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# COMPUTER INVESTIGATION OF CHAOTIC BEHAVIOR OF STATIONARY $\alpha$-STABLE PROCESSES

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*Abstract.* In this paper we present a method of computer investigation of chaotic behavior of stationary $\alpha$-stable stochastic processes, i.e., an important class of processes with cadlag trajectories. Our results are based on spectral representations of such processes and on theorems characterizing their ergodic and mixing properties. Computer simulation techniques, appropriate numerical integration algorithms and computer graphics provide some results useful in applications.

**1. Introduction.** A large number of papers on chaotic properties of stochastic processes have been devoted to Gaussian processes, starting from Maruyama [14], Grenander [6] and Fomin [5]. A chaotic behavior of stable processes was studied by Cambanis et al. [3]. See also Weron [18] and Podgórski and Weron [16].

All infinitely divisible processes (i.e., all $\alpha$-stable processes included) share with the Gaussians the property that pairwise independence implies mutual independence (Maruyama [15]); this follows from the fact that integrals of deterministic functions with respect to a Poisson measure are independent if and only if the functions have disjoint supports. It is reasonable, therefore, to guess that mixing (asymptotic independence) would be equivalent to asymptotic pairwise independence of random variables in an infinitely divisible process. While Maruyama did not state the following result explicitly, it is contained in the proof of his theorem characterizing mixing of infinitely divisible processes.

PROPOSITION 1.1. *A stationary infinitely divisible process $\{X(t)\}$ is mixing if and only if for all $\theta_1, \theta_2 \in \mathbf{R}$ we have*

$$\lim_{t\to\infty} \langle \exp(i\theta_1 X(t)), \exp(i\theta_2 X(0))\rangle = \langle \exp(i\theta_1 X(0)), 1\rangle \langle 1, \exp(i\theta_2 X(0))\rangle.$$

This means that all stationary $\alpha$-stable processes are like the Gaussians in the sense that mixing is determined by the bivariate marginal distributions. With Gaussian processes, however, it suffices to take $\theta_1 = \theta_2 = 1$, as Gaussian processes are mixing if and only if the covariances converge to zero (see [4]). A recent result due to Gross [7] shows that the non-Gaussian $\alpha$-stable processes share this property with their Gaussian counterpart.

In contrast to Maruyama [15], we employ here, as a simple tool, the concept of the dynamical functional and combine it with the spectral representation of stationary $\alpha$-stable stochastic processes developed by Hardin [8], [9]. As a result we are able to present, in a rather simple way, a systematic study of the chaotic behavior of non-Gaussian stationary $\alpha$-stable processes. In this way we obtain a powerful tool for computer investigation of mixing and ergodic properties of such processes.

The significance of these properties for studying and modeling the chaotic behavior of physical systems is discussed in [13].

**2. Stationary $\alpha$-stable processes.** Let us recall that a real random variable $Y$ has a stable distribution if for every $a$, $b > 0$ and independent copies $Y_1, Y_2$ of $Y$ there exists $c > 0$ such that

$$aY_1 + bY_2 = cY.$$

For every stable random variable $Y$ there exists a unique $\alpha \in (0, 2]$ (the *index of stability*) such that the number $c$ which appears in the above definition is uniquely determined by the equality $c = (a^\alpha + b^\alpha)^{1/\alpha}$. If the random variable $Y$ has a *symmetric stable distribution* with index $\alpha$, then its characteristic function is of the form

$$\phi(\theta) = \exp(-c_Y |\theta|^\alpha),$$

where $c_Y$ is some positive constant (for more details we refer to Breiman [2]).

Now our aim is to describe $\alpha$-stable stochastic processes $X = \{X(t)\colon t \in \boldsymbol{R}\}$.

DEFINITION 2.1. A stochastic process $X$ is called *symmetric $\alpha$-stable* or *Lévy $S\alpha S$* or, shortly, *$S\alpha S$ process* for $\alpha \in (0, 2]$ if for every $n \in \boldsymbol{N}$ and any $a_1, \ldots, a_n \in \boldsymbol{R}$, $t_1, \ldots, t_n \in \boldsymbol{R}$, the random variable $Y = \sum_{i=1}^{n} a_i X(t_i)$ has a symmetric stable distribution with index $\alpha$.

Let $X$ be an $S\alpha S$ process, $\alpha \in (0, 2]$. For an $S\alpha S$ random variable $Y$, set $\|Y\|_\alpha = c_Y^{1/\alpha}$. Then $\|\cdot\|_\alpha^{1 \wedge \alpha}$ defines a norm in the case $1 \leqslant \alpha \leqslant 2$ and a quasi-norm in the case $0 < \alpha < 1$ on the space $\operatorname{lin}\{X(t)\colon t \in \boldsymbol{R}\}$, metrizing the convergence in probability. Then, for $Y \in \operatorname{lin}\{X(t)\colon t \in \boldsymbol{R}\}$ we have

$$\mathbf{E} e^{i\theta Y} = \exp(-|\theta|^\alpha \|Y\|_\alpha^\alpha).$$

Let $\boldsymbol{L}^0 = \boldsymbol{L}^0(X)$ denote the closure of the linear span $\operatorname{lin}\{X(t)\colon t \in \boldsymbol{R}\}$ in the space $\boldsymbol{L}^0(\Omega, \mathscr{F}, P)$ of all random variables in $(\Omega, P)$ with respect to the topology of convergence in probability. Taking the closure of $\operatorname{lin}\{X(t)\colon t \in \boldsymbol{R}\}$ in $\boldsymbol{L}^0$ with respect to the norm (quasi-norm) $\|\cdot\|_\alpha$ we obtain the space $\boldsymbol{L}^\alpha = \boldsymbol{L}^\alpha(X)$. If $X$ is a stationary process, then for $Y \in \boldsymbol{L}^\alpha$ and $t \in \boldsymbol{R}$ we have $\|T_t Y\|_\alpha = \|Y\|_\alpha$. Hence $(T_t)_{t \in \boldsymbol{R}}$ is a group of isometries on $\boldsymbol{L}^\alpha$.

DEFINITION 2.2. Let $(E, \mathscr{E}, \lambda)$ be a measure space. Let us introduce the family of sets

$$\mathscr{E}_0 \stackrel{\mathrm{df}}{=} \{A \in \mathscr{E}\colon \lambda(A) < \infty\}.$$

The map $\boldsymbol{Z}\colon \mathscr{E}_0 \ni A \to Z(A) \in \boldsymbol{L}^0$ is called a *stochastic SαS measure with a control measure λ* if:

(i) $Z(\emptyset) = 0$ with probability 1;

(ii) for $A \in \mathscr{E}_0$ the law of $Z(A)$ is described by $\mathrm{E}\exp(i\theta Z(A)) = \exp(-|\theta|^\alpha \lambda(A))$;

(iii) for every sequence $\{A_n\}_{n \in \boldsymbol{N}}$ of pairwise disjoint sets from $\mathscr{E}_0$ the sequence of random variables $\{Z(A_n)\}_{n \in \boldsymbol{N}}$ is independent and such that $Z(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} Z(A_n)$ with probability 1.

According to the definition of stochastic integral of deterministic functions with respect to α-stable random measures (see, e.g., [17]), for every function $f \in \boldsymbol{L}^\alpha(E, \mathscr{E}, \lambda)$ one can define a stochastic integral $\int_E f d\boldsymbol{Z}$ as an α-stable random variable with the law given by

$$\mathrm{E}\exp(i\theta \int_E f d\boldsymbol{Z}) = \exp(-|\theta|^\alpha \|f\|_\alpha^\alpha).$$

The spectral representation theorem for a stationary stochastic SαS process $X = \{X(t)\}_{t \in \boldsymbol{R}}$ (see, e.g., [8], [9]) says that there exist a measure space $(E, \mathscr{E}, \lambda)$ and a group $(U_t)$ of isometries of $\boldsymbol{L}^\alpha(E, \mathscr{E}, \lambda)$ described by a function $f_0 \in \boldsymbol{L}^\alpha(E, \mathscr{E}, \lambda)$ such that

$$X(t) = \int_E U_t f_0 d\boldsymbol{Z} \quad \text{for all } t \in \boldsymbol{R}.$$

If $X$ is measurable, then the above group of isometries is strongly continuous (see [3], Theorem 6).

We find it interesting to end this introduction by computer construction of SαS stationary processes for two different values of the parameter α.

Figs. 2.1 and 2.2 present the visualization of the stationary α-stable Ornstein–Uhlenbeck process $\{X(t)\}$ for $\alpha = 1.7$ and $\alpha = 1.3$, respectively. Both figures show ten typical trajectories of the corresponding stationary Ornstein–Uhlenbeck process $\{X(t)\}$ plotted versus $t \in [0, 1]$. The trajectories are represented by fine lines. The two pairs of quantile lines defined by $p_1 = 0.25$ and $p_2 = 0.35$ are approximately parallel indicating the stationarity of the process.

The method of computer approximation and simulation of stochastic integrals with α-stable measures as integrators is described in detail in Janicki and Weron [11], [12], while a corresponding result concerning convergence of the method is contained in [10].

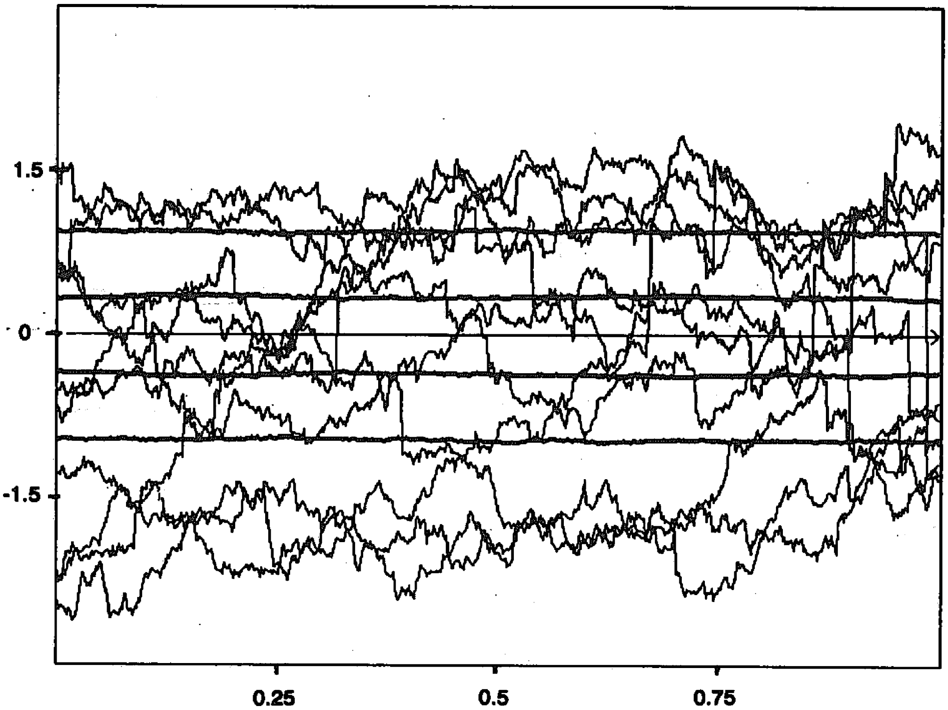


Fig. 2.1. Computer approximation of the $S_{1.7}(1, 0, 0)$-valued stationary Ornstein–Uhlenbeck process

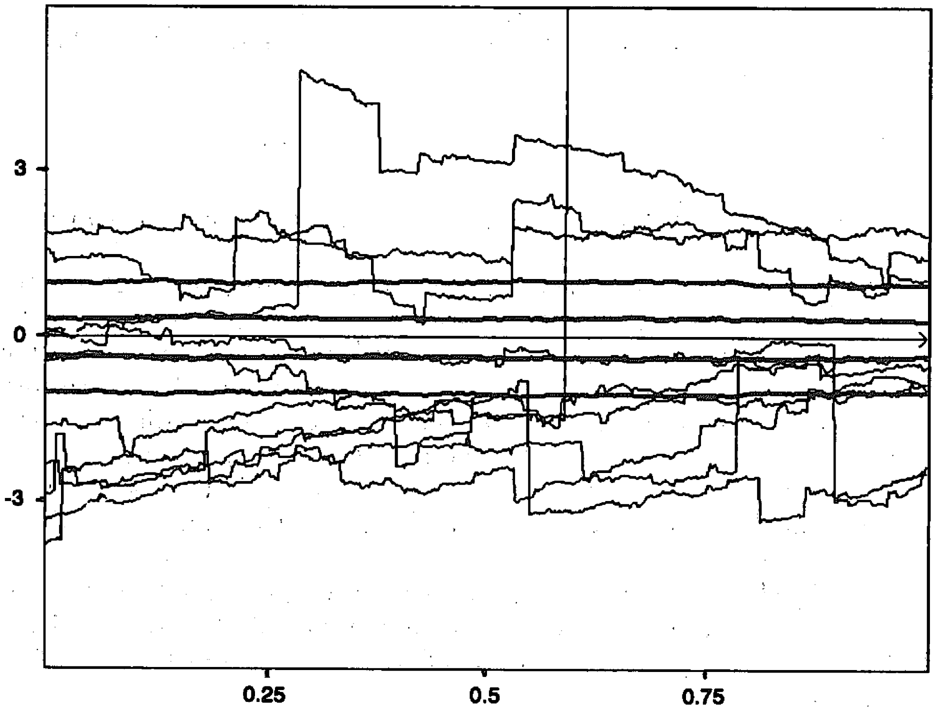


Fig. 2.2. Computer approximation of the $S_{1.3}(1, 0, 0)$-valued stationary Ornstein–Uhlenbeck process

**3. Dynamical functional.** Suppose that $X$ is a measurable stationary stochastic process.

DEFINITION 3.1. The map $\Phi\colon L^0(X)\times \boldsymbol{R}\to \boldsymbol{C}$ defined by

$$\Phi(Y, t) = \mathbf{E}\exp\{i(T_t Y - Y)\} \tag{3.1}$$

is called the *dynamical functional* of the stochastic process $X$.

The dynamical functional was introduced in Podgórski and Weron [16].

For each $Y\in L^0(X)$ the function $\Phi(Y, \cdot)$ is positive definite. If the process $X$ is in addition stochastically continuous, then the group $(T_t)_{t\in\boldsymbol{R}}$ is continuous on $L^0(X)$ with respect to the topology of convergence in probability. Consequently, $\Phi$ is continuous in the product topology on $L^0(X)\times\boldsymbol{R}$. By stationarity we have, for $Y\in L^0(X)$,

$$\Phi(Y, -t) = \mathbf{E}\exp\{i(T_{-t}Y - Y)\} = \mathbf{E}\exp\{i(Y - T_t Y)\} = \overline{\Phi(Y, t)},$$

and thus, if $X$ is symmetric, then $\Phi$ is real and $\Phi(Y, -t) = \Phi(Y, t)$.

It is convenient to characterize ergodicity and mixing in terms of the dynamical functional (see Janicki and Weron [12]).

PROPOSITION 3.1. *Let $X$ be a stationary stochastic process. Then*

(i) *$X$ is ergodic if and only if for each $Y\in\operatorname{lin}\{X(t)\colon t\in\boldsymbol{R}\}$ we have*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \Phi(Y, t)\,dt = |\mathbf{E}e^{iY}|^2;$$

(ii) *$X$ is mixing if and only if for each $Y\in\operatorname{lin}\{X(t)\colon t\in\boldsymbol{R}\}$ we have*

$$\lim_{T\to\infty}\Phi(Y, T) = |\mathbf{E}e^{iY}|^2.$$

We make use of this proposition in next two sections.

EXAMPLE 3.1. *Dynamical functional for the SαS Ornstein–Uhlenbeck process.*

From the spectral representation theorem for $S\alpha S$ stationary processes it follows that the Ornstein–Uhlenbeck process $\{X(t)\colon t\in[0, \infty)\}$ as a moving average process can be represented on the corresponding space $L^\alpha$ by the function $f_0(x) = e^{-x}I_{[0,\infty)}(x)$ and the group of shift operators $U_t g(x) = g(x-t)$. Let $Y$ and $T_t$ correspond via the spectral representation theorem to $h$ and $U_t$, respectively. Then we have

$$\Phi(Y, t) = \mathbf{E}\exp\{(T_t Y - Y)\} = \exp\{-\|U_t h - h\|_\alpha^\alpha\}. \tag{3.2}$$

Define the set of four functions from the linear span $\operatorname{lin}\{U_t f_0(x)\colon t\geqslant 0\}$:

$$\begin{aligned} h_1(x) &= f_0(x),\\ h_2(x) &= f_0(x) - \tfrac{1}{2}U_2 f_0(x),\\ h_3(x) &= f_0(x) - \tfrac{1}{2}U_2 f_0(x) + \tfrac{1}{2}U_4 f_0(x),\\ h_4(x) &= f_0(x) - \tfrac{1}{2}U_2 f_0(x) + \tfrac{1}{2}U_4 f_0(x) - U_6 f_0(x). \end{aligned} \tag{3.3}$$

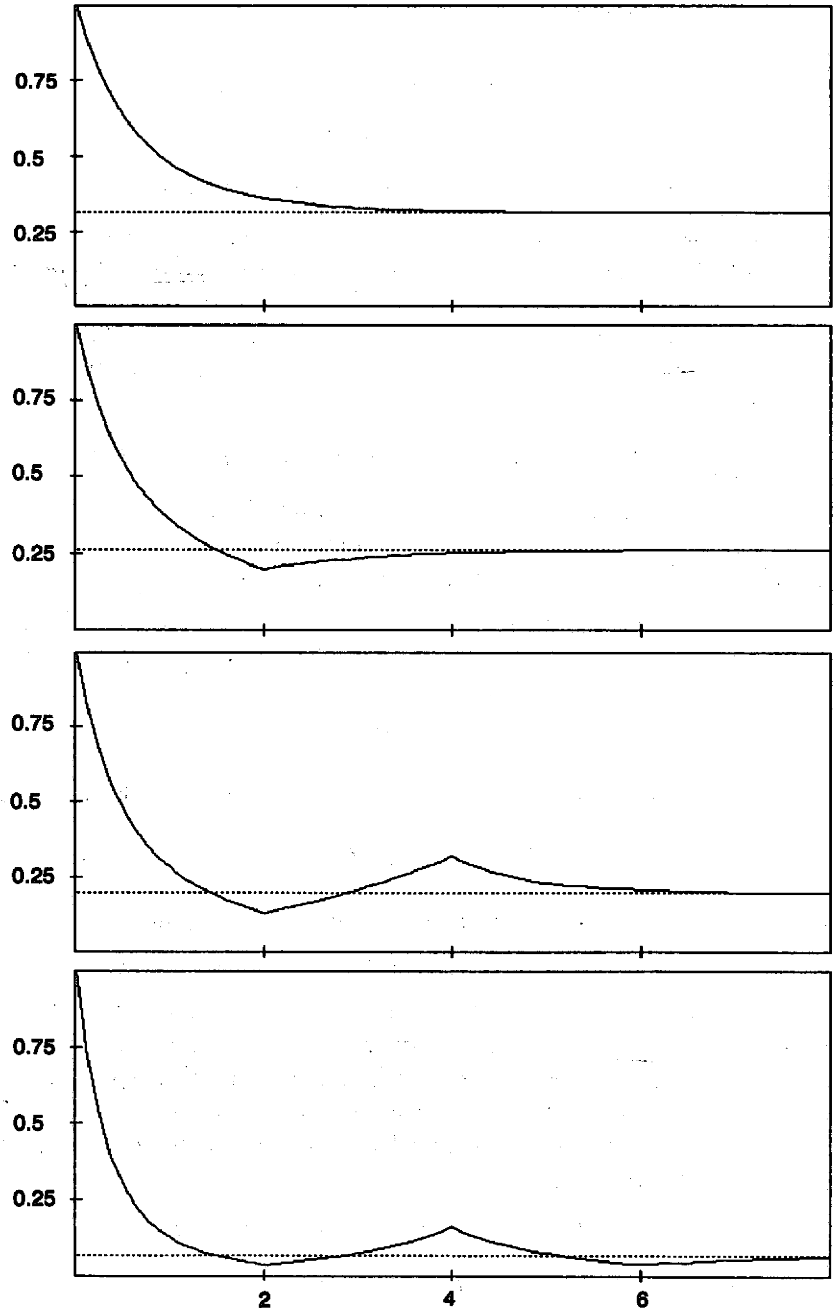


Fig. 3.1. Typical realizations of the dynamical functional for $S1.7S$ Ornstein–Uhlenbeck process. Plotted are $\Phi^*(h_i, t)$ versus $t \in [0, 8]$ for the same set of four functions $h_i$ as in (3.3)

According to (3.2) the dynamical functional $\Phi(Y, t)$ for the $S\alpha S$ Ornstein–Uhlenbeck process $\{X(t)\}$ can be evaluated as

$$\Phi^*(h, t) = \exp\{-\|U_t h - h\|_\alpha^\alpha\},$$

where the functions $h$ correspond to random variables $Y$. Fig. 3.1 contains numerical evaluations of $\Phi^*(h_i, t)$ for $\alpha = 1.7$ and the above-defined functions $h_i(x)$, where $i = 1, \ldots, 4$. These numerical results were obtained with the use of the Richardson method (see Björck and Dahlquist [1]) providing approximate values of appropriate integrals with high order of occuracy. Plotted are $\Phi^*(h_i, t)$ versus $t \in [0, 8]$. Note the different asymptotic values of the dynamical functional represented by dotted lines for the functions $h_i(x)$.

**4. Mixing property of stable processes.** It is clear that mixing is a stronger property than ergodicity. A stationary Gaussian process with the harmonic spectral representation

$$X(t) = \operatorname{Re} \int_{-\infty}^{\infty} e^{it\theta} dW(\theta)$$

is mixing if and only if its covariance $R(u) = \int_{-\infty}^{\infty} e^{iu\theta} d\lambda(\theta)$ tends to 0 as $u \to \infty$. For non-Gaussian stationary stable processes Cambanis et al. [3], Theorem 2, obtained a characterization of mixing $S\alpha S$ processes in the language of spectral representations.

Applying the dynamical functional of an $S\alpha S$ stationary process we can obtain another characterization of the mixing property.

THEOREM 4.1. *Let* $X$ *be a stationary* $S\alpha S$ *process with* $0 < \alpha \leqslant 2$ *and the spectral representation*

$$\{X(t)\colon t \in \boldsymbol{R}\} \stackrel{d}{=} \{\int_E (U_t f_0)(\theta) dZ(\theta)\colon t \in \boldsymbol{R}\}. \tag{4.1}$$

*Then the process* $X$ *is mixing if and only if for any function* $h \in \operatorname{lin}\{U_t f_0\colon t \in \boldsymbol{R}\}$

$$\lim_{t \to \infty} \|U_t h - h\|_\alpha^\alpha = 2\|h\|_\alpha^\alpha. \tag{4.2}$$

Proof. By Proposition 3.1 (ii), the process $X$ is mixing if and only if for any $Y \in \operatorname{lin}\{X_t\colon t \in \boldsymbol{R}\}$ we have

$$\lim_{t \to \infty} \Phi(Y, t) = \mathrm{E}|e^{iY}|^2.$$

From the spectral representation of $X$ given by (4.1) and the definition of the dynamical functional it follows that the above statement is equivalent to the fact that for any $h \in \text{lin}\{U_t f_0 \colon t \in \boldsymbol{R}\}$ we have

$$\lim_{t \to \infty} \exp(-\|U_t h - h\|_\alpha^\alpha) = \exp(-2\|h\|_\alpha^\alpha),$$

which completes the proof. ■

EXAMPLE 4.1. *Numerical illustration of the mixing property for the SαS Ornstein–Uhlenbeck process.*

Taking into account (4.2) it is enough to recall Example 3.1. Consider the same four functions $h_1, \ldots, h_4$. In order to check the mixing property by Theorem 4.1 it is enough to evaluate $\lim_{t \to \infty} \Phi^*(h, t)$.

The numerical evaluation of the dynamical functional for the $S1.7S$ Ornstein–Uhlenbeck process for the given functions $h_i$ is presented in Fig. 3.1. The theoretical limits $\exp(-2\|h_i\|_{1.7}^{1.7})$ are denoted by the dotted lines and their values obtained by numerical integration are the following (from top to bottom):

0.308, 0.259, 0.193, 0.067.

It is clear that in all four cases the curves representing the dynamical functional approach well the theoretical limits even on the interval [0, 8]. Fig. 3.1 illustrates a typical behavior of any mixing SαS stationary process, so the discussed computer method seems to provide useful quantitative information on this property for a given stochastic process.

**5. Ergodicity of stable processes.** The characterization of ergodic processes in terms of their spectral representation plays an important role in the ergodic theory of SαS processes. The characterization given below was established in [3], Theorem 1.

THEOREM 5.1. *Let X be a stationary stochastic SαS process with spectral representation of the form* $\{\int_E U_t f_0 \, dZ\}$. *Then X is ergodic if and only if for every function* $h \in \overline{\text{lin}}\{U_t f_0 \colon t \in \boldsymbol{R}\}$ *we have*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \|U_t h - h\|_\alpha^{2\alpha} dt = 4\|h\|_\alpha^{2\alpha} \tag{5.1}$$

*and*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \|U_t h - h\|_\alpha^\alpha dt = 2\|h\|_\alpha^\alpha. \tag{5.2}$$

An application of Theorem 5.1 allows us to show that any real-valued SαS process with harmonic spectral representation is not ergodic.

Let us recall that an SαS process $\{X(t)\}_{t \in \boldsymbol{R}}$ has a harmonic spectral representation if there exists a complex stochastic measure $W$ defined on

$(\boldsymbol{R}, \mathscr{B}_{\boldsymbol{R}}, \mu)$ with finite control measure $\mu$ such that

$$X(t) = \operatorname{Re} \int_{\boldsymbol{R}} e^{it\theta} dW(\theta), \qquad t \in \boldsymbol{R}.$$

For a stationary $S\alpha S$ process $\{X(t)\}_{t \in \boldsymbol{R}}$ with such a representation there exists a positive constant $c_\alpha$ such that for $n \in N$, $a_1, \ldots, a_n \in \boldsymbol{R}$, $t_1, \ldots, t_n \in \boldsymbol{R}$, we have

$$\left\| \sum_{l=1}^{n} a_l U_{t_l} f_0 \right\|_\alpha^\alpha = c_\alpha \int_{\boldsymbol{R}} \left| \sum_{l=1}^{n} a_l \exp(it_l \theta) \right|^\alpha d\mu(\theta)$$

(see [18]).

EXAMPLE 5.1. *Numerical illustration of the lack of the ergodic property for the SαS harmonizable process.*

In this example we would like to examine the behavior of the $S\alpha S$ harmonizable process. Let us recall that its spectral representation is given by $f_0(x) = I_{[0,\infty)}(x)$ and $U_t g(x) = \cos(tx) g(x)$. Take $dW(x) = e^{-x} dL_\alpha(x)$, where $L_\alpha(\cdot)$ stands for the $S\alpha S$ Lévy motion with $\alpha = 1.7$.

As in Example 3.1 we define the set of four functions $h_1, \ldots, h_4$ from the linear span $\operatorname{lin}\{U_t f_0(x)\colon t \geqslant 0\}$, taking the same linear combinations as in (3.3).

Fig. 5.1 presents the numerical results obtained for the $S\alpha S$ harmonizable process with this set of functions $h_i(x)$. Plotted are time averages corresponding to equations (5.1) and (5.2), versus $T \in [0, 20]$, as indicated. The lower and upper dotted lines represent the theoretical values of limits: $2\|h_i\|_\alpha^\alpha$, $4\|h_i\|_\alpha^{2\alpha}$, respectively. Their values are presented in the following table:

| Theoretical values | | |
|---|---|---|
| function | 1st value | 2nd value |
| $h_1$ | 2.000 | 4.000 |
| $h_2$ | 1.821 | 3.310 |
| $h_3$ | 1.998 | 3.990 |
| $h_4$ | 2.447 | 5.960 |

In all four cases the curves which represent the time averages of the $S1.7S$ harmonizable process do not approach the corresponding theoretical limit values indicated by the dotted lines. As we pointed out, $S\alpha S$ harmonizable processes are never ergodic for $\alpha < 2$. Thus, this example illustrates a typical non-ergodic behavior of the $S\alpha S$ harmonizable processes. Note that in a sharp contrast to the Gaussian case, for $\alpha < 2$ the Ornstein–Uhlenbeck process is never harmonizable since the first one is ergodic and the second is not so.

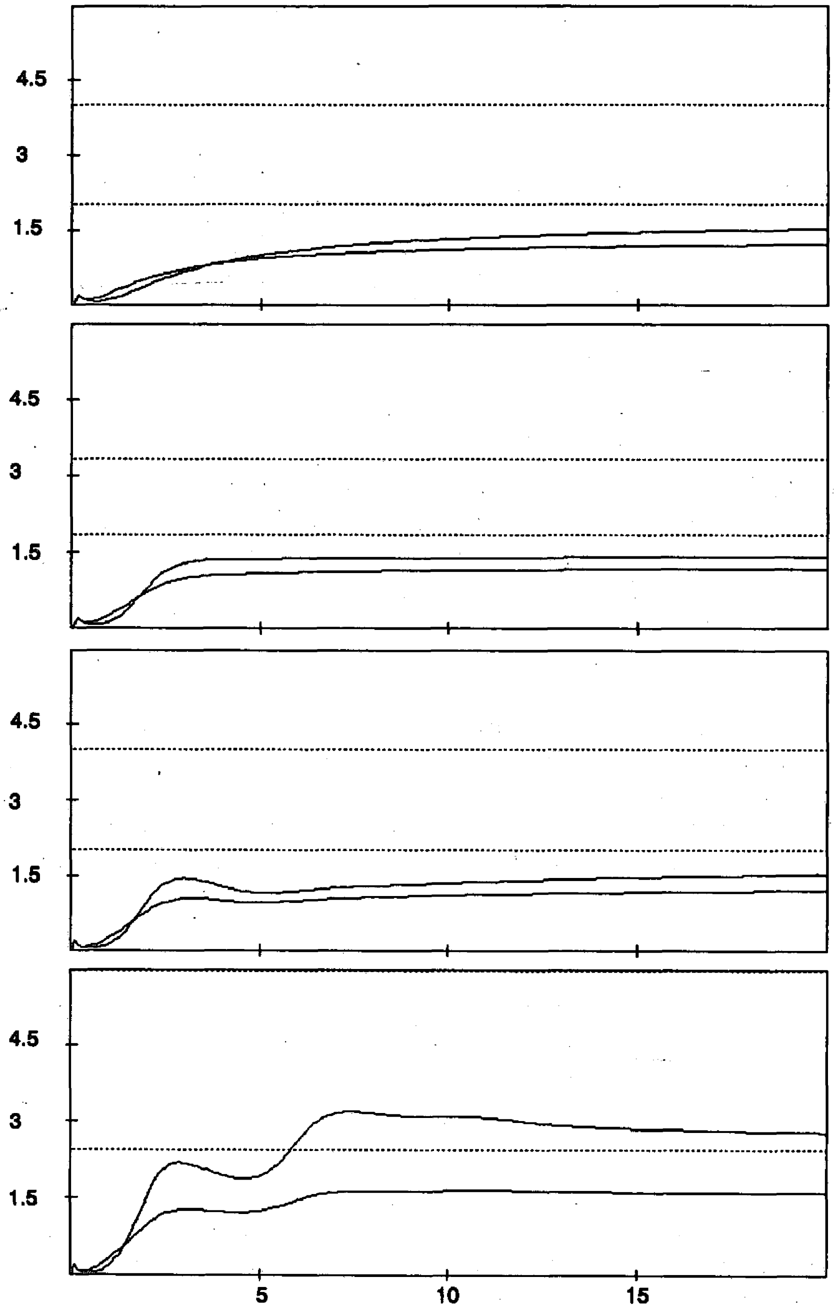


Fig. 5.1. Illustration of the fact that the symmetric 1.7-stable harmonizable process is not ergodic

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