpha}|t|)\big).$$

The function k_{α} satisfies the following inequalities:

$$0 < c_0 = \inf_{t \in \boldsymbol{R}\setminus\{0\}} k_{\alpha}(t) \leqslant \sup_{t \in \boldsymbol{R}\setminus\{0\}} k_{\alpha}(t) = c_1 < \infty, \tag{1.1}$$

and for each $t \in \boldsymbol{R}$

$$k_{\alpha}(r^{-1/\alpha} t) = k_{\alpha}(t). \tag{1.2}$$

Note that in this paper we restrict our attention to symmetric semistable measures, so the abbreviation r-SS(α) will always mean symmetric r-semistable with index α.

A stochastic process $X = (X_t)_{t \in \boldsymbol{R}}$ is called r-SS(α) if all finite linear combinations of the form $\sum_{k=1}^{n} a_k X_{t_k}$ are r-SS(α) random variables or, equivalently, all finite dimensional distributions of X are r-SS(α).

In the sequel, we will exploit the spectral representation of r-SS(α) processes and we need to introduce the notion of an integral with respect to an r-SS(α) random measure. A symmetric random measure M on $\mathscr{B}_\Delta$ is called r-SS(α) if there exists a measure m on $\mathscr{B}_\Delta$ such that $\int_\Delta |s|^\alpha m(ds) < \infty$ and for $A \in \mathscr{B}_\Delta$

$$\phi_{M(A)}(t) = \exp\left(-\int_A |ts|^\alpha k_\alpha(ts) m(ds)\right).$$

A measure m is called the *control measure* of M. Let m_α be a measure on $\mathscr{B}_\Delta$ such that for $s \in \Delta$

$$\frac{dm_\alpha}{dm}(s) = |s|^\alpha.$$

If $f \in L_\alpha(\Delta, m_\alpha)$, then f is integrable with respect to M and

$$\phi_{\int f dM}(t) = \exp\left(-\int_\Delta |tsf(s)|^\alpha k_\alpha\big(tsf(s)\big) m(ds)\right).$$

Let X be a symmetric stochastically continuous r-SS(α) process. It follows from the spectral representation theorem for r-SS(α) processes (see [6]) that there exists an r-SS(α) random measure M on $\mathscr{B}_\Delta$ with the control measure m and a family $(f_t)_{t \in \boldsymbol{R}}$ of elements of $L_\alpha(\Delta, m_\alpha)$ such that

$$(X_t)_{t \in \boldsymbol{R}} \stackrel{\mathrm{d}}{=} \left(\int f_t dM\right)_{t \in \boldsymbol{R}} \tag{1.3}$$

and, consequently, for $n \in \boldsymbol{N}$, $a_1, \ldots, a_n \in \boldsymbol{R}$, $t_1, \ldots, t_n \in \boldsymbol{R}$ and $f = \sum_{k=1}^n a_k f_{t_k}$

$$\mathrm{E} \exp\left(i \sum_{k=1}^n a_k X_{t_k}\right) = \exp\left(-\int_\Delta |sf(s)|^\alpha k_\alpha\big(sf(s)\big)\, m(ds)\right).$$

Let X be additionally a stationary process. *Stationarity* means that for any choice of $t_1, \ldots, t_n \in \boldsymbol{R}$ the distribution of $(X_{t_1+t}, \ldots, X_{t_n+t})$ does not depend on t. Let $\left(\int f_t dM\right)_{t \in \boldsymbol{R}}$ be the spectral representation of X. For $t \in \boldsymbol{R}$ we define U_t by

$$U_t\left(\sum_{k=1}^n \vartheta_k f_{t_k}\right) = \sum_{k=1}^n \vartheta_k f_{t_k+t}.$$

Note that U_t extends to a topological automorphism of $\overline{\mathrm{lin}}\{f_t, t \in \boldsymbol{R}\}$, where the closure is taken in the norm of the space $L^\alpha(\Delta, m_\alpha)$. Indeed, stationarity implies that for each $f \in \mathrm{lin}\{f_t, t \in \boldsymbol{R}\}$

$$\int_\Delta |sU_t f(s)|^\alpha k_\alpha\big(sU_t f(s)\big) m(ds) = \int_\Delta |sf(s)|^\alpha k_\alpha\big(sf(s)\big) m(ds)$$

and inequalities (1.1) yield

$$c_0 c_1^{-1} \int_\Delta |f|^\alpha dm_\alpha \leqslant \int_\Delta |U_t f|^\alpha dm_\alpha \leqslant c_0^{-1} c_1 \int_\Delta |f|^\alpha dm_\alpha.$$

Thus $(U_t)_{t\in \boldsymbol{R}}$ is a group of topological automorphisms of $\overline{\operatorname{lin}}\{f_t,\ t\in \boldsymbol{R}\}$. By (1.3) the group $(U_t)_{t\in \boldsymbol{R}}$ corresponds to the group $(T_t)_{t\in \boldsymbol{R}}$ of shift operators on $\operatorname{lin}\{X_t\colon\ t\in \boldsymbol{R}\}$.

For more information on semistable measures and processes see [2], [6], [7] and references therein.

2. Ergodicity weak mixing and mixing of r-SS(α) processes. The following lemma will play a decisive role in this section. It will allow us to characterize ergodic r-SS(α) processes in terms of their spectral representations and to prove that ergodic r-SS(α) processes are also weak mixing.

LEMMA 2.1. *For a bounded Borel function $\phi\colon\ \boldsymbol{R}\to \boldsymbol{R}$ the following conditions are equivalent*:

(i) *for two different positive numbers* $c_1,\ c_2$

$$\lim_{T\to\infty} T^{-1}\int_0^T \exp(c_1\phi(t))dt = 1$$

and

$$\lim_{T\to\infty} T^{-1}\int_0^T \exp(c_2\phi(t))dt = 1;$$

(ii) *for each* $\varepsilon > 0$

$$\lim_{T\to\infty} T^{-1}\operatorname{Leb}(\{t\in[0,\ T]\colon\ |\phi(t)|>\varepsilon\}) = 0;$$

(iii) *for each positive number* c

$$\lim_{T\to\infty} T^{-1}\int_0^T |e^{c\phi(t)}-1|\,dt = 0;$$

(iv)
$$\lim_{T\to\infty} T^{-1}\int_0^T |\phi(t)|\,dt = 0.$$

Proof. The equivalence of (i), (ii) and (iv) is the contents of Lemma 2 of [5]. Clearly, (iii) implies (i), and (iii) follows from (ii) by the inequality

$$\int_0^T |e^{c\phi(t)}-1|\,dt \leqslant (e^{c\|\phi\|_\infty}+1)\lambda(\{t\in[0,\ T]\colon\ |\phi(t)|>\varepsilon\})$$
$$+(e^{c\varepsilon}-1)\lambda(\{t\in[0,\ T]\colon\ |\phi(t)|\leqslant\varepsilon\}),$$

which holds for each $c,\ \varepsilon > 0$. ∎

From now on, let X be a measurable and stationary stochastic process. Let us recall here (cf. [3], [8]) that the process X is ergodic if and only if for each $Y,\ Z\in \operatorname{lin}_{\boldsymbol{C}}\{X_t\colon\ t\in \boldsymbol{R}\}$

$$\lim_{T\to\infty} T^{-1}\int_0^T \mathrm{E}((T_tY)\bar{Z})dt = \mathrm{E}Y\mathrm{E}\bar{Z};$$

X is weak mixing if and only if for each $Y, Z \in \mathrm{lin}_{\boldsymbol{C}}\{X_t\colon t \in \boldsymbol{R}\}$

$$\lim_{T\to\infty} T^{-1} \int_0^T \left|\mathrm{E}((T_t Y)\bar{Z}) - \mathrm{E}Y\mathrm{E}\bar{Z}\right| dt = 0;$$

and X is mixing if and only if for each $Y, Z \in \mathrm{lin}_{\boldsymbol{C}}\{X_t\colon t \in \boldsymbol{R}\}$

$$\lim_{T\to\infty} \mathrm{E}\big((T_T Y)\bar{Z}\big)dt = \mathrm{E}Y\mathrm{E}\bar{Z}.$$

It is well known (see [9]) that ergodicity can be characterized by the condition saying that for each $Y \in \mathrm{lin}_{\boldsymbol{R}}\{X_t\colon t \in \boldsymbol{R}\}$

$$\lim_{T\to\infty} T^{-1} \int_0^T \mathrm{E}\big((T_t Y)Y\big)dt = |\mathrm{E}Y|^2.$$

Standard approximation arguments (for details see [5]) imply that the process X is ergodic if and only if for each $Y \in \mathrm{lin}\{X_t\colon t \in \boldsymbol{R}\}$

$$\lim_{T\to\infty} T^{-1} \int_0^T \mathrm{E}\exp[i(T_t Y - Y)]\,dt = |\mathrm{E}\exp(iY)|^2. \tag{2.1}$$

Similarly, the process X is weak mixing if and only if for each $Y \in \mathrm{lin}\{X_t\colon t \in \boldsymbol{R}\}$

$$\lim_{T\to\infty} T^{-1} \int_0^T \left|\mathrm{E}\exp[i(T_t Y - Y)] - |\mathrm{E}\exp(iY)|^2\right| dt = 0,$$

and X is mixing if and only if for each $Y \in \mathrm{lin}\{X_t\colon t \in \boldsymbol{R}\}$

$$\lim_{T\to\infty} \mathrm{E}\exp[i(T_T Y - Y)] = |\mathrm{E}\exp(iY)|^2.$$

Now, let X be a stationary and stochastically continuous r-SS(α) process with spectral representation $\left(\int U_t f_0\, dM\right)_{t\in\boldsymbol{R}}$.

THEOREM 2.1. *Let X be a stationary and stochastically continuous r-SS(α) process. Then the following conditions are equivalent*:

(0) *X is ergodic*;

(i) *for each $f \in \mathrm{lin}\{U_t f_0\colon t \in \boldsymbol{R}\}$*

$$\lim_{T\to\infty} T^{-1} \int_0^T e^{\phi(f,t)}\,dt = 1;$$

(ii) *for each $\varepsilon > 0$ and each $f \in \mathrm{lin}\{U_t f_0\colon t \in \boldsymbol{R}\}$*

$$\lim_{T\to\infty} T^{-1}\,\mathrm{Leb}\big(\{t \in [0, T]\colon |\phi(f, t)| > \varepsilon\}\big) = 0;$$

(iii) *for each $c > 0$ and each $f \in \mathrm{lin}\{U_t f_0\colon t \in \boldsymbol{R}\}$*

$$\lim_{T\to\infty} T^{-1} \int_0^T |e^{c\phi(f,t)} - 1|\,dt = 0;$$

(iv) *for each* $f \in \mathrm{lin}\{U_t f_0\colon\ t \in \boldsymbol{R}\}$

$$\lim_{T\to\infty} T^{-1} \int_0^T |\phi(f,t)|\,dt = 0,$$

where

$$\phi(f,t) = \int_\Delta \left[2\,|sf(s)|^\alpha k_\alpha(sf(s)) - |s(f-U_t f)(s)|^\alpha k_\alpha(s(f-U_t f)(s))\right] m(ds).$$

Proof. Direct verifications show that X is ergodic if and only if for each $f \in \mathrm{lin}\{U_t f_0\colon\ t \in \boldsymbol{R}\}$

$$\lim_{T\to\infty} T^{-1} \int_0^T e^{\phi(f,t)}\,dt = 1.$$

By (1.2) and the above condition, X is ergodic if and only if for each $f \in \mathrm{lin}\{U_t f_0\colon\ t \in \boldsymbol{R}\}$

$$\lim_{T\to\infty} T^{-1} \int_0^T \exp[r^{-1}\phi(f,t)]\,dt = 1.$$

Thus we have condition (i) of Lemma 2.1 with $c_1 = 1$ and $c_2 = r^{-1}$, and the equivalence of all conditions follows immediately from this lemma. ∎

Remark 2.1. In [1] a characterization of ergodic SαS (short for symmetric α-stable) processes in terms of their spectral representation was given. By means of Lemma 2.1 this characterization has been extended in [5] to yield conditions similar to (i)–(iv) of Theorem 2.1.

While, in general, weak mixing is a stronger property than ergodicity, they coincide for Gaussian processes (cf. [3]). We show here that, in fact, they coincide for all r-SS(α) processes.

COROLLARY 2.1. *Let X be a stationary and stochastically continuous r-SS(α) process. If X is ergodic, then it is weak mixing.*

Proof. Suppose that X is ergodic and let $(\int U_t f_0\,dM)_{t\in\boldsymbol{R}}$ be a spectral representation of X. Then

$$\mathrm{E}\exp[i(T_t Y - Y)] - |\mathrm{E}\exp(iY)|^2$$
$$= \exp\left[-\int_\Delta 2|sf(s)|^\alpha k_\alpha(sf(s))\,m(ds)\right]\left[e^{\phi(f,t)} - 1\right],$$

where $f \in \mathrm{lin}\{U_t f_0\colon\ t \in \boldsymbol{R}\}$ and $Y = \int f\,dM$. Thus by (2.2) it suffices to prove that for $f \in \mathrm{lin}\{U_t f_0\colon\ t \in \boldsymbol{R}\}$

$$\lim_{T\to\infty} T^{-1} \int_0^T |e^{\phi(f,t)} - 1|\,dt = 0,$$

but this follows at once from condition (iii) of Theorem 2.1. ∎

Since a random variable is SαS if and only if it is r-SS(α) for each $r \in (0, 1)$, and a measurable SαS process is stochastically continuous (Theorem 0 of [1]), we immediately get the following result (see also [4]):

COROLLARY 2.2. *A measurable stationary* SαS *process is ergodic if and only if it is weak mixing.*

Remark 2.2. As symmetric Gaussian processes are exactly S2S processes, the proof of Theorem 2.1 provides an elementary proof of the equivalence under discussion for Gaussian processes.

Since for an r-SS(α) process X with the spectral representation $(\int U_t f_0 \, dM)_{t \in \boldsymbol{R}}$ we have

$$\mathrm{E} \exp\big(i(T_t Y - Y)\big) = e^{\phi(f,t)}, \quad \text{where } f \in \operatorname{lin}\{U_t f_0 \colon t \in \boldsymbol{R}\} \text{ and } Y = \int f \, dM,$$

we obtain at once the following characterization of the mixing property for these processes.

COROLLARY 2.3. *A stationary and stochastically continuous r-SS(α) process X is mixing if and only if for each $f \in \operatorname{lin}\{U_t f_0 \colon t \in \boldsymbol{R}\}$*

$$\lim_{T \to \infty} \phi(f, T) = 0.$$

REFERENCES

[1] S. Cambanis, C. D. Hardin, Jr., and A. Weron, *Ergodic properties of stationary stable processes*, Stochastic Proc. Appl. 25 (1987), pp. 1–18.

[2] D. M. Chung, B. S. Rajput and A. Tortrat, *Semistable laws on topological vector spaces*, Z. Wahrsch. verw. Gebiete 60 (1982), pp. 209–218.

[3] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, *Ergodic Theory*, Springer-Verlag, 1982.

[4] K. Podgórski, *A note on ergodic symmetric stable processes*, Stochastic Proc. Appl. 43 (1992), pp. 355–362.

[5] — and A. Weron, *Characterization of ergodic stable processes via the dynamical functional*, in: *Stable Processes and Related Topics*, Progress in Probability 25, Birkhäuser, Boston 1991.

[6] B. S. Rajput and K. Rama-Murthy, *Spectral representation of semistable processes, and semistable laws on Banach spaces*, J. Multivariate Anal. 21 (1987), pp. 139–157.

[7] — *On sample path properties of semistable processes*, ibidem 21 (1987), pp. 324–336.

[8] Yu. A. Rozanov, *Stationary Random Processes*, Holden-Day, San Francisco 1967.

[9] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, Berlin–Heidelberg 1982.

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