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MOMENTS AND GENERALIZED CONVOLUTIONS. II

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Abstract. For any positive number q a q-equivalence of generalized convolutions is defined in terms of moments of order q. The aim of this paper is to prove that under some natural restrictions on the order q q-equivalent generalized convolutions are identical.

This paper is a continuation of the author's earlier work [8]. We adopt the definitions and notation given in [4] and [8]. In particular, P will denote the space of all Borel probability measures defined on the half-line $[0, \infty)$. The space P is endowed with the topology of weak convergence. For any $a \in (0, \infty)$, T_a will denote the scale change $(T_a \mu)(E) = \mu(a^{-1} E)$ for $\mu \in P$. Further, δ_c will denote the probability measure concentrated at the point c. Two measures μ and ν from P are said to be *similar* if $\mu = T_a \nu$ for a certain $a \in (0, \infty)$. A continuous commutative and associative P-valued binary operation $\circ$ on P is called a *generalized convolution* if it is distributive with respect to the convex combinations of measures and the operations T_a, δ_0 is its unit element and an analogue of the law of large numbers is fulfilled: $T_{c_n} \delta_1^{\circ n} \to \gamma \neq \delta_0$ for a choice of a norming sequence c_n of positive numbers. The power $\delta_1^{\circ n}$ is taken here in the sense of the operation $\circ$. The limit measure $\gamma = \gamma(\circ)$ is called a *characteristic measure* of the generalized convolution in question. It is clear that the characteristic measure is uniquely determined up to the similarity relation.

The set P with the operation $\circ$ and the operations of convex combinations is called a *generalized convolution algebra*. Generalized convolution algebras admitting a non-constant continuous homomorphism into the algebra of real numbers with the operations of multiplication and convex combinations are called *regular*. All generalized convolution algebras under consideration in the sequel will tacitly be assumed to be regular. For regular convolution algebras by Proposition 4.5 in [6] there exists a positive constant $\varkappa = \varkappa(\circ)$ such that

(1) $$T_a \gamma \circ T_b \gamma = T_{g(\varkappa, a, b)} \gamma$$

for any pair $a, b \in (0, \infty)$, where $g(\varkappa, a, b) = (a^\varkappa + b^\varkappa)^{1/\varkappa}$. The constant $\varkappa$ does not depend upon the choice of a characteristic measure and is called the *characteristic exponent* of $\circ$. Moreover, by Proposition 4.4 in [6], every solution γ of equation (1) for all $a, b \in (0, \infty)$ is a characteristic measure of $\circ$. Notice that, by Theorem 4.3 in [6], the pair $\varkappa(\circ), \gamma(\circ)$ determines the generalized convolution $\circ$.

We say that the generalized convolution $\circ$ admits a *characteristic function* if there exists a one-to-one correspondence $\mu \to \hat{\mu}$ between measures μ from P and real-valued bounded continuous functions $\hat{\mu}$ defined on the half-line $[0, \infty)$ commuting with convex combinations and scale changes, i.e. $(T_a\mu)\hat{}\,(t) = \hat{\mu}(at)$ for $a \in (0, \infty)$. Further, the key condition postulates $(\mu \circ \nu)\hat{} = \hat{\mu}\hat{\nu}$ and the convergence $\mu_n \to \mu$ is equivalent to the uniform convergence $\hat{\mu}_n \to \hat{\mu}$ on every compact subset of $[0, \infty)$. It has been proved in [4] (Theorem 6) that a generalized convolution admits a characteristic function if and only if it is regular. By Theorem 2.1 in [5] the characteristic function is unique up to a scale change and is represented by an integral transform

$$\hat{\mu}(t) = \int_0^\infty \Omega(tx)\mu(dx)$$

with a continuous kernel Ω fulfilling the conditions $|\Omega(t)| \leqslant 1$ for $t \in [0, \infty)$ and $\Omega(t) = 1 - t^{\varkappa}L(t)$, where $\varkappa$ is the characteristic exponent of $\circ$ and the function L is slowly varying at the origin.

Many examples of generalized convolutions are to be found in various branches of probability theory ([10], [11]). We shall quote some of them. It is clear that every generalized convolution $\circ$ is uniquely determined by the expressions $\delta_a \circ \delta_b$ with $a, b \in (0, \infty)$.

EXAMPLE 1. α-convolutions $*_\alpha$ $(\alpha > 0)$: $\delta_a *_\alpha \delta_b = \delta_{g(\alpha,a,b)}$. These convolutions correspond to the operations $(X^\alpha + Y^\alpha)^{1/\alpha}$ on independent random variables X and Y. For $\alpha = 1$ we get the ordinary convolution. For any $\alpha > 0$ we have $\varkappa(*_\alpha) = \alpha$ and $\gamma(*_\alpha) = \delta_1$.

EXAMPLE 2. Kingman convolutions $*_{\alpha,\beta}$ $(\alpha > 0, \beta > 1)$: $\delta_a *_{\alpha,\beta} \delta_b$ is the probability measure with the density function equal to

$$4^{-1}a^{-3}b^{-3}B(1/2, \beta/2)^{-1}[x^{\alpha-1}x^{2\alpha}(a^{2\alpha}+b^{2\alpha}) - (a^{2\alpha}-b^{2\alpha})^2 - x^{4\alpha}]^{(\beta-3)/2}$$

in the interval $|a^\alpha - b^\alpha|^{1/\alpha} \leqslant x \leqslant (a^\alpha + b^\alpha)^{1/\alpha}$ and vanishing otherwise, where B is the beta-function. These convolutions have been introduced by Kingman in [3] for the study of spherically symmetric random walk in Euclidean spaces. Here we have $\varkappa(*_{\alpha,\beta}) = 2\alpha$ and

$$\gamma(*_{\alpha,\beta})(dx) = \alpha 4^{1-\beta}\Gamma(\beta - 1/2)^{-1}x^{2\alpha\beta-\alpha-1}\exp(-x^{2\alpha}/4)dx. \tag{2}$$

EXAMPLE 3. Convolutions $\circ_{\alpha,n}$ $(\alpha > 0, n = 1, 2, \ldots)$: for $0 < a \leqslant b$,

$$\delta_a \circ_{\alpha,n} \delta_b(dx) = (1 - a^\alpha b^{-\alpha})\delta_b(dx) + \sum_{k=1}^{n} \alpha(n+1)\binom{n}{k}\binom{n}{k-1}$$
$$\times a^{\alpha(n+1-k)}b^{\alpha k}(x^\alpha - a^\alpha)^{k-1}(x^\alpha - b^\alpha)^{n-k}x^{-2\alpha n-1}1_{[b,\infty)}(x)(dx),$$

where $1_{[b,\infty)}$ denotes the indicator of the half-line $[b, \infty)$ ([5], Example 1.6). Here we have $\varkappa(\circ_{\alpha,n}) = \alpha$ and

$$(3) \qquad \gamma(\circ_{\alpha,n})(dx) = \alpha(n!)^{-1}x^{-1-\alpha(n+1)}\exp(-x^{-\alpha})dx.$$

The case $\alpha = n = 1$ is relevant to work [2] of D. G. Kendall on stationary random closed sets.

Given a number $q \in (0, \infty)$, for any $\mu \in P$ we put

$$m_q(\mu) = \int_0^\infty x^q \mu(dx).$$

Denote by P_q the subset of P consisting of all μ with $m_q(\mu) < \infty$. Further, denote by $Q_q(\circ)$ the subset of P_q consisting of all μ fulfilling the condition $\mu^{\circ n} \in P_q$ for $n = 1, 2, \ldots$ It is clear that both sets P_q and $Q_q(\circ)$ are invariant under the maps T_a $(a > 0)$ and $\delta_0 \in Q_q(\circ)$.

Two generalized convolutions $\circ_1$ and $\circ_2$ are said to be *q-equivalent*, in symbols $\circ_1 \underset{q}{\sim} \circ_2$, if $Q_q(\circ_1) = Q_q(\circ_2)$ and $m_q(\mu^{\circ_1 n}) = m_q(\mu^{\circ_2 n})$ for all $n = 1, 2, \ldots$ and $\mu \in Q_q(\circ_1)$. The aim of this paper is to study the q-equivalence of generalized convolutions. We begin with properties of the sets P_q and $Q_q(\circ)$.

LEMMA 1. *If $\mu \circ \nu \in P_q$, then $\mu \in P_q$.*

Proof. For $q \geqslant \varkappa(\circ)$ we have, by Theorem 1 in [8], the inequality $m_q(\mu \circ \nu) \geqslant m_q(\mu) + m_q(\nu)$, which yields the assertion of Lemma 1. Suppose that $q < \varkappa(\circ)$. Then, by formula (15) in [8], we have for $\lambda \in P$

$$(4) \qquad m_q(\lambda) = c_q \int_0^\infty (1-\hat{\lambda}(t))t^{-q-1}dt,$$

where c_q is a positive constant. Consequently, to prove the relation $\mu \in P_q$ it suffices to show that the integral $\int_0^\infty (1-\hat{\mu}(t))t^{-q-1}dt$ is finite. Since, by Lemma 4.3 in [6], $\hat{\mu}(0) = 1$, we can find a positive number t_0 such that $\hat{\mu}(t) > 0$ for $t \in [0, t_0]$. Moreover, by Lemma 4.4 in [6], $|\hat{\mu}(t)| \leqslant 1$ for $t \in [0, \infty)$, which implies the inequalities

$$(5) \qquad \int_{t_0}^\infty (1-\hat{\mu}(t))t^{-q-1}t < \infty$$

and

$$1-(\mu \circ \nu)\hat{}\,(t) = 1-\hat{\mu}(t)+\hat{\mu}(t)(1-\hat{\nu}(t)) \geqslant 1-\hat{\mu}(t)$$

for $t \in [0, t_0]$. Hence and from (4) we get the inequality

$$\int_0^{t_0} (1-\hat{\mu}(t))t^{-q-1}\,dt \leqslant c_q^{-1} m_q(\mu \circ \nu),$$

which together with (5) completes the proof.

As a consequence of equation (1) we get the following statement:

PROPOSITION 1. *$\gamma(\circ)\in Q_q(\circ)$ if and only if $\gamma(\circ)\in P_q$.*

PROPOSITION 2. *If either $q<\varkappa(\circ)$ or $q>\varkappa(\circ)$ and $Q_q(\circ)\neq\{\delta_0\}$, then $\gamma(\circ)\in Q_q(\circ)$.*

Proof. It has been proved in [1] (Lemma) that $\gamma(\circ)\in P_q$ for $q<\varkappa(\circ)$. Consequently, by Proposition 1, $\gamma(\circ)\in Q_q(\circ)$. In the case $q>\varkappa(\circ)$ and $Q_q(\circ)\neq\{\delta_0\}$ we have, by Theorem 2 in [8], $\gamma(\circ)\in P_q$ which, by Proposition 1, yields the assertion of the proposition.

By Corollary 1 in [8] the set P_q is closed under the convolution $\circ$ for $q\leqslant\varkappa(\circ)$. This yields the following proposition:

PROPOSITION 3. *If $q\leqslant\varkappa(\circ)$, then $Q_q(\circ)=P_q$.*

PROPOSITION 4. *If $(k-1)\varkappa(\circ)<q\leqslant k\varkappa(\circ)$ for a certain $k=2,3,\ldots$ and $Q_q(\circ)\neq\{\delta_0\}$, then $Q_q(\circ)=\{\mu\colon m_q(\mu^{\circ(k-1)})<\infty\}$.*

Proof. First consider the case $k=2$. Then, by Proposition 2, $\gamma(\circ)\in P_q$, which, by Theorem 3 in [8], shows that the set P_q is closed under the convolution $\circ$. This yields the equality $Q_q(\circ)=P_q$.

Now suppose that $k\geqslant 3$. The inclusion $Q_q(\circ)\subset\{\mu\colon m_q(\mu^{\circ(k-1)})<\infty\}$ is evident. In order to prove the converse inclusion we assume that $\mu^{\circ(k-1)}\in P_q$. Hence in particular it follows that $\mu^{\circ(k-1)}\in P_r$, where $r=(k-1)\varkappa(\circ)$. Applying Theorem 4 from [8] we conclude that $\mu^{\circ k}\in P_r$ and, consequently, by Corollary 6 in [8], $\mu^{\circ k}\in P_q$. Applying Theorem 4 from [8] again we get the relation $\mu^{\circ n}\in P_q$ for $n=1,2,\ldots$ Thus $\mu\in Q_q(\circ)$, which completes the proof.

THEOREM 1. *If $\varkappa(\circ_1)=\varkappa(\circ_2)=q$, then $\circ_1\underset{q}{\sim}\circ_2$.*

Proof. Observe that, by Proposition 3, $Q_q(\circ_1)=Q_q(\circ_2)=P_q$ and, by Theorem 1 in [8], $m_q(\mu\circ_j\nu)=m_q(\mu)+m_q(\nu)$ for $j=1,2$, which yields the assertion of the theorem.

THEOREM 2. *If $q>\varkappa(\circ_j)$ and $\gamma(\circ_j)\notin P_q$ for $j=1,2$, then $\circ_1\underset{q}{\sim}\circ_2$.*

Proof. By Proposition 2 we have the equality $Q_q(\circ_1)=Q_q(\circ_2)=\{\delta_0\}$, which yields the assertion of the theorem.

EXAMPLE 4. From (3) we get the formula $m_q(\circ_{\alpha,n})=\infty$ if $q\geqslant\alpha(n+1)$. Since $\varkappa(\circ_{\alpha,n})=\alpha$, the above theorem yields the relation $\circ_{\alpha,n}\underset{q}{\sim}\circ_{\beta,m}$ whenever $q\geqslant\max\big(\alpha(n+1),\beta(m+1)\big)$.

THEOREM 3. *If $q=2\varkappa(\circ_1)=2\varkappa(\circ_2)$, $\gamma(\circ_1)$, $\gamma(\circ_2)\in P_q$ and*

$$m_q\big(\gamma(\circ_1)\big)m_{q/2}^{-2}\big(\gamma(\circ_1)\big)=m_q\big(\gamma(\circ_2)\big)m_{q/2}^{-2}\big(\gamma(\circ_2)\big),\tag{6}$$

then $\circ_1\underset{q}{\sim}\circ_2$.

Proof. As an immediate consequence of Propositions 1, 3 and 4 we get the equality $Q_q(\circ_1) = Q_q(\circ_2) = P_q$. Denoting by a_q the expression (6) we have, by Lemma 2 and Theorem 1 in [8], the formulae

$$m_q(\mu \circ_j \nu) = m_q(\mu) + m_q(\nu) + a_q m_{q/2}(\mu) m_{q/2}(\nu)$$

and

$$m_{q/2}(\mu \circ_j \nu) = m_{q/2}(\mu) + m_{q/2}(\nu)$$

for $j = 1, 2$, which yield the recurrence formula

$$m_q(\mu^{\circ_j n}) = m_q(\mu^{\circ_j (n-1)}) + m_q(\mu) + m_q(\mu) + a_q(n-1) m_{q/2}^2(\mu)$$

for $j = 1, 2$, $n = 1, 2, \ldots$ and $\mu \in P_q$. Using the above formula we obtain the equality $m_q(\mu^{\circ_1 n}) = m_q(\mu^{\circ_2 n})$ for all $n = 1, 2, \ldots$, which completes the proof.

EXAMPLE 5. From Examples 2 and 3 we get the formula $\varkappa(*_{\alpha,n-1/2}) = \varkappa(\circ_{2\alpha,n}) = 2\alpha$. Setting $q = 4\alpha$ and $n \geqslant 2$ we get from (2) and (3), by a standard calculation,

$$m_q(*_{\alpha,n-1/2}) = 16n(n-1), \qquad m_{q/2}(*_{\alpha,n-1/2}) = 4(n-1),$$
$$m_q(\circ_{2\alpha,n}) = 1/(n^2-n), \qquad m_{q/2}(\circ_{2\alpha,n}) = 1/n.$$

It is easy to show that condition (6) is fulfilled. Consequently, by Theorem 3 we have the relation $*_{\alpha,n-1/2} \underset{q}{\sim} \circ_{2\alpha,n}$ for $\alpha > 0$ and $n \geqslant 2$.

THEOREM 4. *If* $\circ_1 \underset{q}{\sim} \circ_1$ *and* $\gamma(\circ_1) \in P_q$, *then* $\varkappa(\circ_1) = \varkappa(\circ_2)$.

Proof. Setting, for simplicity of the notation, $\gamma = \gamma(\circ_1)$ and $r = \varkappa(\circ_1)$ we have, by Proposition 1, $\gamma \in Q_q(\circ_1)$ and, by (1),

$$m_q(\gamma^{\circ_1 n}) = n^{q/r} m_q(\gamma) \quad (n = 1, 2, \ldots).$$

Consequently,

$$m_q(\gamma^{\circ_2 n}) = n^{q/r} m_q(\gamma) \quad (n = 1, 2, \ldots). \tag{7}$$

Further, denoting by $m^*(\mu)$ the greatest median of μ we have the inequality

$$m_q(\mu) \geqslant \int_{m^*(\mu)}^{\infty} x^q \mu(dx) \geqslant 2^{-1}(m^*(\mu))^q,$$

which, by (7), yields $n^{-1/r} m^*(\gamma^{\circ_2 n}) \leqslant 2^{1/q}(m_q(\gamma))^{1/q}$ for all $n = 1, 2, \ldots$ Applying the theorem from [7] on limit behaviour of medians we get the inequality

$$\varkappa(\circ_1) = r \leqslant \varkappa(\circ_2). \tag{8}$$

Since $\gamma \in Q_q(\circ_2)$, we conclude, by Proposition 2, that $\gamma(\circ_2) \in P_q$ for $q \neq \varkappa(\circ_2)$. Consequently, by the first part of the proof, replacing $\circ_1$ by $\circ_2$ we have the inequality $\varkappa(\circ_2) \leqslant \varkappa(\circ_1)$ for $q \neq \varkappa(\circ_2)$, which together with (8) yields the assertion of the theorem in the case $q \neq \varkappa(\circ_2)$. In the remaining case $q = \varkappa(\circ_2)$

we have, by Theorem 1 in [8], $m_q(\gamma^{\circ_2 n}) = nm_q(\gamma)$ for $n = 1, 2, \ldots$, which, by (7), implies the formula $q = r = \varkappa(\circ_1)$. The theorem is thus proved.

LEMMA 2. *If* $\mu_1 \circ_1 \mu_2 \circ_1 \ldots \circ_1 \mu_k \in Q_q(\circ_1)$ *and* $\circ_1 \underset{q}{\sim} \circ_2$, *then*

$$m_q(\mu_1 \circ_1 \mu_2 \circ_1 \ldots \circ_1 \mu_k) = m_q(\mu_1 \circ_2 \mu_2 \circ_2 \ldots \circ_2 \mu_k).$$

Proof. By the assumption we have the relation

$$\mu_1^{\circ_1 n} \circ_1 \mu_2^{\circ_1 n} \circ_1 \ldots \circ_1 \mu_k^{\circ_1 n} \in P_q$$

for every $n = 1, 2, \ldots$ Consequently, by Lemma 1,

$$\mu_1^{\circ_1 r_1} \circ_1 \mu_2^{\circ_1 r_2} \circ_1 \ldots \circ_1 \mu_k^{\circ_1 r_k} \in P_q \tag{9}$$

for every k-tuple $r_1, r_2, \ldots, r_k$ of non-negative integers. Given an arbitrary k-tuple $a_1, a_2, \ldots, a_k$ of non-negative real numbers fulfilling the condition $\sum_{s=1}^{k} a_s = 1$ we put $\lambda = \sum_{s=1}^{k} a_s \mu_s$. Since

$$\lambda^{\circ_j n} = \sum_{r_1+r_2+\ldots+r_k=n} n!(r_1!r_2!\ldots r_k!)^{-1} a_1^{r_1} a_2^{r_2} \ldots \tag{10}$$

$$\ldots a_k^{r_k} \mu_1^{\circ_j r_1} \circ_j \mu_2^{\circ_j r_2} \circ_j \ldots \circ_j \mu_k^{\circ_j r_k}$$

for $j = 1, 2$ and $n = 1, 2, \ldots$, we conclude, by (9), that $\lambda^{\circ_1 n} \in P_q$ for every $n = 1, 2, \ldots$ or, equivalently, $\lambda \in Q_q(\circ_1)$. Thus we have the equality $m_q(\lambda^{\circ_1 k}) = m_q(\lambda^{\circ_2 k})$, which, by the arbitrariness of $a_1, a_2, \ldots, a_k$ and formula (10), yields

$$m_q(\mu_1^{\circ_1 r_1} \circ_1 \mu_2^{\circ_1 r_2} \circ_1 \ldots \circ_1 \mu_k^{\circ_1 r_k}) = m_q(\mu_1^{\circ_2 r_1} \circ_2 \mu_2^{\circ_2 r_2} \circ_2 \ldots \circ_2 \mu_k^{\circ_2 r_k})$$

for any k-tuple $r_1, r_2, \ldots, r_k$ of non-negative integers fulfilling the condition $r_1 + r_2 + \ldots + r_k = k$. Taking $r_1 = r_2 = \ldots = r_k = 1$ we get the assertion of the theorem.

For $\mu_1, \mu_2, \ldots, \mu_k \in P$ with $\mu_1 \circ \mu_2 \circ \ldots \circ \mu_k \in P_q$ we introduce the notation

$$M_{q,k}(\circ, \mu_1, \mu_2, \ldots, \mu_k) = \sum_{r=1}^{k} (-1)^r \sum_{i_1, i_2, \ldots, i_r} m_q(\mu_{i_1} \circ \mu_{i_2} \circ \ldots \circ \mu_{i_r}),$$

where the summation $\sum_{i_1, i_2, \ldots, i_r}$ runs over all r-element subsets $\{i_1, i_2, \ldots, i_r\}$ of the set of indices $\{1, 2, \ldots, k\}$.

As a simple consequence of Lemma 2 we get the following statement:

LEMMA 3. *If* $\nu_1 \circ_1 \nu_2 \circ_1 \ldots \circ_1 \nu_s \circ_1 \mu_2 \circ_1 \ldots \circ_1 \mu_k \in Q_q(\circ_1)$ *and* $\circ_1 \underset{q}{\sim} \circ_2$, *then*

$$M_{q,k}(\circ_1, \nu_1 \circ_1 \ldots \circ_1 \nu_s, \mu_2, \ldots, \mu_k) = M_{q,k}(\circ_2, \nu_1 \circ_2 \ldots \circ_2 \nu_s, \mu_2, \ldots, \mu_k).$$

Now we are in a position to prove a rather unexpected result:

THEOREM 5. *If* $q \neq n\varkappa(\circ_1)$ *for* $n = 1, 2, \ldots, \gamma(\circ_1) \in P_q$ *and* $\circ_1 \underset{q}{\sim} \circ_2$, *then* $\circ_1 = \circ_2$.

Proof. Notice that, by Theorem 4, $\varkappa(\circ_1) = \varkappa(\circ_2) = \varkappa$. For simplicity of the notation we put $\gamma = \gamma(\circ_1)$. Further, denote by k the positive integer fulfilling the condition $(k-1)\varkappa < k\varkappa$. Given $a, b \in (0, \infty)$ we put $c = g(\varkappa, a, b)$ and $\lambda_2 = \lambda_3 = \ldots = \lambda_k = \gamma$. By formula (1) we have

$$T_a\gamma \circ_1 T_a\gamma \circ_1 T_b\gamma \circ_1 T_b\gamma = T_a\gamma \circ_1 T_b\gamma \circ_1 T_c\gamma = T_c\gamma \circ_1 T_c\gamma.$$

Since, by (1) and Proposition 1, $T_{a_1}\gamma \circ_1 \ldots \circ_1 T_{a_s}\gamma \circ_1 \lambda_2 \circ_1 \ldots \circ_1 \lambda_k \in Q_q(\circ_1)$ for any $a_1, \ldots, a_s \in (0, \infty)$, we conclude, by Lemma 3, that

$$(11) \quad M_{q,k}(\circ_2, T_c\gamma \circ_2 T_c\gamma, \lambda_2, \ldots, \lambda_k) - 2M_{q,k}(\circ_2, T_a\gamma \circ_2 T_b\gamma \circ_2 T_c\gamma, \lambda_2, \ldots, \lambda_k)$$
$$+ M_{q,k}(\circ_2, T_a\gamma \circ_2 T_a\gamma \circ_2 T_b\gamma \circ_2 T_b\gamma, \lambda_2, \ldots, \lambda_k)$$
$$= M_{q,k}(\circ_1, T_c\gamma \circ_1 T_c\gamma, \lambda_2, \ldots, \lambda_k) - 2M_{q,k}(\circ_1, T_a\gamma \circ_1 T_b\gamma \circ_1 T_c\gamma, \lambda_2, \ldots, \lambda_k)$$
$$+ M_{q,k}(\circ_1, T_a\gamma \circ_1 T_a\gamma \circ_1 T_b\gamma \circ_1 T_b\lambda, \lambda_2, \ldots, \lambda_k) = 0.$$

Let $\mu \to \hat{\mu}$ be the characteristic function of the convolution $\circ_2$. Applying Lemma 2 and formulae (15) and (17) from [8] we have

$$M_{q,k}(\circ_2, \mu_1, \mu_2, \ldots, \mu_k) = \varkappa\Gamma(-q/\varkappa)^{-1} m_q(\gamma(\circ_2)) \int_0^\infty t^{-q-1} \prod_{j=1}^{k} (1-\hat{\mu}_j(t))dt$$

whenever $\mu_1 \circ_2 \mu_2 \circ_2 \ldots \circ_2 \mu_k \in P_q$. Comparing the above formula with (11) we infer that

$$\int_0^\infty (\hat{\gamma}(ct) - \hat{\gamma}(at)\hat{\gamma}(bt))^2 (1-\hat{\gamma}(t))^{k-1} t^{-q-1} dt = 0.$$

Since, by Lemma 4.4 in [6], $|\hat{\gamma}(t)| \leqslant 1$ and, by Lemma 2.1 in [9], $\hat{\gamma}(t) \neq 1$ for almost every $t \in [0, \infty)$, the integrand is non-negative almost everywhere. This implies the equality $\hat{\gamma}(at)\hat{\gamma}(bt) = \hat{\gamma}(ct)$ for almost every $t \in [0, \infty)$. By the continuity of the characteristic function the above equality holds for all $t \in [0, \infty)$. Consequently, $T_a\gamma \circ_2 T_b\gamma = T_c\gamma$, which together with the equality $\varkappa(\circ_1) = \varkappa(\circ_2) = \varkappa$ shows that the probability measures γ and $\gamma(\circ_2)$ are similar. Now applying Theorem 4.3 from [6] we conclude that $\circ_1 = \circ_2$. The theorem is thus proved.

Notice that, by Theorem 1 and Examples 4 and 5, the assumptions $q \neq \varkappa(\circ_1)$, $q \neq 2\varkappa(\circ_1)$ and $\gamma(\circ_1) \in P_q$ of the above theorem are essential. The problem whether the assumption $q \neq n\varkappa(\circ_1)$ for $n \geqslant 3$ may be omitted is still open. For α-convolutions the following theorem gives an answer to this question:

THEOREM 6. *If* $q \neq \alpha$ *and* $\circ \underset{q}{\sim} *_\alpha$, *then* $\circ = *_\alpha$.

Proof. Since $\varkappa(*_\alpha) = \alpha$, $\gamma(*_\alpha) = \delta_1$ and, consequently, $\gamma(*_\alpha) \in P_q$, it suffices, by Theorem 5, to consider the case $q = k\alpha$ for integers $k \geqslant 2$. It is clear that

$$(12) \qquad Q_{k\alpha}(\circ) = Q_{k\alpha}(*_\alpha) = P_{k\alpha},$$

$$\delta_1^{*_\alpha n} = T_{n^{1/\alpha}}\delta_1 \ (n = 1, 2, \ldots) \quad \text{and} \quad M_{k\alpha,k}(*_\alpha, \delta_1, \delta_1, \ldots, \delta_1) = (-1)^k k!.$$

Applying Lemma 3 for $s = 1$, $\nu_1 = \mu_2 = \ldots = \mu_k = \delta_1$ we get the formula

$$(13) \qquad M_{k,\alpha,k}(\circ, \delta_1, \delta_1, \ldots, \delta_1) = (-1)^k k!.$$

Further, by Theorem 4, $\varkappa(\circ) = \alpha$ and, by Proposition 2, $\gamma(\circ) \in Q_{k\alpha}(\circ)$, which, by (12) and Lemma 2 in [8], yields the formula

$$(-1)^k k!\, m_\alpha(\gamma(\circ))^{-k} m_{k\alpha}(\gamma(\circ)) = M_{k\alpha,k}(\circ, \delta_1, \delta_1, \ldots, \delta_1).$$

Hence and from (13) we get the equality

$$(14) \qquad m_{k\alpha}(\gamma(\circ)) = m_\alpha^k(\gamma(\circ)).$$

Taking into account the assumption $k \geqslant 2$ we have the inequalities

$$m_{k\alpha}(\gamma(\circ))^{1/(k\alpha)} \geqslant m_{2\alpha}(\gamma(\circ))^{1/(2\alpha)} \geqslant m_\alpha(\gamma(\circ))^{1/\alpha},$$

which together with (14) yield $m_{2\alpha}(\gamma(\circ)) = m_\alpha^2(\gamma(\circ))$. Thus

$$\int_0^\infty (x^\alpha - m_\alpha(\gamma(\circ)))^2 \gamma(\circ)(dx) = 0,$$

which shows that the characteristic measure $\gamma(\circ)$ is concentrated at the point $m_\alpha(\gamma(\circ))^{1/\alpha}$. Since $\gamma(\circ) \neq \delta_0$, we conclude that the characteristic measures $\gamma(\circ)$ and $\gamma(*_\alpha)$ are similar. Applying Theorem 4.3 from [6] we get the assertion of the theorem.

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