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TEST FOR ASSOCIATION OF RANDOM VARIABLES IN THE DOMAIN OF ATTRACTION OF MULTIVARIATE STABLE LAW

BY

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Abstract. The problem of estimating the index of stability and the spectral measure of multivariate stable distribution is related to that of evaluating the risk of stable portfolio of financial assets. We show how to solve this problem assuming that the observations are taken from the domain of attraction of a multivariate stable law. Our main results concern tests for association, and estimates of the risk and the covariation of a stable portfolio.

1. Introduction. The problem of estimating multivariate stable distributions has received increasing attention in recent years in modelling portfolio of financial assets (see [20, Chapter 12], [18] and the references therein). Stable laws (with index $\alpha < 2$) for modelling stock returns were proposed in the seminal works of Mandelbrot [15] and Fama [11] (see also [8] and [1]). While the problem of estimating the parameters of the univariate α-stable law seems to have a complete solution (see [8] and [16]), very little is known at present about the statistical procedures for analyzing multivariate stable random samples. Our work was inspired by Press [20], [21] who defined the risk and estimated the parameters of portfolios constituted by assets whose prices follow a stable law with characteristic function (ch.f.) having the parametric form

$$\mathrm{E}\exp(X, t) = \exp\{i(t, \overset{0}{\mu}) - \tfrac{1}{2}\sum_{i=1}^{m} (t^T \Sigma_i t)^{\alpha/2}\}, \quad 1 < \alpha < 2.$$

In this paper, prices are assumed to have a distribution from the domain of attraction of a stable law with a general spectral measure, and so we need not relay on a specific parametric model. The motivation for looking at the domain of attraction rather than the stable law itself comes not only from the reason of the natural generality and more robust modelling. In a recent study of various

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stock returns (cf. [18]) via stable models (among them α-stable, max-, min-stable, random multiplication and their randomized versions), it was shown that the best fit is provided by α-stable ($1 < \alpha < 2$) and geometrical α-stable distributions — both having one and *the same domain of attraction.* Recall that the d-dimensional random vector X has *α-stable distribution* ($0 < \alpha < 2$) if its ch.f. has the form

$$\text{(1.1)} \quad \mathbf{E}\exp i(X, t) = \exp\Big\{-\int_{S_d} |(s, t)|^{\alpha}\big(1 - i\,\mathrm{sign}\,((s, t))\varphi(\alpha; s, t)\big)\Gamma(ds) + i(\overset{0}{\mu}, t)\Big\}, \quad t \in \boldsymbol{R}^d,$$

where S_d is the unit sphere in $\boldsymbol{R}^d$, Γ is a finite Borel measure on S_d, $\overset{0}{\mu} \in \boldsymbol{R}^d$, and

$$\text{(1.2)} \qquad \varphi(\alpha; s, t) = \begin{cases} \tan(\pi\alpha/2) & \text{if } \alpha \neq 1, \\ -(2/\pi)\ln|(s, t)| & \text{if } \alpha = 1. \end{cases}$$

We refer to [30] and [27] for more information on multivariate stable laws. A vector Y is *geometric stable* if the stability property

$$\text{(1.3)} \quad a(p)\sum_{i=1}^{N(p)} (Y_i - b(p)) \xrightarrow{\mathrm{d}} Y \text{ as } p \to 0, \quad a\colon (0, 1) \to [0, \infty), \quad b\colon (0, 1) \to \boldsymbol{R}^d,$$

is preserved up to a geometric random variable, $P(N(p) = k) = p(1-p)^{k-1}$, $k \geqslant 1$, representing the moment of an extreme change in the fundamentals of the portfolio. We can rewrite (1.3) in the form: there exist $a_n > 0$ and $b_n \in \boldsymbol{R}^n$ such that

$$\text{(1.4)} \qquad \frac{1}{a_n}\sum_{i=1}^{n} Y_i - b_n \xrightarrow{\mathrm{d}} X \quad \text{as } n \to \infty,$$

and the one-to-one correspondence between the laws of X and Y is given by

$$\text{(1.5)} \qquad \mathbf{E}\exp i(Y, t) = \frac{1}{1 - \log \mathbf{E}\exp i(X, t)}.$$

In (1.3) and (1.4), Y_i's are independent identically distributed (i.i.d.) random vectors viewed as observations on the vector of per share returns on all assets in the investment portfolio. The assumption of Y_i being i.i.d. is not too restrictive: under some regularity conditions, using the CLT for convergence of martingales to a stable limit (cf. [14] and [23]), the asymptotic results will be preserved even if Y_i's are martingale differences.

The assumption that the vector of returns is in the domain of attraction of a multivariate stable (or geometric stable) law gives rise to the following questions: 1. How should one measure the dependence between the individual returns in the portfolio? 2. What is the value of portfolio risk?

The notion of association seems to be a very natural measure of dependence especially when one deals with heavy tailed multivariate distributions (see [10] and [13]). Recall that $X = (X_1, \ldots, X_d)$ has *positively* *associated components* if for any functions f, g: $\boldsymbol{R}^d \to \boldsymbol{R}$, nondecreasing in each argument, $\operatorname{cov}(f(X), g(X)) \geqslant 0$ whenever the covariance exists. Similarly, X has *negatively associated components* (cf. [2]) if for any $1 \leqslant k < d$, any f: $\boldsymbol{R}^k \to \boldsymbol{R}$, g: $\boldsymbol{R}^{d-k} \to \boldsymbol{R}$, nondecreasing in each argument, $\operatorname{cov}(f(X'), g(X'')) \leqslant 0$ whenever the covariance exists, where X' and X'' are any k- and $(d-k)$-dimensional random vectors, respectively, representing a partition of the X-components into two subsets of sizes k and $d-k$ accordingly. Pitt [19] showed that jointly normal random variables are associated if and only if their covariances are all nonnegative; the corresponding result for negative associated normals was proved by Joag-dev and Proschan [12]. If X is α-stable distributed ($0 < \alpha < 2$), then X is associated if and only if the spectral measure Γ of X (cf. (1.1)) satisfies the condition

$$\Gamma(S_d^-) = 0, \tag{1.6}$$

where $S_d^- = \{(s_1, \ldots, s_d) \in S_d$: for some $i, j \in \{1, \ldots, d\}$, $s_i > 0$ and $s_j < 0\}$; and X is negatively associated if and only if

$$\Gamma(S_d^+) = 0, \tag{1.7}$$

where $S_d^+ = \{(s_1, \ldots, s_d) \in S_d$: for some $i \neq j$, $s_i s_j > 0\}$ (see [13]). Extending this result for X in the domain of attraction of α-stable random vector leads to *test for positive and negative association* (see Section 2, Theorem 1, the central result of our paper). The method we use resembles that developed by Einmahl et al. [9] for estimating a multivariate extreme-value distribution. Recall that X is assumed only to be in the domain of α-stable vectors, and therefore if in modelling portfolio returns X is chosen to be α-stable ($0 < \alpha < 2$) or geometric stable (see (1.1) and (1.5)), then one can perform the test for association we are proposing.

In Section 3 we generalize Press' results (cf. [20] and [21]) by estimating *the risk of a stable portfolio, the parameters* and *the spectral measure* Γ in (1.1), assuming only that the observations are in the domain of attraction of α-stable vectors. Moreover, our estimators are strongly consistent and, under some regularity conditions, asymptotically normal. For completeness of the exposition we collect some technical results in the Appendix.

2. Test for association of stable random variables. Let X be α-stable random vectors with ch.f. (1.1). We consider the bivariate case for the sake of brevity. Write then the ch.f. of $X = (X_1, X_2)$ in the form

$$\begin{aligned} &\mathrm{E}\exp i(X, t) \\ &= \exp\Big\{-|t|^\alpha \int_0^{2\pi} |\cos(t, \theta)^\wedge|^\alpha \big(1 - i\operatorname{sign}(\cos(t, \theta)^\wedge)\tilde{\varphi}(\alpha, \theta, t)\big) d\Phi(\theta) + i(t, \overset{0}{\mu})\Big\}, \end{aligned} \tag{2.1}$$

where

$$\tilde{\varphi}(\alpha, \theta, t) = \begin{cases} \tan(\pi\alpha/2) & \text{for } \alpha \neq 1, \\ -(2/\pi)\ln(\varrho|\cos(t, \theta)^{\wedge}|) & \text{for } \alpha = 1; \end{cases}$$

for $t = (\varrho\cos\varphi, \varrho\sin\varphi)$,

$$\cos(t, \theta)^{\wedge} := \cos\varphi\cos\theta + \sin\varphi\sin\theta;$$

and, finally, Φ is a distribution function (d.f.) on $[0, 2\pi]$ with total mass $\Gamma(S_2)$. Let now $\boldsymbol{Z}, \boldsymbol{Z}_1, \boldsymbol{Z}_2, \ldots$ be i.i.d. random pairs with unknown distribution F from the domain of attraction of an α-stable law G with ch.f. (2.1), that is, for some $a_n > 0$ and $b_n \in \boldsymbol{R}^2$,

$$\frac{1}{a_n}\sum_{i=1}^{n} \boldsymbol{Z}_i - b_n \xrightarrow{d} \boldsymbol{X}. \tag{2.2}$$

We rewrite (2.2) using the polar coordinates of $\boldsymbol{Z}$ (denoted here by $\varrho = |\boldsymbol{Z}|$ and $\Theta = \theta(\boldsymbol{Z})$):

$$\lim_{x\to\infty} \frac{P(\varrho > rx, \Theta < \theta_1)}{P(\varrho > x, \Theta < \theta_2)} = r^{-\alpha}\frac{\Phi(\theta_1)}{\Phi(\theta_2)}, \tag{2.3}$$

and for $\alpha \in (0, 2)$ the latter is equivalent to

$$\lim_{n\to\infty} nP(\varrho > ra_n, \Theta \leqslant \theta) = r^{-\alpha}\Phi(\theta) \tag{2.4}$$

(see [26], [6], and [25]).

Let $k = k_n$ be a sequence of integers satisfying $1 \leqslant k \leqslant n/2$, $n \in \boldsymbol{N}$, and $k \to \infty$, $k/n \to 0$ as $n \to \infty$. We assume that ϱ has a continuous ch.f. and let (ϱ_i, θ_i) be the polar coordinates of $\boldsymbol{Z}_i$. Our test on association of the $\boldsymbol{Z}$-components is based upon the asymptotic properties of the following estimates:

a. *estimator for the index of stability* α:

$$\alpha_n = \alpha_{n,k} = \log 2/(\log\varrho_{n-k+1:n} - \log\varrho_{n-2k+1:n}), \tag{2.5}$$

where $\varrho_{k:n}$ is the k-th order statistic from $(\varrho_1, \ldots, \varrho_n)$;

b. *estimator for the normalized spectral measure* $\varphi(\theta) = \Phi(\theta)/\Phi(2\pi)$:

$$\varphi_n(\theta) = \frac{1}{k}\sum_{i=1}^{n} I(\Theta_i \leqslant \theta, \varrho_i \geqslant \varrho_{n-k+1:n}). \tag{2.6}$$

The proof of the main theorem (the test of association) is based on four lemmas stated below — for their proofs see the Appendix.

LEMMA 1. (A) *If $k/\log\log n \to \infty$ as $n \to \infty$, then $\alpha_n \to \alpha$ a.s.*

(B) *If $k/\log n \to \infty$ as $n \to \infty$, then $\varphi_n(\theta) \to \varphi(\theta)$ a.s. for all points $\theta \in [0, 2\pi]$ of Φ-continuity.* ∎

The next three lemmas concern the asymptotic normality of α_n and φ_n. Let us put

$$S(x) = P(\varrho \geqslant x), \quad R = S(\varrho), \quad R_i = S(\varrho_i),$$

$$F_n(\theta, r) = \frac{n}{k} P\left(\Theta \leqslant \theta,\, R \leqslant \frac{k}{n} r\right), \quad 0 \leqslant \theta \leqslant 2\pi,\ r > 0,$$

(2.7) $$\mathbf{F}_n(\theta, r) = \frac{1}{k} \sum_{i=1}^{n} \mathbf{1}\left(\Theta_i \leqslant \theta,\, R_i \leqslant \frac{k}{n} r\right),$$

(2.8) $$W_{1n}(\theta) = \sqrt{k}\,(\mathbf{F}_n(\theta, 1) - F_n(\theta, 1)),$$

(2.9) $$W_{2n}(r) = \sqrt{k}\left(\frac{1}{k} \sum_{i=1}^{n} \mathbf{1}\left(R_i \leqslant \frac{k}{n} r\right) - r\right).$$

LEMMA 2. *If* (2.4) *holds at* $r = 1$ *uniformly for* $\theta \in [0, 2\pi]$, *then* $(W_{1n}(\theta), W_{2n}(r))$ *converges weakly in* $D[0, 2\pi] \times D[0, 3]$ *to* $(W_1(\theta), W_2(r))$, *which is a mean zero Gaussian process with covariance structure*

$$\mathbf{E}\, W_1(\theta_1) W_1(\theta_2) = \varphi(\theta_1 \wedge \theta_2), \quad \mathbf{E}\, W_2(r_1) W_2(r_2) = r_1 \wedge r_2,$$
$$\mathbf{E}\, W_1(\theta) W_2(r) = (r \wedge 1)\varphi(\theta). \ ■$$

The domain of attraction condition (2.4) gives for $\theta = 2\pi$

(2.10) $$\lim_{n\to\infty} a_n^{-1} S^{\leftarrow}(t/n) = t^{-1/\alpha} \Phi^{1/\alpha}(2\pi),$$

where $S^{\leftarrow}$ is the right continuous inverse of S. The latter implies, for all $x > 0$,

(2.11) $$\lim_{u\downarrow 0} \left(\log S^{\leftarrow}(ux) - \log S^{\leftarrow}(u)\right) = -\alpha^{-1} \log x$$

and therefore, for some positive function $b(t)$, $t > 0$, with $b(0+) = 0$, uniformly on $x \in [\frac{1}{2}, 2\frac{1}{2}]$,

(2.12) $$\log S^{\leftarrow}(ux) - \log S^{\leftarrow}(u) = -\alpha^{-1} \log x + O\big(b(u)\big) \quad \text{as } u \downarrow 0.$$

LEMMA 3. *Suppose the following strengthening of* (2.12) *holds*:

(2.13) $$\lim_{n\to\infty} \sqrt{k}\, b(k/n) = 0.$$

Then

$$\sqrt{k}\,(\alpha_n - \alpha) \xrightarrow{w} \frac{\alpha}{\log 2}\left(W_2(1) - \tfrac{1}{2} W_2(2)\right).$$

Now we turn to the asymptotic normality of $\varphi(\theta)$. Combining (2.4) and (2.10) we can renormalize ϱ in (2.4) to get

(2.14) $$\lim_{n\to\infty} nP\big(\varrho > rS^{\leftarrow}(1/n),\, \Theta \leqslant \theta\big) = \big(r\,\Phi^{1/\alpha}(2\pi)\big)^{-\alpha} \Phi(\theta) = \varphi(\theta) r^{-\alpha},$$

or, uniforming the radial component, we write (with $R = S(\varrho)$)

$$\lim_{n\to\infty} nP(R \leqslant r/n,\ \Theta \leqslant \theta) = r\varphi(\theta). \tag{2.15}$$ ∎

LEMMA 4. *Suppose the following strengthening of* (2.15) *holds: for some* $\delta \in (0, 1)$,

$$\lim_{n\to\infty} \sqrt{k} \sup_{\substack{0\leqslant\theta\leqslant 2\pi \\ 1-\delta\leqslant r\leqslant 1+\delta}} \left|\frac{n}{k} P\left(R \leqslant \frac{k}{n} r,\ \Theta \leqslant \theta\right) - r\varphi(\theta)\right| = 0. \tag{2.16}$$

Then the normalized process φ_n *converges weakly in* $D[0, 2\pi]$ *to a Gaussian process*:

$$\sqrt{k}\,(\varphi_n(\theta) - \varphi(\theta)) \xrightarrow{w} \Lambda(\theta), \tag{2.17}$$

where $\Lambda(\theta) := W_1(\theta) - \varphi(\theta) W_2(1)$, *and* W_i*'s are defined as in Lemma* 2.

Remark 1. $W_2(r)$ and $\Lambda(\theta)$ are not correlated (cf. Lemma 2) and, therefore, the estimators α_n and $\varphi_n(\theta)$ are asymptotically independent. Moreover, again by Lemma 2,

$$\mathbf{E}\,\Lambda(\theta_1)\Lambda(\theta_2) = \varphi(\theta_1 \wedge \theta_2) - \varphi(\theta_1)\varphi(\theta_2). \tag{2.18}$$ ∎

THEOREM 1 (*A test for association*). *Suppose* $\boldsymbol{Z}$ *is in the domain of attraction of* α*-stable random pair with spectral d.f.* $\varphi(\theta) = \Phi(\theta)/\Phi(2\pi)$ (cf. (2.2), (2.4)) *and suppose the* $\boldsymbol{Z}$*-components are associated. Then*

(i) $$\varphi(\pi) - \varphi(\pi/2) + 1 - \varphi(\tfrac{3}{2}\pi) = 0,$$

and if (2.16) *holds, then*

(ii) $$\sqrt{k}\,(\varphi_n(\pi) - \varphi_n(\pi/2) + 1 - \varphi_n(\tfrac{3}{2}\pi)) \xrightarrow{d} N(0, \sigma_\varphi^2),$$

where

$$\begin{aligned}\sigma_\varphi^2 = {} & 1 + \varphi(\tfrac{3}{2}\pi) - \varphi^2(\tfrac{3}{2}\pi) - \varphi(\pi) - \varphi^2(\pi) + \varphi(\pi/2) - \varphi^2(\pi/2) \\ & + 2\varphi(\tfrac{3}{2}\pi)\varphi(\pi) - 2\varphi(\tfrac{3}{2}\pi)\varphi(\pi/2) + 2\varphi(\pi)\varphi(\pi/2).\end{aligned} \tag{2.19}$$

Proof. (i) Take $(\boldsymbol{Z}_i)_{i\geqslant 1}$, independent copies of $\boldsymbol{Z}$; then $(\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n)$ is an associated vector (see, e.g, [3, p. 30]), and so, for $a_n > 0$, $a_n^{-1} \sum_{i=1}^n \boldsymbol{Z}_i - b_n$ is also associated (as the normalized sum is an increasing function on $\boldsymbol{Z}_i$'s). Therefore, the α-stable limit $\boldsymbol{X}$ in (2.2) is associated (see, e.g., [24, p. 301]). Theorem 2.1 of [13] now implies that the sprectral measure Φ in (2.4) is concentrated on the first and third quandrants.

(ii) From Lemma 4 and the assertion (i) we obtain

$$\sqrt{k}\,(\varphi_n(\pi) - \varphi_n(\pi/2) + \varphi_n(2\pi) - \varphi_n(\tfrac{3}{2}\pi)) \xrightarrow{d} \Lambda(\pi) - \Lambda(\pi/2) + \Lambda(2\pi) - \Lambda(\tfrac{3}{2}\pi) =: N,$$

where N is normal with mean zero and variance σ_φ^2 defined by (2.19). ∎

Remark 2. Applying the test, we replace σ_φ in (2.19) by its empirical counterpart $\hat{\sigma}_\varphi := \sigma_{\varphi_n}$ (cf. (2.17)). To estimate the deviation between $N(0, \sigma_\varphi^2)$ and $N(0, \hat{\sigma}_\varphi^2) := N(0, 1)\hat{\sigma}_\varphi$ ($N(0, 1)$ and $\hat{\sigma}_\varphi$ are chosen to be independent), let $l_p(X, Y)$ be the *minimal metric* w.r.t. the average metric

$$\mathscr{L}_p(X, Y) = \{\mathbf{E}|X-Y|^p\}^q, \quad p > 0,\ q = 1 \wedge (1/p),$$

that is, $l_p(X, Y) = \min\{\mathscr{L}_p(\hat{X}, \hat{Y}) \colon \hat{X} \overset{d}{=} X,\ \hat{Y} \overset{d}{=} Y\}$ (see, e.g., [5] and [22, Chapter 5]). Recall now the well-known upper bound for the Prokhorov metric $\pi(X, Y) := \inf\{\varepsilon > 0 \colon P(X \in A) \leqslant P(Y \in A^\varepsilon) + \varepsilon$ for all Borel $A \subset \mathbf{R}\}$:

$$\pi^{(1+p)q} \leqslant l_p, \quad p > 0.$$

Making use of the last inequality we get

$$\pi^{(1+p)/q}(N(0, \sigma_\varphi^2), N(0, \hat{\sigma}_\varphi^2)) \leqslant l_p(N(0, \sigma_\varphi^2), N(0, \hat{\sigma}_\varphi^2))$$
$$\leqslant (\mathbf{E}|N(0, 1)|^p)^q (\mathbf{E}|\sigma_\varphi - \sigma_{\varphi_n}|^p)^q =: \Delta_n.$$

Applying (ii) again we have $\Delta_n = O(k^{-pq/2})$. As for the uniform metric ϱ between the d.f.'s of $N(0, \sigma_\varphi^2)$, $N(0, \hat{\sigma}_\varphi^2)$, using Berry–Esseen type smoothing inequalities (see [22], Chapter 14]) one can get an alternative bound for $\varrho = O(k^{-pq/2})$. ∎

As in Theorem 1 we readily obtain

COROLLARY 1 (*A test for negative association*). *Suppose* $\mathbf{Z}$ *is in the domain of attraction of α-stable random pair with spectral d.f.* $\varphi(\theta)$ *and suppose the* $\mathbf{Z}$*-components are negatively associated. Then*

$$\varphi(\pi/2) + \varphi(\tfrac{3}{2}\pi) - \varphi(\pi) = 0,$$

and if (2.16) *holds, then*

$$\sqrt{k}\,(\varphi_n(\pi/2) + \varphi_n(\tfrac{3}{2}\pi) - \varphi_n(\pi)) \xrightarrow{d} N(0, \tilde{\sigma}_\varphi^2),$$

where

$$\tilde{\sigma}_\varphi^2 = \varphi(\pi/2) - \varphi(\pi/2)^2 + \varphi(\tfrac{3}{2}\pi) - \varphi(\tfrac{3}{2}\pi)^2 - \varphi(\pi) - \varphi(\pi)^2$$
$$+ 2\varphi(\pi/2)\varphi(\tfrac{3}{2}\pi) - 2\varphi(\pi/2)\varphi(\pi) - 2\varphi(\pi)\varphi(\tfrac{3}{2}\pi). \ ∎$$

3. Estimating the risk of a stable portfolio. Assume that the vector $\mathbf{X}$ representing the per share returns on all assets in the investment portfolio follows a multivariate stable distribution with ch. f. (1.1). Press [20, Chapter 12] defined the risk $r(c)$, $c \in \mathbf{R}^d$, of a stable portfolio with vector of returns $\mathbf{X}$, having ch.f.

$$\mathbf{E}\exp(t, \mathbf{X}) = \exp\{-\tfrac{1}{2}\sum_{i=1}^{m} (t^{\mathrm{T}}\Sigma_i t)^{\alpha/2} + i(t, \overset{0}{\mu})\} \tag{3.1}$$

(with matrices $\Sigma_i > 0$ and $\overset{0}{\mu} \in \mathbf{R}^d$), as the scaled parameter of the law of $\mathbf{X}$, that is,

$$r(c) = \tfrac{1}{2}\sum_{j=1}^{m} (c^{\mathrm{T}}\Sigma_j c)^{\alpha/2}, \qquad c \in \mathbf{R}_+^d. \tag{3.2}$$

The reason for such a definition of the risk comes from the normal case: for $\alpha = 2$, $r(c)$ is an increasing function of the covariances. Similar arguments lead to the following generalization of (3.2) (assuming an arbitrary α-stable law of returns (1.1)):

$$r(c) := \int_{S_d} |(s, c)|^\alpha \Gamma(ds), \quad c \in \boldsymbol{R}^d_+. \tag{3.3}$$

Press [20, p. 344] wrote that to permit positive and negative price changes to be weighted in the same way, $X - \overset{0}{\mu}$ should be "symmetric", which in the more recent terminology means that $X - \overset{0}{\mu}$ follows a (strictly) α-stable law with ch.f.

$$\mathbf{E} \exp(t, X) = \exp\{-\int_{S_d} |(t, s)|^\alpha \Gamma(ds) + (t, \overset{0}{\mu})\}. \tag{3.4}$$

In this case, for a scalar variable $v \geq 0$ and $t = vc \in \boldsymbol{R}^d_+$, the right-hand side of (3.4) becomes $\exp\{-|v|^\alpha r(c) + v(c, \overset{0}{\mu})\}$, motivating the choice of (3.3) as the scalar parameter in the distribution of the return on the portfolio.

To estimate $r(c)$ we need only an estimator for the spectral measure. To this end we assume that the i.i.d. observations $\boldsymbol{Z}, \boldsymbol{Z}_1, \boldsymbol{Z}_2, \ldots$ are taken from the *normal* domain of attraction of the α-stable vector $\boldsymbol{X}$, that is

$$\frac{1}{\lambda n^{1/\alpha}} \sum_{i=1}^{n} \boldsymbol{Z}_i - b_n \xrightarrow{\mathrm{d}} \boldsymbol{X}. \tag{3.5}$$

Without loss of generality we may and do assume that the scaling parameter $\lambda = 1$. Note that, in the most interesting cases of $\boldsymbol{Z}$ being α-stable ($\boldsymbol{Z} \overset{\mathrm{d}}{=} \boldsymbol{X}$ with ch.f. (1.1)) or α-geometric stable ($\boldsymbol{Z} \overset{\mathrm{d}}{=} Y$ with ch.f. (1.5)), the limit relation (3.5) holds with $\lambda = 1$. As in Section 2 we consider the case $d = 2$ for the sake of simplicity.

Using the same notation as in Section 2 define the following *estimator for* $\Phi(\theta)$:

$$\Phi_n(\theta) = \varphi_n(\theta)\Phi_n(2\pi). \tag{3.6}$$

In (3.6), φ_n is determined by (2.6) and for $\Phi_n(2\pi)$ we invoke the α-estimator (2.5) and define

$$\Phi_n(2\pi) = (k/n)(\varrho_{n-k:n})^{\alpha_n}. \tag{3.7}$$

The next lemma deals with the asymptotic normality of Φ_n. Recall first that from (2.4) we obtain

$$\lim_{n\to\infty} nS(rn^{1/\alpha}) = \Phi(2\pi) r^{-\alpha}$$

and inverting the latter we get

$$\lim_{n\to\infty} \frac{k}{n} \frac{(S^{\leftarrow}(k/n))^\alpha}{\Phi(2\pi)} = 1. \tag{3.8}$$

LEMMA 5. *Suppose that, together with the assumptions* (2.13) *and* (2.16) *in Lemmas* 3 *and* 4, *respectively, the following strengthening of* (3.8) *holds*:

$$\frac{k}{n}\frac{(S^{\leftarrow}(k/n))^{\alpha}}{\Phi(2\pi)} = 1+O\left(\frac{1}{\sqrt{k}}\right) \quad \text{as } n\to\infty. \tag{3.9}$$

Then, in $D[0, 2\pi]$, *the normalized* Φ_n *weakly converges to a degenerate Gaussian process with zero mean, namely*

$$\frac{\sqrt{k}}{\log(n/k)}(\Phi_n(\theta)-\Phi(\theta)) \xrightarrow{w} N\Phi(\theta), \tag{3.10}$$

where N *is normal with zero mean and variance* $(\log 2)^{-2}$.

Proof. We shall prove that Φ_n converges to Φ with a rate that is slower than that in $\varphi_n \xrightarrow{w} \varphi$. This together with Lemma 4 leads to (3.10).

CLAIM. *Assuming* (2.13), *as in Lemma* 3, *and* (3.9), *we have*

$$\frac{\sqrt{k}}{\log(n/k)}\left(\frac{\Phi_n(2\pi)}{\Phi(2\pi)}-1\right) \xrightarrow{d} N. \tag{3.11}$$

Proof of the Claim. By (2.13) we obtain

$$\frac{\varrho_{n-k:n}}{S^{\leftarrow}(k/n)} = \frac{S^{\leftarrow}(R_{k:n})}{S^{\leftarrow}(k/n)} = 1+O\left(\frac{1}{\sqrt{k}}\right).$$

Invoking Lemma 3, observe that $\sqrt{k}(\alpha_n-\alpha) \xrightarrow{d} \alpha N$. Combining the above two limit relations with (3.9) we determine the asymptotic behavior of $\Phi_n(2\pi)$:

$$\begin{aligned}\frac{\Phi_n(2\pi)}{\Phi(2\pi)} &= \frac{(k/n)(S^{\leftarrow}(k/n))^{\alpha}}{\Phi(2\pi)}\left(\frac{\varrho_{n-k:n}}{S^{\leftarrow}(k/n)}\right)^{\alpha_n} S^{\leftarrow}\left(\frac{k}{n}\right)^{\alpha_n-\alpha}\\ &= [1+O(1/\sqrt{k})][1+O(1/\sqrt{k})]^{\alpha_n}\exp\{(\alpha_n-\alpha)\log S^{\leftarrow}(k/n)\}\\ &= [1+O(1/\sqrt{k})][1+(N/\sqrt{k})(\log(n/k)+O(1))],\end{aligned}$$

where the last equality follows from the relation

$$\log S^{\leftarrow}(k/n) = \log(n/k)^{1/\alpha}+\alpha^{-1}\log\Phi(2\pi)+O(1/\sqrt{k}).$$

This proves the Claim.

Recall now Lemma 4: under (2.16), $\sqrt{k}(\varphi_n-\varphi) \xrightarrow{w} \Lambda$ (the convergence is in $D[0, 2\pi]$). The latter rate is faster than that in (3.10) and, therefore, the rate in $\Phi_n \xrightarrow{w} \varphi\Phi(2\pi)$ will be determined by that in (3.11). In fact, with $\Phi = \varphi\Phi(2\pi)$,

$$\Phi_n(\theta) = \varphi_n(\theta)\Phi_n(2\pi) = \Phi(\theta)+\Phi(\theta)\frac{\log(n/k)}{\sqrt{k}}N+o\left(\frac{\log(n/k)}{\sqrt{k}}\right),$$

as desired in (3.11). ■

Lemma 5 provides the asymptotic normality of the following estimator for $r(c)$:

$$(3.12)\qquad r_n(c) = \int_0^{2\pi} \xi(c,\theta)^{\alpha_n} d\Phi_n(\theta), \quad c \in \boldsymbol{R}_+^2,$$

where $\xi(c,\theta) := |c|\,|\cos(c,\theta)\hat{}\,|$.

THEOREM 2. *Under the regularity assumptions* (2.13), (2.16) *and* (3.9),

$$(3.13)\qquad \frac{\sqrt{k}}{\log(n/k)}(r_n(c)-r(c)) \xrightarrow{d} N\left(0, \left(\frac{r(c)}{\log 2}\right)^2\right), \; c \in \boldsymbol{R}_+^2, \quad \text{as } n \to \infty.$$

Proof. From Lemmas 3 and 5 we obtain

$$\sqrt{k}(\alpha_n - \alpha) \xrightarrow{d} \alpha N \quad \text{and} \quad \frac{\sqrt{k}}{\log(n/k)}(\Phi_n - \Phi) \xrightarrow{w} N\Phi,$$

where N is normal with mean zero and variance $(\log 2)^{-2}$. Since the second limit relation has a slower rate, it will play a dominant role in the convergence of $r_n(c)$ to $r(c)$. Indeed, as $n \to \infty$,

$$r_n(c) = \int_0^{2\pi}\left(1+\frac{1}{\sqrt{k}}(\alpha N)\log\xi(c,\theta)\right)\xi(c,\theta)^\alpha d\Phi(\theta)$$
$$+\frac{\log(n/k)}{\sqrt{k}}\int_0^{2\pi}\left(1+\frac{1}{\sqrt{k}}(\alpha N)\log\xi(c,\theta)\right)\xi(c,\theta)^\alpha dN\Phi(\theta)$$
$$+o\left(\frac{\log(n/k)}{\sqrt{k}}\right),$$

and therefore

$$r_n(c)-r(c) = \frac{\log(n/k)}{\sqrt{k}} N r(c) + o\left(\frac{\log(n/k)}{\sqrt{k}}\right),$$

as desired. ∎

Remark 3. The moment estimators for the risk $r(c)$ (in the form (3.2)) proposed by Press [20], [21] have different structures (cf. (3.12) and [20, Section 12.6.1]) and cannot be used in our general framework.

Remark 4. With estimates for the index α and the spectral distribution function $\Phi(\theta)$, the only parameter left to be estimated in (2.1) is the shift $\overset{0}{\mu}$. Since the assumption that $\alpha > 1$ is in general agreement with the empirical evidence (see [20, p. 344], and especially [1], where α's for 200 stocks are estimated), the vector of sample means $\overset{0}{\mu}_n = n^{-1}\sum_{i=1}^n \boldsymbol{Z}_i$ (cf. (3.5)) is the most plausible estimator for this finite mean case. Nevertheless, the problem of estimating $\overset{0}{\mu}$ for $0 < \alpha < 2$ from observations in the domain of attraction of α-stable law is of interest.

Remark 5. In the stable portfolio analysis the problem of optimization of efficient portfolios consists of minimizing the risk subject to the restriction that the fractions of funds allocated to each asset must total unity [20, p. 350]. Having estimated the risk (Theorem 2) the problem is reduced to the following:

Find $c \in \boldsymbol{R}_+^d$ *such that* $r_n(c)$ *is minimal on the simplex* $c^{\mathrm{T}} \boldsymbol{e} = 1$.

Here $\boldsymbol{e}$ stands for the vector of ones in $\boldsymbol{R}^d$.

Remark 6. Suppose the portfolio consists of only two stocks, and the return $\boldsymbol{X} = (X_1, X_2)$ follows a bivariate symmetric α-stable distribution ($1 < \alpha \leqslant 2$), that is,

$$\mathbf{E}\exp\{i(t, X)\} = \exp\{-\int_{S_2} |(t, s)|^{\alpha} \Gamma(ds)\}, \tag{3.14}$$

where Γ is a symmetric finite measure on S_2. In this case, as an alternative notion of risk one can use the *covariation* of X_1, X_2, defined as

$$[X_1, X_2]_{\alpha} = \int_{S_2} s_1 s_2^{\langle \alpha - 1 \rangle} \Gamma(ds),$$

where $s^{\langle p \rangle} := |s|^p \operatorname{sign} s$. Indeed, if the ch.f. of $\boldsymbol{X}$ is given by (3.4), we define the covariation as $[X_1 - \overset{0}{\mu}, X_2 - \overset{0}{\mu}]_{\alpha}$, and we can assume further that $\overset{0}{\mu} = \boldsymbol{0}$.

We list some of the covariation properties (see Samorodnitsky and Taqqu [27] for their proofs and more facts on covariation):

(P1) $[X_1, X_2]_2 = \frac{1}{2}\operatorname{cov}(X_1, X_2)$.

(P2) If (X_1, X_2, Y) are jointly symmetric α-stable, then $[X_1 + X_2, Y]_{\alpha} = [X_1, Y]_{\alpha} + [X_2, Y]_{\alpha}$; note, however, that the covariation is not additive in the second argument.

(P3) $[aX_1, bX_2]_{\alpha} = ab^{\langle \alpha - 1 \rangle}[X_1, X_2]_{\alpha}$.

(P4) If X_1 and X_2 are independent, then $[X_1, X_2]_{\alpha} = 0$; however, for $1 < \alpha < 2$, it is possible that $[X_1, X_2]_{\alpha} = 0$ for dependent X_1 and X_2.

(P5) $[X_1, X_2]_{\alpha} = 0$ if and only if X_2 is James orthogonal to X_1, i.e. for every $\lambda > 0$,

$$[\lambda X_1 + X_2, \lambda X_1 + X_2]_{\alpha} \geqslant [X_2, X_2]_{\alpha}.$$

To estimate the covariation $[X_1, X_2]$ of $\boldsymbol{X}$, having observations $\boldsymbol{Z}_1, \boldsymbol{Z}_2, \ldots$ from the normal domain of attraction of $\boldsymbol{X}$, we shall use again Lemmas 3 and 5. Define the following estimator for $[X_1, X_2]_{\alpha}$:

$$[X_1, X_2]_{\alpha}^{(n)} = \int_0^{2\pi} (\cos\theta)(\sin\theta)^{\langle \alpha_n - 1 \rangle} d\Phi_n(\theta).$$

THEOREM 3. *Under the regularity assumptions* (2.13), (2.16) *and* (3.9) *and assuming* (3.14),

$$(3.15)\qquad \frac{\sqrt{k}}{\log(n/k)}([X_1, X_2]_\alpha^{(n)} - [X_1, X_2]_\alpha) \xrightarrow{\mathrm{d}} N\left(0, \left(\frac{[X_1, X_2]_\alpha}{\log 2}\right)^2\right)$$

as $n \to \infty$. ∎

The proof is similar to that of Theorem 2 and thus omitted.

APPENDIX

Proof of Lemma 1.

(A) We start with the following result from [29].

CLAIM 1. *If* $nb_n/\log\log n \to \infty$, $b_n \downarrow 0$ *and* F_n *is the empirical d.f. based upon a random sample of n uniforms, then a.s.*

$$\lim \sup_{b_n \leqslant t \leqslant 1} F_n(t)/t = \lim \sup_{b_n \leqslant t \leqslant 1} t/F_n(t) = 1.$$

From the Claim and the assumptions on $k = k_n$ and ϱ we have

$$R_{2k:n} := S(\varrho_{2k:n}) \to 0 \quad \text{and} \quad R_{k:n}/R_{2k:n} \to \tfrac{1}{2}.$$

Therefore, applying (2.11) we get

$$\log \varrho_{n-k+1:n} - \log \varrho_{n-2k+1:n}$$
$$= \log S^{\leftarrow}\left(R_{2k:n}\frac{R_{k:n}}{R_{2k:n}}\right) - \log S^{\leftarrow}(R_{2k:n}) \to -\frac{1}{\alpha}\log\tfrac{1}{2} = \frac{1}{\alpha}\log 2,$$

as desired in (A).

(B) Recall that $R_i = S(\varrho_i)$, where (ϱ_i, Θ_i) are the polar coordinates of the random sample $\boldsymbol{Z}_i = (Z_{i1}, Z_{i2})$, $i = 1, \ldots, n$. Recall also the definition of

$$F_n(\theta, r) = \frac{n}{k}P\left(\Theta \leqslant \theta, R \leqslant \frac{k}{n}r\right),$$

and its corresponding empirical counterpart $\boldsymbol{F}_n(\theta, r)$ (cf. (2.7)). Then

$$\varphi_n(\theta) = \frac{1}{k}\sum_{i=1}^{n} 1(\Theta_i \leqslant \theta, \varrho_i \geqslant \varrho_{n-k+1:n})$$
$$= \frac{1}{k}\sum_{i=1}^{n} 1(\Theta_i \leqslant \theta, R_i \leqslant R_{k:n}) = \boldsymbol{F}_n\left(\theta, \frac{n}{k}R_{k:n}\right).$$

Put $e_n = (n/k)R_{k:n}$ and let us show that $F_n(\theta, e_n) \to \varphi(\theta) = \Phi(\theta)/\Phi(2\pi)$ a.s. It is enough to see that

$$D_{1,n} = \sup_{\substack{0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant s \leqslant 2}} |F_n(\theta, s_n) - F_n(\theta, s_n)| \quad \text{and} \quad D_{2,n} = |F_n(\theta, e_n) - \varphi(\theta)|$$

vanish as $n \to \infty$. To estimate $D_{1,n}$ we use the following multivariate analogue of the exponential bounds in [29].

CLAIM 2 (Einmahl (1987)). *Let $(Z_i)_{i=1,\ldots,n}$ be a random sample from a probability law C on $\mathbf{R}^2$, and C_n the empirical counterpart of C. Let $\mathscr{A}$ be the ring of rectangles $(a_1, b_1] \times (a_2, b_2] \subset \mathbf{R}^2$. Take $A \in \mathscr{A}$ with $0 < C(A) < \frac{1}{2}$ and $0 < \delta < 1$. There exists a function $\psi(\lambda) > 0$ with $\psi(\lambda) \uparrow 1$ as $\lambda \downarrow 0$ such that for all $\lambda > 0$*

$$P\big(\sup_{\tilde{A} \subset A} |n^{1/2}(C_n(\tilde{A}) - C(\tilde{A}))| \geqslant \lambda\big)$$
$$\leqslant K(\delta) \exp\left\{\frac{-(1-\delta)\lambda^2}{2C(A)} \psi\left(\frac{\lambda}{n^{1/2} C(A)}\right)\right\}, \qquad \lambda > 0,$$

where $K(\delta)$ is an absolute positive constant.

From Claim 2 for any $\varepsilon > 0$ we obtain

$$P(D_{1,n} > \varepsilon) = P\left(\sup_{\substack{0 \leqslant \theta \leqslant 2\pi \\ 0 \leqslant s \leqslant 2}} \sqrt{n} \left| \frac{1}{n} \sum_{i=1}^{n} 1\left(\Theta_i \leqslant \theta, R_i < \frac{k}{n} s\right) \right.\right.$$
$$\left.\left. - \frac{1}{n} P\left(\Theta \leqslant \theta, R \leqslant \frac{k}{n} s\right) \right| \geqslant \varepsilon \frac{k}{\sqrt{n}}\right)$$
$$\leqslant K\left(\frac{1}{2}\right) \exp\left\{\frac{-\varepsilon^2 k^2/n}{4P(A)} \psi\left(\frac{\varepsilon k/\sqrt{n}}{\sqrt{n}\, P(A)}\right)\right\}$$
$$= K\left(\frac{1}{2}\right) \exp\left\{-\frac{k\varepsilon^2}{8} \psi(\varepsilon/2)\right\} < n^{-1},$$

where $A = \{\theta \leqslant 2\pi, R \leqslant 2k/n\}$. As for $D_{2,n}$, we use (2.14) and that $e_n \to 1$ a.s., and so $D_{2,n} \to 0$ as $n \to \infty$. ∎

Proof of Lemma 2. To check the convergence of the finite dimensional distributions take $u_1, \ldots, u_m$, $v_1, \ldots, v_l$ real, $\theta_1, \ldots, \theta_m$ from $[0, 2\pi)$ and $r_1, \ldots, r_l$ from $[0, 3]$. Set

$$\xi_i(\theta) = 1(\Theta_i \leqslant \theta, R_i \leqslant k/n) - P(\Theta \leqslant \theta, R \leqslant k/n),$$
$$\eta_i(r) = 1\big(R_i \leqslant (k/n)r\big) - (k/n)r,$$
$$\Delta_i = \sum_{\alpha=1}^{m} u_\alpha \xi_i(\theta_\alpha) + \sum_{\alpha=1}^{l} v_\alpha \eta_i(r_\alpha), \qquad S_n = (1/\sqrt{k}) \sum_{i=1}^{n} \Delta_i.$$

Observe that Δ_i, $i = 1, \ldots, n$, are identically distributed and independent of each other with zero mean:

$$\mathbf{E}\Delta_i^2 = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{m} u_\alpha u_\beta P(\Theta \leqslant \theta_\alpha \wedge \theta_\beta,\, R \leqslant k/n) + \sum_{\alpha=1}^{l}\sum_{\beta=1}^{l} v_\alpha v_\beta (k/n)(r_\alpha \wedge r_\beta)$$

$$+2\sum_{\alpha=1}^{m}\sum_{\beta=1}^{l} u_\alpha v_\beta P(\Theta \leqslant \theta_\alpha,\, R \leqslant (k/n)(r_\beta \wedge 1)) + O((k/n)^2) = O(k/n),$$

and $\mathbf{E}|\Delta_i|^3 = O(k/n)$. Then the Lyapunov condition holds:

$$\frac{\mathbf{E}|S_n|^3}{(\mathbf{E}S_n^2)^{3/2}} = O\left(\frac{(n/k^{3/2})\mathbf{E}|\Delta_1|^3}{((n/k)\mathbf{E}\Delta_1^2)^{3/2}}\right) = O\left(\frac{1}{\sqrt{k}}\right) \to 0.$$

The proof of the tightness of the marginal processes $(W_{1,n})_{n\geqslant 1}$ is based on the weak compactness criterion in $D[0, 2\pi]$ and $D[0, 3]$, respectively (see [4, Theorem 15.2], for details we refer to [9]). ∎

Proof of Lemma 3. Recall that $\alpha_n = \log 2/(\log \varrho_{n-k+1:n} - \log \varrho_{n-2k+1:n})$, and $R_i = S(\varrho_i)$. From (2.8) we obtain

$$\log S^{\leftarrow}(R_{[kr]:n}) - \log S^{\leftarrow}\left(\frac{k}{n}\right) = -\frac{1}{\alpha}\log\left(\frac{rR_{[kr]:n}}{rk/n}\right) + o(k^{-1/2})$$

$$= -\frac{1}{\alpha}\log r - \frac{1}{\alpha}\log\left(\frac{R_{[kr]:n}}{rk/n}\right) + o(k^{-1/2})$$

and since $R_{[kr]:n}/(rk/n) \xrightarrow{w} 1$, we get

$$\sqrt{k}(\log S^{\leftarrow}(R_{[kr]:n}) - \log S^{\leftarrow}(k/n) + \alpha^{-1}\log r)$$

$$= -\sqrt{k}\,\alpha^{-1}\log(R_{[kr]:n})/([kr]/n) + o(1)$$

$$= (\alpha r)^{-1}\sqrt{k}((n/k)R_{[kr]:n} - r) + o(1) \to (\alpha r)^{-1}(-W_2(r)).$$

The last limit relation follows from Lemma 2. In fact, from Lemma 2 and the Skorokhod–Dudley theorem (see [7, Theorem 11.7.1]) there exist a probability space and a sequence of processes $\tilde{W}_{2,n} \stackrel{d}{=} W_{2,n}$, $\tilde{W}_2 \stackrel{d}{=} W_2$, such that

$$\sup_{0\leqslant r\leqslant 3} |\tilde{W}_{2,n}(r) - \tilde{W}_2(r)| \to 0 \quad \text{as } n \to \infty \text{ a.s.}$$

Invoking [28] we see that the last limit relation implies

$$\sup_{1/2\leqslant r\leqslant 2^1/_2} |\sqrt{k}((n/k)R_{[kr]:n} - r) + \tilde{W}_2| \to 0 \text{ a.s.}$$

Combining the above bounds, we obtain

$$\sqrt{k}\left(\frac{\alpha}{\alpha_n}-1\right) = \sqrt{k}\left(\alpha\frac{\log S^{\leftarrow}(R_{k:n})-\log S^{\leftarrow}(R_{2k:n})}{\log 2}-1\right)$$
$$\xrightarrow{w} (-W_2(1)+\tfrac{1}{2}W_2(2))/\log 2. \blacksquare$$

Proof of Lemma 4. Recall the definition of

$$\varphi_n(\theta) = F_n(\theta, e_n) = k^{-1}\sum_{i=1}^{n} 1(\Theta_i \leqslant \theta,\ R_i \leqslant R_{k:n});$$

cf. (2.6) and (2.7).

CLAIM. *As* $n \to \infty$,

$$\sqrt{k}\,(\varphi_n(\theta)-(n/k)P(\Theta \leqslant \theta,\ R \leqslant R_{k:n})) \xrightarrow{w} W_1(\theta) \quad \text{in } D[0, 2\pi].$$

Proof of the Claim. From Lemma 2 we obtain $W_{1,n} \xrightarrow{w} W_1$. Let us show that

(a) $$\sup_{\substack{0\leqslant\theta\leqslant 2\pi\\ |r-1|\leqslant k_n^{-1/4}}} |\sqrt{k}\,(F_n(\theta, r)-F_n(\theta, r))-W_{1,n}(\theta)| \xrightarrow{P} 0$$

and

(b) $$\sqrt{k}\,(e_n-1) = O_p(1)$$

(and so (b) implies $P\big(k_n^{1/4}(e_n-1)\geqslant 1\big) < \varepsilon$ for n large).

For (a), using Claim 2 in the proof of Lemma 1, we get the bound

$$P\Big(\sup_{\substack{0\leqslant\theta\leqslant 2\pi\\ 1-k^{-1/4}\leqslant r\leqslant 1+k^{-1/4}}} |\sqrt{k}\,(F_n(\theta, r)-F_n(\theta, r))-W_{1,n}(\theta)| > \varepsilon\Big)$$
$$\leqslant P\Big(\sup_{\substack{0\leqslant\theta\leqslant 2\pi\\ 1\leqslant r\leqslant 1+k^{-1/4}}} |\ldots| > \varepsilon\Big)+P\Big(\sup_{\substack{0\leqslant\theta\leqslant 2\pi\\ 1-k^{-1/4}\leqslant r\leqslant 1}} |\ldots| > \varepsilon\Big) =: \Delta_1+\Delta_2,$$

where, as $n\to\infty$, $\Delta_1 \leqslant \text{const}\exp\big(-\varepsilon^2 k_n^{1/4}\psi(\varepsilon/\text{const}\,k_n^{1/4})\big)\to 0$, and similarly $\Delta_2 \to 0$.

For (b) we use Lemma 2 and [28]. In a similar fashion as in the proof of Lemma 3 we get

$$\sqrt{k}\,(e_n-1) = \sqrt{k}\,\big((n/k)R_{k:n}-1\big) \xrightarrow{d} -W_2(1),$$

which completes the proof of the Claim.

By the Claim and $e_n \to 1$ a.s., we obtain

$$\sqrt{k}\,(\varphi_n(\theta)-e_n\varphi(\theta)) = \sqrt{k}\,(\varphi_n(\theta)-(n/k)P(\Theta\leqslant\theta,\ R\leqslant R_{k:n}))$$
$$+\sqrt{k}\,\big((n/k)P(\Theta\leqslant\theta,\ R\leqslant(n/k)e_n)-e_n\varphi(\theta)\big)\to W_1(\theta);$$

the second term vanishes as $n \to \infty$ due to our assumption (2.16). Using Lemma 2, we now have

$$\sqrt{k}\,(\varphi_n(\theta)-\varphi(\theta)) = \sqrt{k}\,(\varphi_n(\theta)-(n/k)R_{k:n}\varphi(\theta)) \\ +\sqrt{k}\,((n/k)R_{k:n}\varphi(\theta)-\varphi(\theta)) \xrightarrow{w} W_1(\theta)-\varphi(\theta)W_2(1). \blacksquare$$

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