PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 16, Fasc. 1 (1996), pp. 157–176

NEGATIVE DEFINITE FUNCTIONS AND CONVOLUTION SEMIGROUPS OF PROBABILITY MEASURES ON A COMMUTATIVE HYPERGROUP

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Abstract. Corresponding to the definitions of positive definite functions there are various approaches to defining negative definite functions on hypergroups. These range from the obvious "pointwise" definition to axiomatization via the Schoenberg duality. Researchers in this area have used definitions best suited to their immediate purposes. In this paper we present a comprehensive treatment of negative definite functions on commutative hypergroups, leading to convolution semigroups of probability measures and their Lévy–Khintchine representation within the framework of commutative hypergroups on subsets of Euclidean space.

Throughout this paper the analysis will be carried out on a commutative hypergroup K for which our fundamental reference is [17]. In general, we follow Jewett's notation except that the point measure at $x \in K$ will be denoted by ε_x, and ω_K, π_K will denote respectively Haar measure on K and its associated Plancherel measure. For an overview of probability theory on a hypergroup the reader is referred to [15]. Important to our treatment will be the knowledge of the different spaces of positive definite functions which were considered in detail in [8], and which we introduce in Section 1. Negative definite functions on K and $K^\wedge$ are the subjects of Sections 2 and 3 respectively, and in Sections 4 and 5 we present a range of examples to support the theory.

1. POSITIVE DEFINITENESS ON HYPERGROUPS AND THEIR DUALS

Since negative definite functions are in each framework defined in duality to positive definite functions, we shall briefly report on the various concepts of positive definiteness which in the case of commutative hypergroups have been compared in [8].

* This article was prepared in part while the first-named author held an Alexander von Humboldt fellowship at the University of Tübingen.

By analogy with the group case we introduce, for any hypergroup K, *positive definite functions* as complex-valued functions f on K that are measurable, locally bounded and satisfy

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{a}_j f(x_i * x_j^-) \geqslant 0$$

whenever $a_1, a_2, \ldots, a_n \in C$ and $x_1, x_2, \ldots, x_n \in K$. Clearly, every semicharacter of K is positive definite. It can be easily seen that positive definite functions on K need be neither bounded nor continuous. On the other hand, continuity of a positive definite function on K at e implies continuity everywhere. By $P(K)$ and $P_b(K)$ we denote the sets of continuous positive definite and bounded continuous positive definite functions on K, respectively. In the sequel we shall apply among other properties the fact that the set $P_b(K)$ is closed in the topology of compact convergence, and that Bochner's theorem holds in the form of the equality (in fact, a homeomorphism)

$$P_b(K) = M_+^b(K^\wedge)^\vee .$$

There are three other types of positive definiteness appearing in the literature. Let S be a subspace of $M^b(K)$. A locally bounded measurable function f on K is said to be *S-positive definite* if

$$\int_K f d(\mu * \mu^\sim) \geqslant 0$$

for all $\mu \in S$, and f is said to be *S-strongly positive definite* if

$$\int_K f d\mu \geqslant 0$$

whenever $\mu \in S$ with $\hat{\mu} \geqslant 0$. In addition we shall consider *strongly positive definite* functions f on K in the sense that $f = \check{\sigma}$ for some $\sigma \in M_+^b(K^\wedge)$.

We use the suggestive notation $P_S(K)$, $P_S^{(s)}(K)$ and $P_B(K)$ for the bounded continuous elements in the three classes defined above.

Prominent choices of S are S_0, the space of finitely supported measures, and $S_1 = M_a(K)$, the space of ω_K-absolutely continuous measures. Further choices proposed in [13], [26] and [18] are

$$S_2 := \{\mu \in M^b(K) \colon \mu = g\omega_K, \, g \in C_c(K)\},$$

$$S_3 := \{\mu \in M^b(K) \colon \mu = c\varepsilon_e + g\omega_K \text{ with } c \in C \text{ and } g \in C_c(K)\},$$

and $S_4 := M^b(K)$, respectively. The following results are straightforward once Bochner's theorem is known.

1.1. THEOREM. *For $i = 0, 1, 2, 3, 4$ the sets $P_{S_i}(K)$ coincide.*

1.2. THEOREM. *For $i = 0, 1, 2, 3, 4$,*

$$P_B(K) = P_{S_i}(K) = P_{S_i}^{(s)}(K).$$

Next we shall introduce positive definiteness for functions on the dual space $K^\wedge$ of a commutative hypergroup K.

For any subspace T of $M^b(K^\wedge)$ containing ε_1 (where 1 is the unit character) a locally bounded measurable function f on K is called *T-strongly positive definite* if for every $\mu \in T$ with $\check{\mu} \geqslant 0$ we have

$$\int_{K^\wedge} f d\mu \geqslant 0,$$

and f is called *strongly positive definite* if there exists a measure $\mu \in M^b_+(K)$ such that $\hat{\mu} = f$. The collections of bounded continuous T-strongly positive definite and strongly positive definite functions on $K^\wedge$ will be abbreviated by $P_T^{(s)}(K^\wedge)$ and $P_B(K^\wedge)$, respectively. Again we present the prominent choices T_0 of finitely supported measures on $K^\wedge$, $T_1 := M^b(K^\wedge)$ and

$$T_2 := \{\mu \in M^b(K^\wedge)\colon \mu = c\varepsilon_1 + g\pi_K \text{ with } c \in \boldsymbol{C},\ g \in C_c(K^\wedge)\}.$$

The latter two classes have been introduced in [18] and [27], respectively.

Concerning the relationship between these classes of positive definite functions on $K^\wedge$ we quote the following results.

1.3. Theorem. $P_B(K^\wedge) \subset P_{T_1}^{(s)}(K^\wedge) \subset P_{T_2}^{(s)}(K^\wedge)$, *where equality holds if* $\operatorname{supp}(\pi_K) = K^\wedge$.

1.4. Theorem. *If K is strong we have*

$$P_{T_1}^{(s)}(K^\wedge) \subset P_{T_0}(K^\wedge),$$

and in the case where K is compact, equality holds if and only if K is Pontryagin.

2. NEGATIVE DEFINITE FUNCTIONS ON HYPERGROUPS

2.1. Definition. A complex-valued function f on K will be called *negative definite* if it is measurable, locally bounded and satisfies $f(e) \geqslant 0$, $f^- = \bar{f}$, and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{a}_j f(x_i * x_j^-) \leqslant 0$$

for every $n \in \boldsymbol{N}$, every choice of $a_1, a_2, \ldots, a_n \in \boldsymbol{C}$ satisfying $\sum_{i=1}^n a_i = 0$, and every choice of $x_1, x_2, \ldots, x_n \in K$.

Negative definite functions need be neither bounded nor continuous. The set of negative definite functions on K will be denoted by $N(K)$. Clearly, each constant function $c1 \in N(K)$ for all $c \geqslant 0$.

It is quite standard to show that a locally bounded measurable function f is negative definite if and only if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{a}_j \big(f(x_i) + \overline{f(x_j)} - f(x_i * x_j^-)\big) \geqslant 0 \tag{2.1.1}$$

for all choices of $n \in \mathbf{N}$, $a_1, a_2, \ldots, a_n \in \mathbf{C}$, and $x_1, x_2, \ldots, x_n \in K$; see for example [6], Chapter 4, Proposition 1.9.

For the following two results we do not assume K to be commutative.

2.2. PROPOSITION. *Let $f \in N(K)$. For all $x, y \in K$*

(a) $f(x * x^-) \in \mathbf{R}$.

(b) $f(x * x^-) + f(y * y^-) \leqslant 2\mathrm{Re}(f(x^- * y))$.

(c) $f(x * x^-) + f(e) \leqslant f(x) + f(x^-)$.

(d) $f(x * x^-) \geqslant 0$ *if and only if* $\mathrm{Re}(f) \geqslant 0$.

(e) $|f(x) + \overline{f(y)} - f(x * y^-)|^2 \leqslant (f(x) + \overline{f(x)} - f(x * x^-))(f(y) + \overline{f(y)} - f(y * y^-))$.

(f) $f - f(e) \in N(K)$.

(g) *If* $\mathrm{Re}(f)$ *is bounded below, then* $\mathrm{Re}(f) \geqslant f(e) \geqslant 0$.

(h) *If* $\mathrm{Re}(f) \geqslant 0$, *then* $|f(x * y)|^{1/2} \leqslant |f(x)|^{1/2} + |f(y)|^{1/2}$.

If $g \in P(K)$, then $g(e) - g \in N(K)$.

Proof. The proofs of (a)–(d), (f) and the last statement can be shown directly. To prove (e) take $x_1 = x$ and $x_2 = y$ in (2.1.1), which implies that the matrix

$$\begin{pmatrix} f(x) + \overline{f(x)} - f(x * x^-) & f(x) + \overline{f(y)} - f(x * y^-) \\ f(y) + \overline{f(x)} - f(y * x^-) & f(y) + \overline{f(y)} - f(y * y^-) \end{pmatrix}$$

is positive definite, and hence has nonnegative determinant as required. For (g) we first note that [20], Proposition 1.3, gives the result when $\mathrm{Re}(f) \geqslant 0$ is assumed. In general, if we only know that $\mathrm{Re}(f) \geqslant c$, then $f - c$ satisfies $\mathrm{Re}(f - c) \geqslant 0$. We also observe that $f - c = f - f(e) + f(e) - c \in N(K)$ using (f), and applying the first part we see that $\mathrm{Re}(f) - c = \mathrm{Re}(f - c) \geqslant f(e) - c$, and again $\mathrm{Re}(f) \geqslant f(e)$. For (h) see [20], Proposition 1.4. Finally, we just appeal to Definition 2.1 to complete the proof of the theorem. ∎

It is of independent interest that the continuity of negative definite functions is determined by their continuity at e. This can be proved using Proposition 2.2 (e); see [9], Theorem 1.10.

2.3. THEOREM. *A negative definite function f that is continuous at e is continuous everywhere.*

It is convenient to note that $(M_c^1(K), {}^{\sim})$ is an Abelian semigroup with neutral element ε_e and involution ${}^{\sim}$ in which K can be naturally embedded via the mapping $x \to \varepsilon_x$. Then each continuous negative definite function f has a natural extension to a negative definite function F on $M_c^1(K)$ given by $F(\mu) := \int_K f \, d\mu$ for all $\mu \in M_c^1(K)$. Using this technique many results on negative definite functions on semigroups can be transferred to hypergroups. For example, it can easily be shown that for every bounded continuous negative definite function f there exists $c \in \mathbf{R}$ such that $c - f$ is positive definite. For details of the method see [9].

2.4. DEFINITION. Let S be a subspace of $M^b(K)$. A locally bounded measurable function f is called *S-weakly negative definite* if $f(e) \geqslant 0$, $f^{-} = \bar{f}$, and $\int_K f d\nu \leqslant 0$ for all $\nu \in S$ with $\hat{\nu} \geqslant 0$ and $\hat{\nu}(1) = 0$.

We denote by $N_S^{(w)}(K)$ the space of continuous S-weakly negative definite functions on K.

2.5. DEFINITION. Let S be a subspace of $M^b(K)$. A locally bounded measurable function f is called *S-weakly′ negative definite* if $f(e) \geqslant 0$, $f^{-} = \bar{f}$, and $\int_K f d(\nu * \nu^{\sim}) \leqslant 0$ for all $\nu \in S$ with $\hat{\nu}(1) = 0$.

We denote by $N_S^{(w')}(K)$ the space of continuous S-weakly′ negative definite functions on K. Note that $N_{S_0}^{(w')}(K) = N(K) \cap C(K)$, where S_0 is the subspace of $M^b(K)$ consisting of all finitely supported measures.

2.6. DEFINITION. Let S be a subspace of $M^b(K)$. A locally bounded measurable function f is called *S-negative definite* if $f(e) \geqslant 0$ and $\exp(-tf) \in P_S(K)$ for all $t > 0$, and f is called *S-strongly negative definite* if $f(e) \geqslant 0$ and $\exp(-tf) \in P_S^{(s)}(K)$ for all $t > 0$.

We denote by $N_S(K)$ (respectively, $N_S^{(s)}(K)$) the space of continuous S- (respectively, S-strongly) negative definite functions on K. Clearly, if $S \subset S'$ then

$$N_{S'}^{(w)}(K) \subset N_S^{(w)}(K), \qquad N_{S'}^{(w')}(K) \subset N_S^{(w')}(K),$$

$$N_{S'}(K) \subset N_S(K) \quad \text{and} \quad N_{S'}^{(s)}(K) \subset N_S^{(s)}(K).$$

It is also easy to see that

$$N_S^{(s)}(K) \subset N_S(K) \quad \text{and} \quad N_S^{(w)}(K) \subset N_S^{(w')}(K).$$

Note that for both S-negative definite and S-strongly negative definite functions f it is part of the definition that $\exp(-tf)$ be bounded, which is equivalent to $\mathrm{Re}(f)$ being bounded below.

2.7. LEMMA. *If $S * C_c(K)\omega_K \subset S'$, then $N_{S'}^{(w)}(K) \subset N_S^{(w)}(K)$ and $N_{S'}^{(w')}(K) \subset N_S^{(w')}(K)$.*

Proof. We deal only with the first of the two inclusions as the proof of the second is similar. Choose (k_V) in $C_c^+(K)$, a bounded approximate unit for $L^1(K)$ with $\hat{k}_V \geqslant 0$. Then for $\nu \in S$ satisfying $\hat{\nu}(1) = 0$ and $\hat{\nu} \geqslant 0$ it is the case that $\nu_V := (\nu * k_V)\omega_K \in S'$ (by assumption) with $\hat{\nu}_V(1) = 0$ and $\hat{\nu}_V \geqslant 0$. Hence for $f \in N_{S'}^{(w)}(K)$

$$\int_K f d\nu = \lim_V \int_K f d\nu_V \leqslant 0,$$

so that $f \in N_S^{(w)}(K)$. ∎

Lasser's pointwise definition of negative definiteness (see [19] and [20]) is just S_0-weak negative definiteness together with continuity. Other choices of

S treated in the literature correspond to those given for positive definiteness in Section 1.

S-negative definiteness and S-strong negative definiteness have been introduced via an axiomatization of the Schoenberg duality. A particularly interesting choice in the latter case is when $\exp(-tf) \in P_B(K)$ (cf. [7], Definition 2.1). We denote by $N_B^{(s)}(K)$ the corresponding space of continuous negative definite functions.

We have the following general properties of negative definite functions.

2.8. PROPOSITION. (a) *For* $f \in N_{S_0}^{(w)}(K)$,

$$\operatorname{Re}(f) \geqslant f(e) \geqslant 0.$$

(b) *If* $f, g \in N_S^{(w)}(K)$ *(respectively,* $N_S^{(w')}(K)$, $N_B^{(s)}(K)$*),* $c > 0$, *then*

$$\bar{f},\ cf,\ f+g,\ \operatorname{Re}(f) \in N_S^{(w)}(K) \text{ (respectively, } N_S^{(w')}(K),\ N_B^{(s)}(K)).$$

(c) *If* $f \in N_S^{(w)}(K)$ *(respectively,* $N_S^{(w')}(K)$, $N_B^{(s)}(K)$*), then*

$$f - f(e) \in N_S^{(w)}(K) \text{ (respectively, } N_S^{(w')}(K),\ N_B^{(s)}(K)).$$

(d) *If* $f \in P_S^{(s)}(K)$ *(respectively,* $P_S(K)$*), then*

$$f(e) - f \in N_S^{(w)}(K) \text{ (respectively, } N_S^{(w')}(K)).$$

(e) *If* (f_n) *in* $N_S^{(w)}(K)$ *(respectively,* $N_S^{(w')}(K)$, $N_B^{(s)}(K)$*) converges uniformly on compact subsets of* K *to* f, *then*

$$f \in N_S^{(w)}(K) \text{ (respectively, } N_S^{(w')}(K),\ N_B^{(s)}(K)).$$

Proof. (a) Fix $x \in K$ and consider $\mu := 2\varepsilon_e - \varepsilon_x - \varepsilon_{x^-} \in S_0$. We have for all $\chi \in K^\wedge$

$$\hat{\mu}(\chi) = 2 - \overline{\chi(x)} - \chi(x) \geqslant 0 \quad \text{and} \quad \hat{\mu}(1) = 0.$$

Thus $\int_K f d\mu \leqslant 0$, and since $f^- = \bar{f}$, we have

$$2f(e) \leqslant f(x) + f(x^-) = 2\operatorname{Re}(f(x)).$$

The proofs of (b)–(e) are straightforward. ∎

2.9. PROPOSITION. *Let* $f \in N_{S_3}^{(w)}(K)$ *and* $\lambda > 0$. *Then* $R_\lambda := (\lambda + f)^{-1} \in P_B(K)$.

Proof. The function R_λ is continuous and, by Proposition 2.8 (a), bounded. In view of Theorem 1.2 we need only show that $R_\lambda \in P_{S_2}^{(s)}(K)$. Consider $\mu := g\omega_K \in S_2$ $(g \in C_c(K))$ with $\hat{\mu} \geqslant 0$. Using $(\mu^\sim)^\wedge = (\hat{\mu})^- = \hat{\mu}$ and the uniqueness of the Fourier transform we have $\mu = \mu^\sim$, and hence $g = g^\sim$. Note that since f and R_λ satisfy $f = f^\sim$ and $R_\lambda = R_\lambda^\sim$ respectively, all the integrals that appear in the remaining part of this proof are real-valued. Also in view of Proposition 2.8 (c) we can assume that $f(e)=0$.

Suppose that $\int_K R_\lambda d\mu < 0$. Now $R_\lambda \mu \in S_2$ and

$$h := (R_\lambda \mu)^\wedge = \left(\frac{g}{\lambda+f}\right)^\wedge \in C_0(K^\wedge).$$

Since by our assumption $h(1) < 0$, there exists $\chi_0 \in K^\wedge$ such that the real-valued function h takes its minimum on $K^\wedge$ at χ_0. Define $\nu \in S_3$ by $\nu := h(\chi_0)\varepsilon_e - \overline{\chi_0} R_\lambda \mu$. Then ν satisfies $\hat{\nu}(1) = 0$, $\nu = \nu^\sim$, and for all $\chi \in K^\wedge$

$$\hat{\nu}(\chi) = h(\chi_0) - \int_K \overline{\chi\chi_0} R_\lambda d\mu = h(\chi_0) - \int_{K^\wedge} \int_K \bar{\eta} R_\lambda d\mu (\varepsilon_\chi * \varepsilon_{\chi_0})(d\eta)$$
$$= h(\chi_0) - \int_{K^\wedge} h d(\varepsilon_\chi * \varepsilon_{\chi_0}) \leqslant 0.$$

Therefore, as $f \in N_{S_3}^{(w)}(K)$ with $f(e) = 0$,

$$0 \leqslant \int_K f d\nu = -\int_K f \overline{\chi_0} R_\lambda d\mu$$

and

$$\lambda \int_K R_\lambda d\mu = \lambda h(1) \geqslant \lambda h(\chi_0) = \int_K \lambda \overline{\chi_0} R_\lambda d\mu \geqslant \int_K (\lambda+f) R_\lambda \overline{\chi_0} d\mu = \hat{\mu}(\chi_0) \geqslant 0,$$

contradicting our assumption, and thus the proposition is proved. ∎

2.10. THEOREM. $N_{S_3}^{(w)}(K) \subset N_B^{(s)}(K)$.

Proof. Let $f \in N_{S_3}^{(w)}(K)$. By Proposition 2.9, $R_\lambda := (\lambda+f)^{-1} \in P_B(K)$ for each $\lambda > 0$. Write

$$f_\lambda := \lambda f(\lambda+f)^{-1} = \lambda^2(\lambda^{-1} - (\lambda+f)^{-1}).$$

Now

$$\exp(-tf_\lambda) = \exp(-t\lambda)\exp(t\lambda^2 R_\lambda)$$
$$= \exp(-t\lambda)\left(1 + t\lambda^2 R_\lambda + \tfrac{1}{2}(t\lambda^2)^2 R_\lambda^2 + \ldots\right) \in P_B(K).$$

Since $\lim_\lambda f_\lambda = f$ on K, we have $\lim_\lambda \exp(-tf_\lambda) = \exp(-tf)$ for all $t > 0$. Now $\exp(-tf) \in C_b(K)$, and hence $\exp(-tf) \in P_B(K)$ for all $t > 0$. Appealing to [8], Theorems 1.7 and 1.17, there exist $\mu_t \in M_+^b(K^\wedge)$ with $\check{\mu}_t = \exp(-tf)$, which just says that $f \in N_B^{(s)}(K)$. ∎

2.11. THEOREM. *We have*

$$N_B^{(s)}(K) = N_{S_i}^{(s)}(K) = N_{S_j}(K) = N_{S_k}^{(w)}(K) \subset N_{S_0}^{(w)}(K) \subset N_{S_0}^{(w')}(K) = N_{S_l}^{(w')}(K)$$

for all $i, j \in \{0, 1, 2, 3, 4\}$ and $k, l \in \{1, 2, 3, 4\}$. Furthermore, there exist hypergroups for which the second inclusion is proper.

Proof. The equalities $N_B^{(s)}(K) = N_{S_i}^{(s)}(K) = N_{S_j}(K)$ for $i, j \in \{0, 1, 2, 3, 4\}$ follow immediately from [8], Theorem 1.17, which is just the corresponding statement for the underlying spaces of positive definite functions. Now the obvious inclusions between the test spaces give

$$N_{S_4}^{(w)}(K) \subset N_{S_1}^{(w)}(K) \subset N_{S_2}^{(w)}(K), \quad N_{S_4}^{(w)}(K) \subset N_{S_3}^{(w)}(K) \subset N_{S_2}^{(w)}(K)$$

and

$$N_{S_4}^{(w)}(K) \subset N_{S_0}^{(w)}(K).$$

Appealing to Lemma 2.7 we infer from $S_0 * C_c(K)\omega_K \subset S_2$ and $S_3 * C_c(K)\omega_K \subset S_2$ that

$$N_{S_2}^{(w)}(K) \subset N_{S_0}^{(w)}(K) \quad \text{and} \quad N_{S_2}^{(w)}(K) \subset N_{S_3}^{(w)}(K).$$

Putting these inclusions together gives

$$N_{S_4}^{(w)}(K) \subset N_{S_1}^{(w)}(K) \subset N_{S_2}^{(w)}(K) = N_{S_3}^{(w)}(K) \subset N_{S_0}^{(w)}(K).$$

A similar argument gives the analogous statement for S-weakly$'$ negative definite functions.

We now show that

$$N_B^{(s)}(K) \subset N_{S_4}^{(w)}(K).$$

First consider $f \in N_B^{(s)}(K)$ so that for each $t > 0$ there exists $\nu_t \in M_+^b(K^\wedge)$ such that $\check{\nu}_t = \exp(-tf)$. An immediate consequence is that $f^- = \bar{f}$. Moreover, for $\mu \in S_4$ with $\hat{\mu} \geqslant 0$ and $\hat{\mu}(\mathbb{1}) = 0$,

$$\int_K f\,d\mu = \int_K \lim_{t\to 0} t^{-1}[1-\exp(-tf)]\,d\mu = \lim_{t\to 0} t^{-1} \int_K [1-\exp(-tf)]\,d\mu$$

$$= -\lim_{t\to 0} t^{-1} \int_K \check{\nu}_t\,d\mu = -\lim_{t\to 0} t^{-1} \int_K \hat{\mu}\,d\nu_t^- \leqslant 0,$$

which gives $f \in N_{S_4}^{(w)}(K)$.

To complete the equalities between $N_B^{(s)}(K)$ and the spaces of S_k-weakly definite functions for $k \in \{1, 2, 3, 4\}$ just use Theorem 2.10.

We next show that $N_{S_0}^{(w')}(K) \subset N_{S_4}^{(w')}(K)$ which will give equality of the spaces of S_l-weakly$'$ negative definite functions. Indeed, let $f \in N_{S_0}^{(w')}(K)$ and choose $\mu \in M^b(K)$ with $\hat{\mu}(\mathbb{1}) = 0$. Now there exists (μ_α) in S_0 with $\lim_\alpha \mu_\alpha = \mu$, and replacing μ_α by $\mu_\alpha - \hat{\mu}_\alpha(\mathbb{1})\varepsilon_e$ if necessary, we can further assume that $\hat{\mu}_\alpha(\mathbb{1}) = 0$ for all α. By the choice of f we have

$$\int_K f\,d(\mu * \mu^\sim) = \lim_\alpha \int_K f\,d(\mu_\alpha * \mu_\alpha^\sim) \leqslant 0$$

so that $f \in N_{S_4}^{(w')}(K)$.

Finally, we observe that by Proposition 2.8 (a), $\mathrm{Re}(f) \geqslant f(e) \geqslant 0$ for all $f \in N_{S_0}^{(w)}(K)$. It is known (see [28], Remark 4.8) that for every polynomial hypergroup $(\boldsymbol{Z}_+, *)$ there exists $f \in N_{S_0}^{(w')}(\boldsymbol{Z}_+)$ for which $f(n) < 0$ for all $n \in N$ (in fact, $f(n) = -P_n'(x)$ is such a function). This shows that $N_{S_0}^{(w)}(K)$ could be a proper subset of $N_{S_0}^{(w')}(K)$, and this completes the proof of the theorem. ∎

2.12. Remark. In view of the preceding result it is not possible in general to prove that

$$\sum_{i,j=1}^{n} c_i \bar{c}_j \exp(-tf(x_i * x_j^-)) = \sum_{i,j=1}^{n} c_i \bar{c}_j \exp(-tf)(x_i * x_j^-)$$

(cf. [28], Remark 2.6 (c)) which would give $N_B^{(s)}(K) = N_{S_0}^{(w')}(K)$, and hence equality of all the spaces in Theorem 2.11.

3. NEGATIVE DEFINITE FUNCTIONS ON THE DUAL SPACE OF A COMMUTATIVE HYPERGROUP

Negative definiteness has an analogous interpretation for the dual $K^\wedge$ when K is commutative.

3.1. Definition. Let T be any subspace of $M^b(K^\wedge)$ containing ε_1. A locally bounded measurable function f on $K^\wedge$ is called *T-weakly negative definite* if $f(\mathbf{1}) \geqslant 0, f^- = \bar{f}$ and $\int_{K^\wedge} f d\nu \leqslant 0$ for all $\nu \in T$ with $\check{\nu} \geqslant 0$ and $\check{\nu}(e) = 0$.

We denote by $N_T^{(w)}(K^\wedge)$ the space of continuous T-weakly negative definite functions on K. It should be noted that membership of f in $N_T^{(w)}(K^\wedge)$ (see below) is only affected by the behaviour of f on $\mathrm{supp}(T)$.

3.2. Definition. Let T be any subspace of $M^b(K^\wedge)$ containing ε_1. A locally bounded measurable function f on $K^\wedge$ is called *T-strongly negative definite* if $f(\mathbf{1}) \geqslant 0$ and $\exp(-tf) \in P_T^{(s)}(K^\wedge)$ for all $t > 0$.

We denote by $N_T^{(s)}(K^\wedge)$ the space of continuous T-strongly negative definite functions on $K^\wedge$. Note that $\mathbf{1} \in N_T^{(s)}(K^\wedge)$ for any such subspace T. Also $N_B^{(s)}(K^\wedge)$ denotes the corresponding strong space with $P_T^{(s)}(K^\wedge)$ replaced by $P_B(K^\wedge)$. T-strongly negative definite functions have been studied for the various choices of $T \subset M^b(K^\wedge)$ appearing in Section 1.

3.3. Proposition. *We have*

$$N_B^{(s)}(K^\wedge) \subset N_{T_1}^{(s)}(K^\wedge) \subset N_{T_2}^{(s)}(K^\wedge),$$

$$N_{T_1}^{(w)}(K^\wedge) \subset N_{T_2}^{(w)}(K^\wedge) \quad \textit{and} \quad N_T^{(s)}(K^\wedge) \subset N_T^{(w)}(K^\wedge).$$

Proof. The first line of inclusions follows from [8], Proposition 2.4. The inclusion $N_{T_1}^{(w)}(K^\wedge) \subset N_{T_2}^{(w)}(K^\wedge)$ follows from the inclusion $T_2 \subset T_1$. To prove that $N_T^{(s)}(K^\wedge) \subset N_T^{(w)}(K^\wedge)$ consider $\nu \in T$ with $\check{\nu} \geqslant 0$ and $\check{\nu}(e) = 0$, and let

$f \in N_T^{(s)}(K^\wedge)$. Then

$$\int_{K^\wedge} f\,dv = \int_{K^\wedge} \lim_{t\to 0} t^{-1}(1-\exp(-tf))\,dv$$
$$= \lim_{t\to 0} t^{-1} \int_{K^\wedge} (1-\exp(-tf))\,dv = -\lim_{t\to 0} t^{-1} \int_{K^\wedge} \exp(-tf)\,dv \leqslant 0,$$

the last equality and the inequality using $\check{v}(e) = 0$ and $\exp(-tf) \in P_T^{(s)}(K^\wedge)$, respectively. ∎

3.4. Proposition. (a) *For $f \in N_{T_2}^{(w)}(K^\wedge)$ and $\gamma \in \operatorname{supp}(\pi_K)$,*

$$\operatorname{Re}(f(\gamma)) \geqslant f(1) \geqslant 0.$$

(b) *If $f, g \in N_T^{(s)}(K^\wedge)$ (respectively, $N_T^{(w)}(K^\wedge)$), and $c > 0$, then*

$$\bar{f},\ cf,\ f+g,\ \operatorname{Re}(f) \in N_T^{(s)}(K^\wedge) \ (\textit{respectively, } N_T^{(w)}(K^\wedge)).$$

(c) *If $f \in N_T^{(s)}(K^\wedge)$ (respectively, $N_T^{(w)}(K^\wedge)$), then*

$$f - f(1) \in N_T^{(s)}(K^\wedge) \ (\textit{respectively, } N_T^{(w)}(K^\wedge)).$$

(d) *If $f \in P_T^{(s)}(K^\wedge)$ (respectively, $P_B(K^\wedge)$), then*

$$f(1) - f \in N_T^{(w)}(K^\wedge) \ (\textit{respectively, } N_B^{(s)}(K^\wedge)).$$

(e) *If (f_n) in $N_T^{(s)}(K^\wedge)$ (respectively, $N_T^{(w)}(K^\wedge)$) converges uniformly on compact subsets of $K^\wedge$ to f, then*

$$f \in N_T^{(s)}(K^\wedge) \ (\textit{respectively, } N_T^{(w)}(K^\wedge)).$$

Proof. The proof of (a) is given in [27], Lemma 3.5. The other parts are straightforward. ∎

3.5. Remark. Proposition 3.4 (a) should be compared with the following result: For $f \in N_{T_1}^{(s)}(K^\wedge)$ and for all $\gamma \in K^\wedge$, $\operatorname{Re}(f(\gamma)) \geqslant f(1) \geqslant 0$. Indeed, $\exp(-tf) \in P_{T_1}^{(s)}(K^\wedge)$ so that

$$\exp\big(-t\operatorname{Re}(f(\gamma))\big) \leqslant \|\exp(-tf)\|_\infty \leqslant \exp(-tf(1)) \quad \text{for all } \gamma \in K^\wedge,\ t > 0,$$

and the desired inequality follows. Note that $N_{T_1}^{(s)}(K^\wedge)$ is the smaller space, giving a stronger result, in which case it is probably true that the behaviour of f off $\operatorname{supp}(\pi_K) \cup \{1\}$ is already determined.

3.6. Proposition. *Let $f \in N_{T_2}^{(w)}(K^\wedge)$ and $\lambda > 0$. Then $R_\lambda := (\lambda + f)^{-1}$ is continuous and bounded on $\operatorname{supp}(\pi_K) \cup \{1\}$, and every continuation of R_λ to a bounded continuous function on $K^\wedge$ belongs to $P_{T_2}^{(s)}(K^\wedge)$.*

The proof of Proposition 3.6 can be found in [27], Lemma 3.6.

3.7. Remark. As in Proposition 3.6, for $f \in N_{T_1}^{(s)}(K^\wedge)$ and $\lambda > 0$ we have $(\lambda+f)^{-1} \in P_{T_1}^{(s)}(K^\wedge)$. The proof which is somewhat more direct goes as follows. Let $\nu \in M^b(K^\wedge)$ and assume $f(1) > 0$. Then, using Remark 3.5, we obtain

$$\int_{K^\wedge}\int_{\mathbf{R}_+} |e^{-tf}|\, dt\, d|\nu| \leqslant \int_{K^\wedge}\int_{\mathbf{R}_+} e^{-tf(1)}\, dt\, d|\nu| = f(1)^{-1}|\nu|(K^\wedge) < \infty.$$

Applying Fubini's theorem gives for $\check{\nu} \geqslant 0$

$$\int_{K^\wedge} f^{-1}\, d\nu = \int_{K^\wedge}\int_{\mathbf{R}_+} e^{-tf}\, dt\, d\nu = \int_{\mathbf{R}_+}\int_{K^\wedge} e^{-tf}\, d\nu\, dt \geqslant 0$$

as $e^{-tf} \in P_{T_1}^{(s)}(K^\wedge)$, so that $f^{-1} \in P_{T_1}^{(s)}(K^\wedge)$. Now consider arbitrary $f \in N_{T_1}^{(s)}(K^\wedge)$ and $\lambda > 0$. By Proposition 3.4 (b) we have $\lambda+f \in N_{T_1}^{(s)}(K^\wedge)$, and using the argument above we obtain $(\lambda+f)^{-1} \in P_{T_1}^{(s)}(K^\wedge)$.

A key result in the theory of negative definite functions is the Schoenberg correspondence which was given in [27], Theorem 3.7.

3.8. Theorem. (a) *To each continuous convolution semigroup $(\mu_t)_{t\geqslant 0}$ on K there corresponds a uniquely determined function $f \in N_B^{(s)}(K^\wedge)$ with $\hat{\mu}_t = \exp(-tf)$ for all $t > 0$.*

(b) *To each $f \in N_{T_2}^{(w)}(K^\wedge)$ there corresponds a uniquely determined continuous convolution semigroup $(\mu_t)_{t\geqslant 0}$ on K satisfying*

$$\hat{\mu}_t|_{\operatorname{supp}(\pi_K)} = \exp(-tf)|_{\operatorname{supp}(\pi_K)} \quad \textit{and} \quad \hat{\mu}_t(1) \leqslant \exp(-tf(1)) \quad \textit{for all } t > 0.$$

We can use Theorem 3.8 to refine Proposition 3.3. The latter result shows that of the five spaces of negative definite functions in common use $N_B^{(s)}(K^\wedge)$ is the smallest, and $N_{T_2}^{(w)}(K^\wedge)$ the largest. However, these two spaces, and hence all five, agree when restricted to $\operatorname{supp}(\pi_K)$.

3.9. Proposition. *We have*

$$N_B^{(s)}(K^\wedge)|_{\operatorname{supp}(\pi_K)} = N_{T_1}^{(s)}(K^\wedge)|_{\operatorname{supp}(\pi_K)}$$
$$= N_{T_2}^{(s)}(K^\wedge)|_{\operatorname{supp}(\pi_K)} = N_{T_1}^{(w)}(K^\wedge)|_{\operatorname{supp}(\pi_K)} = N_{T_2}^{(w)}(K^\wedge)|_{\operatorname{supp}(\pi_K)}.$$

Proof. In view of the inclusions in Proposition 3.3 it suffices to prove that

$$N_{T_2}^{(w)}(K^\wedge)|_{\operatorname{supp}(\pi_K)} \subset N_B^{(s)}(K^\wedge)|_{\operatorname{supp}(\pi_K)}.$$

Let $f \in N_{T_2}^{(w)}(K^\wedge)$. By Theorem 3.8 (b) there exists a unique convolution semigroup $(\mu_t)_{t\geqslant 0}$ with

$$\hat{\mu}_t|_{\operatorname{supp}(\pi_K)} = \exp(-tf)|_{\operatorname{supp}(\pi_K)}.$$

Now use Theorem 3.8 (a) to deduce the existence of a uniquely determined function $g \in N_B^{(s)}(K^\wedge)$ with $\hat{\mu}_t = \exp(-tg)$ so that

$$\exp(-tf)|_{\operatorname{supp}(\pi_K)} = \exp(-tg)|_{\operatorname{supp}(\pi_K)}.$$

Since the function exp is one-to-one, we have

$$f|_{\mathrm{supp}(\pi_K)} = g|_{\mathrm{supp}(\pi_K)} \in N_B^{(s)}(K^\wedge)|_{\mathrm{supp}(\pi_K)},$$

which just proves that $N_{T_2}^{(w)}(K^\wedge)|_{\mathrm{supp}(\pi_K)} \subset N_B^{(s)}(K^\wedge)|_{\mathrm{supp}(\pi_K)}$ as required. ∎

4. APPLICATION: POSITIVE AND NEGATIVE DEFINITENESS WITH RESPECT TO A SUBSET OF THE DUAL SPACE

In the application of negative definite functions to continuous convolution semigroups and their associated additive (Lévy) processes further restricted versions of the original definitions occur (see [5]); we sketch one of these.

We are remaining within the framework of a commutative hypergroup K with sets K^* and $K^\wedge$ of continuous semicharacters and characters, respectively. Let L denote a nonempty subset of K^*.

4.1. Definition. A continuous function f on K is said to be *positive definite with respect to* L if for all $\mu \in M_c(K)$ with $\hat{\mu}(\chi) \geqslant 0$ whenever $\chi \in L$ we have

$$\int_K f d\mu \geqslant 0,$$

and f is said to be *negative definite with respect to* L provided $f(e) \geqslant 0, f^- = \bar{f}$ and for all $\mu \in M_c(K)$ with $\mu(K) = 0$ and $\hat{\mu}(\chi) \geqslant 0$ whenever $\chi \in L$ we have

$$\int_K f d\mu \leqslant 0.$$

The classes of functions on K that are positive or negative definite with respect to L will be denoted by

$$P^{(s)}(K, L) := P_{M_c(K)}(K, L) \quad \text{and} \quad N^{(w)}(K, L) := N^{(w)}_{M_c(K)}(K, L),$$

respectively. For the special choice $L := K^*$ we obtain

$$P^{(s)}(K, K^*) \subset P(K) \cap C(K) \quad \text{and} \quad P^{(s)}(K, K^*) \subset N(K) \cap C(K).$$

As for the reverse inclusions only partial results are available (see [26] and [28]).

4.2. Polynomial hypergroups in one variable. This class of discrete hypergroups together with its most known subclasses appears for example in [16].

For real-valued functions f on a polynomial hypergroup $(\boldsymbol{Z}_+, *(Q_n))$ the notions of positive and negative definiteness with respect to $L := K^* \simeq \boldsymbol{R}$ coincides with the unrestricted ones. Moreover, defining

$$T_f(Q_n) := f(n) \quad \text{for all } n \in \boldsymbol{Z}_+$$

we see that

(a) $f \in P^{(s)}(\boldsymbol{Z}_+, \boldsymbol{R})$ if and only if T_f is a positive linear functional on $\boldsymbol{R}[x]$ if and only if there exists a (not necessarily unique) measure $\mu \in M_+(\boldsymbol{R})$ satisfying

$$f(n) = \int_{\boldsymbol{R}} Q_n(x)\,\mu(dx) \quad \text{for all } n \in \boldsymbol{Z}_+.$$

(b) f with $f(0) = 0$ belongs to $N^{(w)}(\boldsymbol{Z}_+, \boldsymbol{R})$ if and only if T_f is a linear functional on $\boldsymbol{R}[x]$ satisfying

$$T_f(Q) \leqslant 0$$

for all $Q \in \boldsymbol{R}[x]$ with $Q(1) = 0$ and $Q(x) \geqslant 0$ whenever $x \in \boldsymbol{R}$.

In the special case where $L := K^\wedge \simeq D_S := [-1, 1]$ and $f(0) = 0$ the following conditions are equivalent:

(i) $f \in N^{(w)}(\boldsymbol{Z}_+, [-1, 1])$.

(ii) $T_f(Q) \leqslant 0$ for all $Q \in \boldsymbol{R}[x]$ with $Q(1) = 0$ and $Q(x) \geqslant 0$ whenever $x \in [-1, 1]$.

(iii) The mapping

$$n \to T\left(\frac{1-Q_n}{1-\mathrm{Id}}\right) := f(n)$$

from $\boldsymbol{Z}_+$ into $\boldsymbol{R}$ gives rise to a linear function T on $\boldsymbol{R}[x]$ satisfying $T(Q) \leqslant 0$ for all $Q \in \boldsymbol{R}[x]$ with $Q \geqslant 0$ on $[-1, 1]$.

(iv) There exists $\mu \in M_+([-1, 1])$ such that for all $n \in \boldsymbol{Z}_+$

$$f(n) = \int_{[-1,1[} \frac{1-Q_n(x)}{1-x}\,\mu(dx).$$

In fact, the equivalence (i)$\Leftrightarrow$(iii) remains true for $L := K^\wedge \simeq D_S$ (not necessarily coinciding with $[-1, 1]$) and real-valued functions f on $\boldsymbol{Z}_+$ satisfying $f(0) \geqslant 0$. Under these assumptions it is shown in [28] that (i) implies

(v) $f \in N(\boldsymbol{Z}_+)$ and $f(n) \geqslant f(0) \geqslant 0$.

For the converse implication (v)$\Rightarrow$(i) additional assumptions are needed, for example:

(A1) $D_S \supset [-1, 1]$

and

(A2) For every $n \in \boldsymbol{Z}_+$ there exist $\gamma_{n,k} \geqslant 0$ with $k \in \boldsymbol{Z}_+$, $k \leqslant n$, such that

$$x^n = \sum_{k=0}^{n} \gamma_{n,k}\,Q_k.$$

Examples of polynomial hypergroups $(\boldsymbol{Z}_+, *(Q_n))$ satisfying (A1) and (A2) are the ultraspherical hypergroups (of the form $(\boldsymbol{Z}_+, *(Q_n^{\alpha,\beta}))$ with $\alpha = \beta \geqslant -\frac{1}{2}$), the generalized Chebyshev (polynomial) hypergroups $(\boldsymbol{Z}_+, *(Q_n^{\alpha,\beta}))$ with $\alpha - 1$

$\geqslant \beta \geqslant -\frac{1}{2}$, and the Cartier hypergroups $(Z_+, *(Q_n^a))$ with $a \in N$, $a \geqslant 2$. On the other hand, the implication (v) $\Rightarrow$ (i) for general Jacobi polynomial hypergroups $(Z_+, *(Q_n^{\alpha,\beta}))$, where $\alpha > \beta$, remains an open problem.

5. APPLICATION: REPRESENTATIONS OF NEGATIVE DEFINITE FUNCTIONS

At first we note that bounded (continuous) negative definite functions on hypergroups are of little interest in Lévy–Khintchine type representation theory. In fact, any function $f \in N(K) \cap C_b(K)$ is of the form

$$f(x) = f(e) + \int_{K^\wedge} (1 - \chi(x)) \eta(d\chi)$$

for all $x \in K$, where $\eta \in M_+^b(K^\wedge)$ (see [28]). In particular, such representations are readily available once the underlying hypergroup is compact.

On the other hand, establishing Lévy–Khintchine representations for unbounded negative definite functions on hypergroups K and their dual spaces $K^\wedge$ in the sense of the various definitions given in Sections 1 and 2 remains an involved matter unless at least one-sided boundedness conditions are satisfied.

In the subsequent listing of examples we shall emphasize the relationship of the definitions of negative definiteness treated in the literature to the hierarchy of classes discussed in the previous sections.

5.1. One-dimensional hypergroups.

5.1.1. *Jacobi polynomial hypergroups* of the form $(Z_+, *(Q_n^{\alpha,\beta}))$ with $\alpha \geqslant \beta > -1$ and ($\beta \geqslant -\frac{1}{2}$ or $\alpha + \beta \geqslant 0$). These hypergroups are hermitian and Pontryagin with $Z_+^\wedge \simeq D_S = [-1, 1]$. There are several sources where the following representation occurs ([10], [14], [20], [9]).

Any $f \in N(Z_+)$ satisfying $f(0) \geqslant 0$ admits a representation

$$f(n) = f(0) + q(n) + \int_{[-1,1[} (1 - Q_n^{\alpha,\beta})\, d\eta \qquad \text{for all } n \in Z_+,$$

where q is a nonnegative quadratic form on Z_+ given by

$$q(n) = a \frac{n(n+\alpha+\beta+1)}{\alpha+\beta+2}$$

for all $n \in Z_+$ and some $a \geqslant 0$, and $\eta \in M_+([-1, 1[)$.

5.1.2. *Sturm–Liouville hypergroups* $([0, \pi/2], *(A))$ *of compact type.* We consider the dual space $[0, \pi/2]^\wedge$ of $[0, \pi/2]$ which can be identified with the set $\{\phi_n : n \in Z_+\}$ of normed eigenfunctions ϕ_n associated with the countably many simple eigenvalues $\lambda_n \leqslant 0$ of the Sturm–Liouville operator L_A defining the given hypergroup. In [1] it is shown that a real-valued function f on the dual $Z_+ \simeq \{\phi_n : n \in Z_+\}$ of $[0, \pi/2]$ belongs to the class $N_{T_0}^{(w)}(Z_+)$ (with

$T_0 := M_c(\boldsymbol{Z}_+)$) if and only if there exist a constant $a \geqslant 0$ and $\eta \in M_+(]0, \pi/2])$ satisfying

$$\int_{]0,\pi/2]} (1-\phi_n)\, d\eta < \infty$$

such that

$$f(n) = f(0) + a\lambda_n + \int_{]0,\pi/2]} (1-\phi_n)\, d\eta \qquad \text{for all } n \in \boldsymbol{Z}_+.$$

We note that the characters ϕ_n of $[0, \pi/2]$ are related to the Jacobi polynomials $Q_n^{\alpha,\beta}$, $n \in \boldsymbol{Z}_+$, via the equality

$$\phi_{4n(n+\alpha+\beta+1)}(x) = Q_n^{\alpha,\beta}(\cos 2x)$$

valid for all $x \in [0, \pi/2]$. Consequently, in this case $\big([0, \pi/2], *(A)\big)$ is isomorphic to the dual Jacobi polynomial hypergroup $\big([-1, 1], *(Q_n^{\alpha,\beta})\big)$.

5.1.3. *Sturm–Liouville hypergroups of noncompact type.* Let $(\boldsymbol{R}_+, *(A))$ be a Sturm–Liouville hypergroup of noncompact type with dual space

$$\boldsymbol{R}_+^\wedge = \boldsymbol{R}_+ \cup i[0, \varrho] \simeq \{\phi_\lambda \colon \lambda \in \boldsymbol{R}_+ \cup i[0, \varrho]\},$$

where ϱ denotes the index of the Sturm–Liouville operator L_A defining the underlying hypergroup, and ϕ_λ is the eigenfunction associated with the eigenvalue $\lambda^2 + \varrho^2$. By [23] every ϕ_λ can be expanded on $\boldsymbol{R}_+$ as

$$\phi_\lambda = \sum_{k \geqslant 0} (-1)^k b_k (\lambda^2 + \varrho^2)^k,$$

where the functions b_k are determined by

$$b_0 :\equiv 1, \qquad L_A b_{k+1} = -b_k,$$

$$b_{k+1}(0) = \frac{d}{dx} b_{k+1}(x)\big|_{x=0} = 0 \qquad (k \geqslant 0).$$

In particular, b_1 is given by

$$b_1(x) = -\frac{d}{ds}\psi_s\big|_{s=0}, \qquad \text{where } s := \lambda^2 + \varrho^2 \text{ and } \psi_s := \phi_\lambda.$$

The following representations have been given in [3] and [11], [12], respectively. For every $f \in N_{S_2}(\boldsymbol{R}_+)$ there exist a constant $a \geqslant 0$ and $\eta \in M_+(\boldsymbol{R}_+^\wedge \setminus \{i\varrho\})$ such that $\eta|_{\boldsymbol{R}_+}$ is bounded and

$$\int_{i]0,\varrho[} (\lambda^2 + \varrho^2)\, \eta(d\lambda) < \infty$$

satisfying

$$f = f(0) + ab_1 + \int_{\boldsymbol{R}_+^\wedge \setminus \{i\varrho\}} (1-\phi_\lambda)\, \eta(d\lambda).$$

On the other hand, given a function $f \in N_B^{(s)}(\boldsymbol{R}_+^\wedge)$ there exist a constant $a \geqslant 0$ and $\eta \in M_+(\boldsymbol{R}_+ \backslash \{0\})$ satisfying

$$\int_{\boldsymbol{R}_+} \frac{x^2}{1+x^2} \eta(dx) < \infty$$

such that

$$f(\lambda) = f(0) + a(\lambda^2 + \varrho^2) + \int_{\boldsymbol{R}_+ \backslash \{0\}} (1 - \phi_\lambda(x)) \eta(dx) \qquad \text{for all } \lambda \in \boldsymbol{R}_+^\wedge.$$

While the first cited representation can be also obtained from a general result in [9], the second one remains still subject to *ad hoc* methods.

5.2. Higher dimensional hypergroups.

5.2.1. *The disc polynomial hypergroups* $((\boldsymbol{Z}_+^2, *(Q_{m,n}^\alpha))$ *with* $\alpha > 0)$. These are (non-hermitian) Pontryagin hypergroups with dual hypergroup $(\boldsymbol{Z}_+^2)^\wedge$ identified with the unit disc $\boldsymbol{D}$. In [22] and [2] we find the following representation, which for functions with lower bounded real part can also be obtained from [9].

A complex-valued function ψ on $\boldsymbol{Z}_+^2$ belongs to the class $N_{T_0}^{(w)}(\boldsymbol{Z}_+^2)$ if and only if there exist $a, b, c, d \in \boldsymbol{R}$, $a, b, d \geqslant 0$, and $\eta \in M_+(\boldsymbol{D} \backslash \{(1, 0)\})$ satisfying

$$\int_{\boldsymbol{D} \backslash \{(1,0)\}} (1-x)\eta(d(x, y)) < \infty$$

such that

$$\begin{aligned} \psi(m, n) &= d + a(m-n)^2 - ic(m-n) + b\left(m+n+\frac{2mn}{\alpha+1}\right) \\ &\quad + \int_{\boldsymbol{D} \backslash \{(1,0)\}} (1 - Q_{m,n}^\alpha(x, y) + iy(m-n))\eta(d(x, y)) \end{aligned}$$

whenever $(m, n) \in \boldsymbol{Z}_+^2$.

5.2.2. *Product hypergroups.* We consider the product hypergroups $(\boldsymbol{R}^d \times \boldsymbol{R}_+, *)$, where the first factor denotes the Euclidean group, and the second one the Bessel–Kingman hypergroup with parameter $\alpha := (d-1)/2$ for $d \geqslant 1$. This hypergroup is again Pontryagin. In fact, its dual $(\boldsymbol{R}^d \times \boldsymbol{R}_+)^\wedge$ can be written in the form

$$\Gamma := (\boldsymbol{R}^d \times \boldsymbol{R}_+) \cup \{(\lambda, \mu) \in \boldsymbol{R}^d \times i\boldsymbol{R}_+ : \mu \leqslant i\|\lambda\|\}$$

which in turn is identifiable with $\boldsymbol{R}^d \times \boldsymbol{R}_+$ under the homeomorphism $B: \Gamma \to \boldsymbol{R}^d \times \boldsymbol{R}_+$ given by

$$B(\lambda, \mu) := \left(\lambda, \sqrt{\|\lambda\|^2 + \mu^2}\right)$$

for all $(\lambda, \mu) \in \Gamma$. Inspired by [25] we find the following representation in [4].

A complex-valued function f on Γ belongs to $N_B^{(s)}(K^\wedge)$ with $K := \boldsymbol{R}^d \times \boldsymbol{R}_+$ if and only if there exist $b, b_1, \dots, b_d \in \boldsymbol{R}$, $b \geqslant 0$, a quadratic form q on Γ defined by

$$q(\lambda, \mu) := \sum_{j,k=1}^{d} a_{jk} \lambda_j \lambda_k + c\mu^2 \quad \text{for all } (\lambda, \mu) \in \Gamma,$$

where $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_d) \in \boldsymbol{R}^d$, (a_{jk}) is a positive semidefinite matrix in $M(d, \boldsymbol{R})$, $c \geqslant 0$, and $\eta \in M_+(K \setminus \{(0, 0)\})$ satisfying

$$\int_{K \setminus \{(0,0)\}} \frac{\|x\|^2 + r^2}{1 + \|x\|^2 + r^2} \eta(d(x, r)) < \infty$$

such that

$$f(\lambda, \mu) = b + i \sum_{j=1}^{d} b_j \lambda_j + q(\lambda, \mu)$$
$$+ \int_{K \setminus \{(0,0)\}} \left(1 - \phi_{\lambda,\mu}(x, r) + \frac{i\langle \lambda, x \rangle}{1 + \|x\|^2 + r^2}\right) \eta(d(x, r))$$

whenever $(\lambda, \mu) \in \Gamma$. The characters $\phi_{\lambda,\mu}$ of K appearing in the integral term have a product form and can be written as

$$\phi_{\lambda,\mu}(x, r) = e^{i\langle \lambda, x \rangle} j_{(d-1)/2}\left(r\sqrt{\|\lambda\|^2 + \mu^2}\right) \quad \text{for all } (x, r) \in \boldsymbol{R}^d \times \boldsymbol{R}_+,$$

where $j_{(d-1)/2}$ denotes the modified Bessel function of the first kind and of order $(d-1)/2$.

5.2.3. *Mixed Jacobi hypergroups.* Finally, we consider the mixed Jacobi hypergroups $(\boldsymbol{R}_+ \times [-\pi, \pi], *)$, where the convolution $*$ is introduced as follows. For $\alpha \geqslant 0$, $\lambda \in \boldsymbol{Z}$, and $\mu \in \boldsymbol{C}$ let $\phi_\mu^{(\alpha,\lambda)}$ denote the Jacobi function associated with parameters α, λ and μ. The functions $\phi_{\lambda,\mu}^{(\alpha)}$ defined by

$$\phi_{\lambda,\mu}^{(\alpha)}(y, \theta) := e^{i\lambda\theta} (\cosh y)^\lambda \phi_\mu^{(\alpha,\lambda)}(y) \quad \text{for all } (y, \theta) \in \boldsymbol{R}_+ \times [-\pi, \pi]$$

satisfy the product formula

$$T_{(y,\theta)}^{(\alpha)} \phi_{\lambda,\mu}^{(\alpha)}(t, \tau) = \phi_{\lambda,\mu}^{(\alpha)}(y, \theta) \phi_{\lambda,\mu}^{(\alpha)}(t, \tau),$$

where $T_{(y,\theta)}^{(\alpha)}$ is the generalized translation operator given explicitly in [24]. Now the convolution of two Dirac measures on $K := \boldsymbol{R}_+ \times [-\pi, \pi]$ is given by

$$\varepsilon_{(y,\theta)} * \varepsilon_{(t,\tau)}(f) := T_{(y,\theta)}^{(\alpha)} f(t, \tau)$$

whenever $(y, \theta), (t, \tau) \in K$ and $f \in C_c(K)$; it can be extended to all bounded measures on K. The hypergroup $(K, *)$ has an involution $(y, \theta) \to (y, \theta)^- := (y, -\theta)$, and hence

$$(y, \theta) = (t, \tau)^- \Leftrightarrow y = t \text{ and } \cos(\theta + 1) = 1.$$

It follows that $(K, *)$ is not hermitian. Much of the harmonic analysis on K can be developed from the more general discussion in [21]. In particular,

$$\begin{aligned} K^\wedge &:= \{(\lambda, \mu)\in \boldsymbol{Z}\times \boldsymbol{C}\colon \ \|\phi^{(\alpha)}_{\lambda,\mu}\|_\infty \leqslant 1\} \\ &= \{(\lambda, \mu)\in \boldsymbol{Z}\times \boldsymbol{C}\colon \ |\mathrm{Im}(\varrho)| \leqslant \alpha+1\} \\ &\quad \cup \{(\lambda, \mu)\in \boldsymbol{Z}\times \boldsymbol{C}\colon \ \mu = iv,\ v \geqslant -(\alpha+1),\ \lambda = \pm(\alpha+2m+1+n),\ n\in \boldsymbol{N}\} \\ &=: \Gamma. \end{aligned}$$

Moreover,

$$\omega_K = A_\alpha \lambda_{\boldsymbol{R}_+} \otimes \sigma$$

with

$$A_\alpha(y) := 2^{2(\alpha+1)}(\sinh y)^{2\alpha+1}\cosh y \quad \text{for all } y\in \boldsymbol{R}_+,$$

$$\sigma(d\theta) := \frac{1}{2\pi} d\theta$$

and

$$\pi_K(d(\lambda, \mu)) = \frac{1}{(2\pi)^2}|C_1(\lambda, \mu)|^{-2}\,\mathbf{1}_{\boldsymbol{Z}\times\boldsymbol{R}_+}(\lambda, \mu)\,\Lambda(d\lambda)\,d\mu$$

$$+\frac{1}{(2\pi)^2} C_2(\lambda, \mu)\,\mathbf{1}_{\tilde{D}}(\lambda, \mu)\,\Lambda(d\lambda)\,d\mu,$$

where

$$C_1(\lambda, \mu) := \frac{2^{\alpha-i\mu+1}\,\Gamma(i\mu)\,\Gamma(\alpha+1)}{\Gamma((\alpha+\lambda+1+i\mu)/2)\ \Gamma((\alpha-\lambda+1+i\mu)/2)},$$

$$C_2(\lambda, \mu) := \mathrm{Res}_{\mu_0=\mu}[C_1(\lambda, \mu_0)\,C_1(\lambda, -\mu_0)]^{-1},$$

$$\Lambda := \varepsilon_\lambda,$$

and

$$\tilde{D} := \bigcup_{m\in\boldsymbol{N}} \{(\lambda, \mu)\in \boldsymbol{Z}\times\boldsymbol{C}\colon \ \mu = iv,\ v>0,\ \lambda = \pm(\alpha+2m+1+\eta)\}.$$

Obviously, the unit character $\mathbf{1}$ of K corresponds to the element $(0, i(\alpha+1))$ of Γ, and

$$(\boldsymbol{Z}\times\boldsymbol{R}_+)\cup\tilde{D} = \mathrm{supp}(\pi_K).$$

The necessary Fourier analysis is carried out on the subset $\mathrm{supp}(\pi_K)\cup\{\mathbf{1}\}$ of $K^\wedge$. From [21] we know that any $f\in N^{(s)}_B(K^\wedge)$ is of the form

$$f(\lambda, \mu) = a+ib\lambda+d\lambda^2+\frac{c}{2(\alpha+1)}(\lambda^2+\mu^2+(\alpha+1)^2)$$

$$+\int_{K\setminus\{(0,0)\}} \left(1-\phi^{(\alpha)}_{\lambda,\mu}(y, \theta)+i\lambda\theta u(y, \theta)\right)\eta(d(y,\theta))$$

for all $(\lambda, \mu) \in \operatorname{supp}(\pi_K) \cup \{\mathbb{1}\}$, where $a, b, c, d \in \boldsymbol{R}$, $a, b, d \geqslant 0$, u is a function in the class

$$D_*(K):$$

$$= \{f \in C^\infty(\boldsymbol{R} \times (]-\pi, 0[\cup]0, \pi[)): \ y \to f(y, \theta) \text{ is even with compact support,} \\ \theta \to f(y, \theta) \text{ is } 2\pi\text{-periodic}\}$$

such that $0 \leqslant u \leqslant 1$ and $u = 1$ on a neighbourhood of $(0, 0)$, and $\eta \in M_+(K \backslash \{(0, 0)\})$ with

$$\int_{K \backslash \{(0,0)\}} \frac{y^2+\theta^2}{1+y^2+\theta^2} \eta(d(y, \theta)) < \infty.$$

It is clear that the above representations yield canonical decompositions of continuous convolution semigroups on K once the Schoenberg correspondence in terms of negative definite functions on $K^\wedge$ has been established. Theorem 3.8 provides the required information for the class $N_B^{(s)}(K^\wedge)$.

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Received on 23.8.1994