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PREDICTION OF INFINITE VARIANCE FRACTIONAL ARIMA

BY

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Abstract. We establish conditions for the existence and invertibility of fractionally differenced ARIMA time series whose innovations are in the domain of attraction of an α-stable law with $\alpha < 2$ and consequently have infinite variance. More importantly, we study the effect of truncation on the minimum dispersion linear predictor of X_{n+k} based on the infinite past $X_n, X_{n-1}, \ldots$ We verify that the truncated predictor $\hat{X}_{n+k}$ based on the *finite* past $X_n, \ldots, X_0$ is asymptotically efficient, and derive asymptotic bounds on the rate of convergence to 1 of the efficiency of $\hat{X}_{n+k}$. The bounds are shown to decay like power functions with the rate of decay depending on the index of stability α and the difference parameter d.

1. Introduction. This paper studies prediction of a fractional ARIMA (FARIMA) time series $\{X_n\}$ defined by the equations

$$\Phi_p(B)X_n = \Theta_q(B)\Delta^{-d}(B)Z_n, \qquad n = \ldots, -1, 0, 1, \ldots, \tag{1.1}$$

with the innovations Z_n having *infinite variance*. More specifically, we assume that the Z_n's are i.i.d. and belong to the domain of attraction of an α-stable law with $0 < \alpha < 2$, i.e. satisfy

$$P\{|Z_n| > x\} = x^{-\alpha}L(x), \tag{1.2}$$

where L is slowly varying at infinity, and

$$\lim_{x\to\infty} P\{Z_n > x\}/P\{Z_n < -x\} = c_1/c_2, \tag{1.3}$$

where c_1 and c_2 are non-negative constants satisfying $c_1+c_2 > 0$.

We also impose an additional restriction on the distribution of the Z_n, viz.

$$\begin{array}{ll} EZ_n = 0 & \text{if } \alpha > 1, \\ EZ_n = 0 \text{ or } Z_n \text{ is symmetric} & \text{if } \alpha = 1. \end{array} \tag{1.4}$$

Notice that condition (1.4) implies that $0 < c_1/c_2 < \infty$ if $1 \leqslant \alpha < 2$.

An introduction to the theory of stable laws and their domains of attraction is given, e.g., in Laha and Rohatgi [20]. In (1.1), B is the backward shift operator defined by $BX_n = X_{n-1}$, and Δ is the difference operator defined by $\Delta X_n = X_n - X_{n-1}$. The difference parameter d is allowed to take fractional values.

We show in Section 2 that under standard assumptions on the polynomials Φ_p and Θ_q there is a unique moving average

$$X_n = \sum_{j=0}^{\infty} c_j Z_{n-j} \tag{1.5}$$

satisfying equations (1.1) whenever $d < 1-1/\alpha$, and that the solution is invertible if $\alpha > 1$ and $|d| < 1-1/\alpha$. In Section 3, we study asymptotic efficiency of a predictor of X_{n+k} based on the past values $X_n, \ldots, X_0$.

The FARIMA time series with *finite* variance innovations Z_n were introduced by Granger and Joyeux [11] and Hosking [12] to provide convenient finite parameter models for modeling *long range dependence*, a phenomenon drawing increasing attention in the last two decades (see, e.g., Beran [2]). On the other hand, there has lately been growing interest in modeling real-world phenomena by time series whose constituent random variables exhibit *high variability*. Consequently, many facets of the theory of *infinite variance* stochastic processes have been investigated in recent years. To name a few related contributions, let us mention here Cline and Brockwell [8], Davis and Resnick [9], Bhansali [3], Kokoszka and Taqqu [17]–[19], Mikosch et al. [22], and Klüppelberg and Mikosch [14]–[16] ([1]). Even though all the above papers study moving averages with infinite variance innovations, they impose comparatively strong summability conditions on the coefficients, requiring at least absolute summability. Such assumptions are appropriate for the study of ARMA models but, as is well known, the coefficients in the moving average representation of FARIMA time series are, in general, not absolutely summable and decay slowly like a power function. In this paper, we study infinite variance FARIMA time series, i.e. models exhibiting both long range dependence and high variability. The theory of FARIMA time series with *symmetric stable* innovations has been developed in Kokoszka and Taqqu [17]. The dependence structure of moving averages of the form (1.5) with symmetric stable innovations Z_n and not necessarily absolutely summable coefficients c_j has also been investigated in Kokoszka and Taqqu [19]. Whereas stable distributions are important archetypes, in applications it is often desirable to consider innovations from the domain of attraction of a stable law. Recall that if the Z_n's are i.i.d. and satisfy (1.2) and (1.3), then for some norming constants a_N

([1]) Futher references can be found in Samorodnitsky and Taqqu [23], Janicki and Weron [13], and in Section 13.3 of Brockwell and Davis [4].

$$\lim_{N\to\infty} a_N^{-1} \sum_{n=0}^{N} Z_n \xrightarrow{d} X,$$

where X is a stable random variable and "d" denotes convergence in distribution (see, e.g., Laha and Rohatgi [20]).

In Section 2 we show that equations (1.1) have a unique solution of the form (1.5) whenever $d < 1-1/\alpha$, which is invertible if $\alpha > 1$ and $|d| < 1-1/\alpha$. As is well known (see, e.g., Section 13.2 of Brockwell and Davis [4]), the corresponding conditions under the assumption of finite variance are $d < \frac{1}{2}$ and $|d| < \frac{1}{2}$. The proofs in the finite variance case utilize the spectral representation of stationary stochastic processes and Hilbert space methods. Our proofs rely on Theorem 2.1 which establishes sufficient conditions for two time-invariant linear filters whose coefficients may not be absolutely summable to commute. Theorem 2.1 extends Theorem 2.2 of Kokoszka and Taqqu [17] which is proved under the assumption of symmetric α-stable innovations. Theorem 2.1 plays a crucial role in our set-up, as neither the Box-Jenkins theory, which requires the absolute summability of the coefficients of linear filters, nor the classical L^2-theory is applicable.

Section 3 is devoted to the main subject of the paper, the study of the asymptotic efficiency of the linear predictor

$$\hat{X}_{n+k} = \sum_{j=0}^{n} a_j X_{n-j} \tag{1.6}$$

of X_{n+k} based on the finite past $X_n, \ldots, X_0$.

A natural and convenient criterion for the choice of the best linear predictor for the moving averages (1.5) is to minimize the *dispersion* $\operatorname{disp}(\hat{X}_{n+h} - X_{n+h})$ (see Cline and Brockwell [8] and references therein). The dispersion is defined as follows:

Definition 1.1. For any moving average (1.5) with the Z_n's satisfying (1.2) and (1.3) define

$$\operatorname{disp}(X_n) = \sum_{j=0}^{\infty} |c_j|^{\alpha}. \tag{1.7}$$

If the innovations Z_n are α-stable with the scale parameter σ, then $\sigma\big(\operatorname{disp}(X_n)\big)^{1/\alpha}$ is the scale parameter of X_n, i.e.

$$|\mathrm{E}\exp\{itX_n\}| = \exp\{-\sigma^{\alpha}\operatorname{disp}(X_n)|t|^{\alpha}\}.$$

In the case of finite variance innovations we have, of course,

$$\operatorname{Var}(X_n) = \operatorname{disp}(X_n)\operatorname{Var}(Z_1). \tag{1.8}$$

In Section 3 we verify that $\hat{X}_{n+k}-X_{n+k}=\sum_{j=0}^{\infty}u_j Z_{n+k-j}$ for some coefficients u_j and that

$$(1.9)\quad \operatorname{disp}(\hat{X}_{n+k}-X_{n+k})=\sum_{j=0}^{\infty}|u_j|^{\alpha}=\sum_{j=0}^{k-1}|c_j|^{\alpha}+\sum_{j=1}^{\infty}\Big|c_{k+n+j}-\sum_{j=0}^{n}a_j c_{n+j-1}\Big|^{\alpha}.$$

As demonstrated in Cline and Brockwell [8], the right most expression in (1.9) can be effectively minimized only for very special choices of the coefficients c_j, e.g., those appearing in the moving average representation of $\mathrm{AR}(p)$ and $\mathrm{ARMA}(1, 1)$ processes. In the general case, the only feasible procedure to follow seems to be to find the minimum dispersion linear predictor

$$(1.10)\qquad X_{n+k}^{*}=\sum_{j=0}^{\infty}a_j X_{n-j}$$

based on the *infinite* past $X_n, X_{n-1}, \ldots$, and then use the truncated predictor (1.6). As will be shown in Section 3 (see also [8]) the coefficients a_j of the predictor (1.10) are easy to determine.

The above procedure gives rise to the following question: Suppose $\tilde{X}_{n+k}$ is a linear predictor based on the finite past $X_n, \ldots, X_0$. (We do not know whether there is a unique predictor $\tilde{X}_{n+k}$ minimizing the right most expression in (1.9) and what the minimum is.) Assuming that

$$\operatorname{disp}(\tilde{X}_{n+k}-X_{n+k})\leqslant \operatorname{disp}(\hat{X}_{n+k}-X_{n+k})$$

we define the efficiency $\tilde{e}_n$ of $\hat{X}_{n+k}$ with respect to $\tilde{X}_{n+k}$ by

$$(1.11)\qquad \tilde{e}_n=\frac{\operatorname{disp}(\tilde{X}_{n+k}-X_{n+k})}{\operatorname{disp}(\hat{X}_{n+k}-X_{n+k})}.$$

Since, by (1.9),

$$\operatorname{disp}(\tilde{X}_{n+k}-X_{n+k})\geqslant \sum_{j=0}^{k-1}|c_j|^{\alpha},$$

we know, in general, only the *upper* bound on $1-\tilde{e}_n$, namely

$$1-\tilde{e}_n\leqslant 1-\frac{\sum_{j=0}^{k-1}|c_j|^{\alpha}}{\operatorname{disp}(\hat{X}_{n+k}-X_{n+k})}.$$

Therefore, it is natural to define the efficiency e_n of $\hat{X}_{n+k}$ by

$$(1.12)\qquad e_n=\frac{\sum_{j=0}^{k-1}|c_j|^{\alpha}}{\operatorname{disp}(\hat{X}_{n+k}-X_{n+k})}.$$

Consequently, any upper bound on $1-e_n$ will also be an upper bound on $1-\tilde{e}_n$.

It is verified in Section 3 that for FARIMA time series, e_n tends to 1, as $n \to \infty$, showing that $\hat{X}_{n+k}$ is asymptotically efficient with respect to any linear predictor based on $X_n, \ldots, X_0$. More importantly, effective asymptotic upper bounds on $1-e_n$ are established. It is demonstrated that $1-e_n$ tends to zero like a power function and the rates of convergence are given, which depend on the index of stability α and the difference parameter d. The above results are then contrasted with corresponding results for ARMA processes, where $1-e_n$ tends exponentially to zero.

We conclude this introduction by remarking that a number of interesting approaches to solving the prediction problem for more general infinite variance processes have been proposed. An interested reader is referred to Urbanik [24]–[26], Cambanis and Miller [6], Cambanis and Soltani [7], Cambanis et al. [5], and Miamee and Pourahmadi [21].

2. Existence and invertibility. In the remainder of the paper we assume that $\{Z_n, n = \ldots, -1, 0, 1, \ldots\}$ is a sequence of i.i.d. random variables satisfying conditions (1.2)–(1.4).

If there is $v < \alpha$ such that

$$\sum_{j=0}^{\infty} |c_j|^v < \infty, \tag{2.1}$$

then the series in (1.5) converges a.s. and in L^v, and the following inequality holds:

$$\mathrm{E}|X_n|^v \leqslant 2\mathrm{E}|Z_0|^v \sum_{j=0}^{\infty} |c_j|^v \tag{2.2}$$

(see Avram and Taqqu [1]). Notice that condition (1.2) implies $\mathrm{E}|Z_n|^v < \infty$ whenever $v < \alpha$ and $\mathrm{E}|Z_n|^p = \infty$ if $p > \alpha$.

Theorem 2.1 below is used to establish conditions for the existence and invertibility of the solution of FARIMA equations (1.1). It is also used in Section 3. The proof relies on inequality (2.2) and the inequality of W. H. Young

$$\|\psi * c\|_v \leqslant \|\psi\|_1 \|c\|_v \quad (v \geqslant 1), \tag{2.3}$$

which holds for sequences $\psi := \{\psi_0, \psi_1, \ldots\} \in l^1$ and $c := \{c_0, c_1, \ldots\} \in l^v$ (see, e.g., Chapter 13 of Edwards [10]). Even though the proof of inequality (2.3) is not trivial, it can be readily verified that, for $0 < v \leqslant 1$,

$$\|\psi * c\|_v \leqslant \|\psi\|_v \|c\|_v \quad (v \leqslant 1) \tag{2.4}$$

whenever both ψ and c are in l^v.

THEOREM 2.1. *Suppose $\{c_0, c_1, \ldots\}$ and $\{\psi_0, \psi_1, \ldots\}$ are sequences of real numbers such that for some $v \in [1, \alpha)$ if $1 < \alpha \leqslant 2$, and for some $v \in (0, \alpha)$ if $0 < \alpha \leqslant 1$*

$$\sum_{j=0}^{\infty} |c_j|^v < \infty, \tag{2.5}$$

(2.6) $$\sum_{j=0}^{\infty} |\psi_j|^{\nu\wedge 1} < \infty.$$

Define

(2.7) $$X_n = \sum_{j=0}^{\infty} c_j Z_{n-j},$$

(2.8) $$Y_n = \sum_{j=0}^{\infty} \psi_j Z_{n-j},$$

(2.9) $$A_n = \sum_{j=0}^{\infty} a_j Z_{n-j},$$

where

(2.10) $$a_j = (\psi * c)_j = \sum_{k=0}^{j} \psi_k c_{j-k}.$$

Then, for every n,

(2.11) $$\lim_{m\to\infty} \sum_{k=0}^{m} \psi_k X_{n-k} = A_n,$$

(2.12) $$\lim_{m\to\infty} \sum_{j=0}^{m} c_j Y_{n-j} = A_n.$$

The convergence in (2.12) *is in* L^ν*-norm, and in* (2.11) *both in* L^ν*-norm and absolutely a.s.*

Proof. First of all notice that it follows as an immediate consequence of (2.3) and (2.4) that the random variables A_n in (2.9) are well defined.

Suppose first that $P(z) = \sum_{k=0}^{m} p_k z^k$ is a polynomial. We shall verify that

(2.13) $$\sum_{k=0}^{m} p_k X_{n-k} = \sum_{j=0}^{\infty} \Big(\sum_{k=0}^{m\wedge j} p_k c_{j-k}\Big) Z_{n-j} \text{ a.s.}$$

Consider the set

$$\Omega_0 = \{\omega\colon \forall n,\ \sum_{j=0}^{\infty} c_j Z_{n-j}(\omega) \text{ converges}\}.$$

Note that $P(\Omega_0) = 1$ and, for any fixed $\omega_0 \in \Omega_0$,

$$\sum_{k=0}^{m} p_k X_{n-k}(\omega_0) = \sum_{k=0}^{m} p_k\Big(\sum_{j=0}^{\infty} c_j Z_{n-k-j}(\omega_0)\Big) = \lim_{s\to\infty} \sum_{k=0}^{m} p_k\Big(\sum_{j=0}^{s-k} c_j Z_{n-k-j}(\omega_0)\Big)$$

$$= \lim_{s\to\infty} \sum_{j=0}^{s} \Big(\sum_{k=0}^{m\wedge j} p_k c_{j-k}\Big) Z_{n-j}(\omega_0) = \sum_{j=0}^{\infty} \Big(\sum_{k=0}^{m\wedge j} p_k c_{j-k}\Big) Z_{n-j}(\omega_0).$$

To see that the series in (2.11) converges absolutely, notice that for $\alpha > 1$

$$\mathrm{E}\{\sum_{k=0}^{\infty} |\psi_k X_{n-k}|\} = \mathrm{E}|X_0| \sum_{k=0}^{\infty} |\psi_k| < \infty,$$

and for $\alpha \leqslant 1$ (implying $\nu < 1$)

$$\mathrm{E}\{\sum_{k=0}^{\infty} |\psi_k X_{n-k}|^{\nu}\} = \mathrm{E}|X_0|^{\nu} \sum_{l=0}^{\infty} |\psi_k|^{\nu} < \infty.$$

Note that $\mathrm{E}|X_0|^{\nu\wedge 1} < \infty$, by (2.2).

To verify the L^{ν}-convergence in (2.11) write, using (2.13) and (2.2),

$$\begin{aligned}\mathrm{E}|\sum_{k=0}^{m} \psi_k X_{n-k} - A_n|^{\nu} &= \mathrm{E}|\sum_{j=0}^{\infty} (\sum_{k=0}^{m\wedge j} \psi_k c_{j-k}) Z_{n-j} - \sum_{j=0}^{\infty} a_j Z_{n-j}|^{\nu} \\ &= \mathrm{E}|\sum_{j=0}^{\infty} [(\sum_{k=0}^{m\wedge j} \psi_k c_{j-k}) - (\sum_{k=0}^{j} \psi_k c_{j-k})]\, Z_{n-j}|^{\nu} \\ &\leqslant 2\mathrm{E}|Z_0|^{\nu} \sum_{j=0}^{\infty} |\sum_{k=(m\wedge j)+1}^{j} \psi_k c_{j-k}|^{\nu} = 2\mathrm{E}|Z_0|^{\nu}\, \|\psi^{(m+1)} * c\|_{\nu}^{\nu},\end{aligned}$$

where $\psi^{(k)}$ is the truncated sequence $\{0, \ldots, 0, \psi_k, \psi_{k+1}, \ldots\}$. Clearly, $\|\psi^{(m+1)}\|_1 \to 0$ if $\alpha > 1$ and $\|\psi^{(m+1)}\|_{\nu} \to 0$ if $\alpha \leqslant 1$, so the L^{ν}-convergence follows from (2.3) if $\alpha > 1$ and from (2.4) if $\alpha \leqslant 1$.

The proof of (2.12) is similar. ∎

Suppose $\{\xi_n, \ldots, n = \ldots, -1, 0, 1, \ldots\}$ is a random sequence, and $\{h_0, h_1, \ldots\}$ a sequence of real numbers such that the random series $\sum_{j=0}^{\infty} h_j \xi_{n-j}$ converges a.s. Then we define

$$H(B)\xi := \sum_{j=0}^{\infty} h_j \xi_{n-j},$$

where B stands, as usual, for the backward shift operator. Using this notation we have

COROLLARY 2.1. *Under the assumptions of Theorem* 2.1

(2.14) $$\Psi(B)C(B)Z_n = A(B)Z_n = C(B)\Psi(B)Z_n \quad a.s.$$

(*Notice that A is defined by $A(z) = \Psi(z)C(z)$.*)

Theorem 2.2 below gives conditions for equations (1.1) to have a solution of the form (1.5). The random variables $\Delta^{-d}(B)Z_n$ appearing in (1.1) are defined as follows:

(2.15) $$\Delta^{-d}(B)Z_n := Z_0 + \sum_{j=1}^{\infty} \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} Z_{n-j}.$$

Note that the coefficients of Z_{n-j} in (2.15) are the corresponding coefficients of z^j in the series expansion of $(1-z)^{-d}$, whenever d is not an integer. *Unless stated otherwise, we consider in this paper only non-integer values of d.* Note that, for $d = \ldots, -2, -1, 0$, $\Delta^{-d}(B) Z_n$ is a finite moving average. Since $\Gamma(j+d)/\Gamma(j+1) \sim j^{d-1}$ (2), the random series (2.15) converges a.s., provided $(d-1)\alpha < -1$.

THEOREM 2.2. *Suppose the polynomials $\Phi_p(z)$ and $\Theta_q(z)$ have no roots in common and $\Phi_p(z)$ has no roots in the closed unit disk $\{z: |z| \leqslant 1\}$. Define*

$$C_d(z) := \frac{\Theta_q(z)(1-z)^{-d}}{\Phi_p(z)} = \sum_{j=0}^{\infty} c_j z^j, \quad |z| < 1. \tag{2.16}$$

If

$$d < 1 - 1/\alpha, \tag{2.17}$$

then the sequence

$$X_n := C_d(B) Z_n = \sum_{j=0}^{\infty} c_j Z_{n-j} \tag{2.18}$$

is the unique solution of equations (1.1) *of the form* (1.5).

Proof. It is shown in Kokoszka and Taqqu [17] that the coefficients c_j in (2.16) satisfy

$$\lim_{j\to\infty} \frac{c_j}{j^{d-1}} = \frac{\Theta_q(1)}{\Phi_p(1)\Gamma(d)}. \tag{2.19}$$

Consequently, since $(d-1)\alpha < -1$, there is $\nu < \alpha$ which can be chosen arbitrarily close to α such that (2.1) holds. To verify that (2.18) is a solution of equations (1.1), set

$$A(z) = \Theta_q(z)(1-z)^{-d} = \Phi_p(z) C_d(z), \quad |z| < 1,$$

and, using (2.14), write

$$\Phi_p(B) X_n = \Phi_p(B) C_d(B) Z_n = \Theta_q(B) \Delta^{-d} Z_n \text{ a.s.}$$

Suppose $X'_n = \sum_{j=0}^{\infty} c'_j Z_{n-j}$ is another solution. Then

$$\Phi_p(B) X'_n = \Phi_p(B) X_n. \tag{2.20}$$

Applying $\Phi_p^{-1}(B)$ to both sides of (2.20) (the coefficients of $\Phi_p^{-1}(z)$ tend exponentially to zero), we get $X'_n = X_n$. ∎

Remark. Since ν in the proof of Theorem 2.2 can be chosen arbitrarily close to α, it follows that the partial sums $\sum_{j=0}^{m} c_j Z_{n-j}$ converge to $C_d(B) Z_n$ in L^p for each $p \in (0, \alpha)$ (and a.s.).

(2) $a_j \sim b_j$ means that a_j/b_j tends to 1.

Our next theorem establishes sufficient conditions for the invertibility of the solution (2.18) of equations (1.1).

THEOREM 2.3. *Suppose the polynomials* $\Phi_p(z)$ *and* $\Theta_q(z)$ *have no roots in common and neither has roots in the closed unit disk* $\{z\colon |z| \leqslant 1\}$. *Define*

$$H_d(z) = C_d^{-1}(z) = \frac{\Phi_p(z)(1-z)^d}{\Theta_q(z)} = \sum_{j=0}^{\infty} h_j z^j, \quad |z| < 1. \tag{2.21}$$

If

$$\alpha > 1 \quad \text{and} \quad |d| < 1 - 1/\alpha, \tag{2.22}$$

then

$$\lim_{m\to\infty} \sum_{j=0}^{m} h_j X_{n-j} = Z_n \tag{2.23}$$

(succinctly, $C_d^{-1}(B)X_n = Z_n$*), where* X_n *is given by* (2.18). *The convergence in* (2.23) *is in* L^p *for* $p \in (0, \alpha)$, *and if* $d \in (0, 1-1/\alpha)$, *also absolutely a.s.*

Proof. Observe that, for $d \in (1/\alpha - 1, 0)$ and $v \in ((1+d)^{-1}, \alpha)$, $\sum |c_j| < \infty$ and $\sum |h_j|^v < \infty$, and for $d \in (0, 1-1/\alpha)$ and $v \in ((1-d)^{-1}, \alpha)$, $\sum |c_j|^v < \infty$ and $\sum |h_j| < \infty$. Now it remains to apply Corollary 2.1 and Theorem 2.1. ∎

3. Prediction. In this section we assume that $1 < \alpha < 2$, $|d| < 1-1/\alpha$, and the polynomials Φ_p and Θ_q satisfy the assumptions of Theorem 2.3. Recall that $\{Z_n\}$ is a sequence of i.i.d. random variables satisfying (1.2)–(1.4). Our results remain valid (with $\alpha = 2$) if $\{Z_n\}$ is a finite variance white noise sequence.

We start by determining the minimum dispersion linear predictor of X_{n+k} based on the infinite past $X_n, X_{n-1}, \ldots$

THEOREM 3.1. *There is a unique sequence* $\{a_0, a_1, \ldots\}$ *such that*

$$\operatorname{disp}\Big(X_{n+k} - \sum_{j=0}^{\infty} a_j X_{n-j}\Big) = \min_{u_0, u_1, \ldots} \operatorname{disp}\Big(X_{n+k} - \sum_{j=0}^{\infty} u_j X_{n-j}\Big), \tag{3.1}$$

where the minimum is taken over all sequences $\{u_0, u_1, \ldots\}$ *satisfying* $\sum |u_j| < \infty$ *if* $d > 0$ *and* $\sum |u_n|^v$, *for some* $1 \leqslant v < \alpha$, *if* $d < 0$. *The sequence* $\{a_0, a_1, \ldots\}$ *is given by*

$$a_j = -\sum_{t=0}^{k-1} c_t h_{j+k-t}, \tag{3.2}$$

where the c_j*'s and* h_j*'s are defined by* (2.16) *and* (2.21), *respectively. Moreover,*

$$\operatorname{disp}\Big(X_{n+k} - \sum_{j=0}^{\infty} a_j X_{n-j}\Big) = \sum_{j=0}^{k-1} |c_j|^{\alpha}. \tag{3.3}$$

Proof. The summability conditions on $\{u_j\}$ ensure that the series $\sum_{j=0}^{\infty} u_j X_{n-j}$ is well defined.

Setting

$$C_d^{(k)}(z) = \sum_{j=0}^{\infty} c_{j+k} z^j, \tag{3.4}$$

we have

$$X_{n+k} = \sum_{j=0}^{k-1} c_j Z_{n+k-j} + C_d^{(k)}(B) Z_n,$$

$$\sum_{j=0}^{\infty} u_j X_{n-j} = U(B) X_n = U(B) C_d(B) Z_n.$$

By Corollary 2.1,

$$\operatorname{disp}\left(X_{n+k} - \sum_{j=0}^{\infty} u_j X_{n-j}\right) = \sum_{j=0}^{k-1} |c_j|^\alpha + \sum_{j=0}^{\infty} |v_j|^\alpha, \tag{3.5}$$

with the v_j's defined by

$$\sum_{j=0}^{\infty} v_j z^j = C_d^{(k)}(z) - U(z) C_d(z).$$

It follows that the minimum is attained if $v_j = 0$, $j = 0, 1, \ldots$, i.e. $U(z) = C_d^{-1}(z) C_d^{(k)}(z)$, yielding

$$u_j = \sum_{s=0}^{j} h_s c_{j+k-s} = -\sum_{s=j+1}^{j+k} h_s c_{j+k-s} = -\sum_{t=0}^{k-1} c_t h_{j+k-t}. \blacksquare$$

Remarks. 1. Theorem 3.1 can be rephrased as follows:

The minimum dispersion linear predictor of X_{n+k} based on the infinite past $X_n, X_{n-1}, \ldots$ is given by

$$X_{n+k}^* = C_d^{-1}(B) C_d^{(k)}(B) X_n = C_d^{(k)}(B) Z_n. \tag{3.6}$$

We see that the minimum dispersion predictor X_{n+k}^* coincides with the minimum variance linear predictor for finite variance FARIMA processes (cf. relation (13.2.42) of Brockwell and Davis [4]).

2. In view of (1.8), relation (3.3) extends the well-known formula for the variance of the prediction error.

3. Theorem 3.1 extends Theorem 2.2 of Cline and Brockwell [8] which states an analogous result for ARMA processes.

4. If the innovations Z_n are *symmetric α-stable*, then

$$X_{n+k}^* = \mathrm{E}\{X_{n+k} | X_n, X_{n-1}, \ldots\},$$

i.e. X_{n+k}^* is actually the regression predictor. The random variable $X_{n+k} - X_{n+k}^*$ also minimizes the L^p-distance, $1 < p < \alpha$, from X_{n+k} to the closed linear subspace spanned by $X_n, X_{n-1}, \ldots$ (See Cambanis and Miller [6], Corollaries 5.3 and 5.7.)

In practice only a finite number of past observations $X_n, X_{n-1}, \ldots, X_0$ are given, and since, as remarked in the Introduction, in general, an explicit formula for the values $u_n, \ldots, u_0$ minimizing $\operatorname{disp}(X_{n+k}-\sum_{j=0}^{n} u_j X_{n-j})$ is not available, one has to use the truncated predictor

$$\hat{X}_{n+k} = \sum_{j=0}^{n} a_j X_{n-j}, \tag{3.7}$$

with the a_j's given by (3.2). The following proposition gives a convenient expression for $\operatorname{disp}(X_{n+k}-\hat{X}_{n+k})$.

PROPOSITION 3.1. *For the truncated predictor $\hat{X}_{n+k}$ given by* (3.7) *we have*

$$\operatorname{disp}(X_{n+k}-\hat{X}_{n+k}) = \sum_{j=0}^{k-1} |c_j|^\alpha + r_n, \tag{3.8}$$

where

$$r_n = \sum_{m=n+1}^{\infty} |S_1(m)+S_2(m)|^\alpha, \tag{3.9}$$

and

$$S_1(m) = \sum_{j=0}^{k-1} c_j\Big(\sum_{u=0}^{k-j-1} h_u c_{k-j+m-u}\Big) - c_{k+m}, \tag{3.10}$$

$$S_2(m) = \sum_{j=0}^{k-1} c_j\Big(\sum_{u=k-j+n+1}^{k-j+m} h_u c_{k-j+m-u}\Big). \tag{3.11}$$

Proof. Write

$$\begin{aligned} X_{n+k}-\hat{X}_{n+k} &= \sum_{j=0}^{\infty} c_j Z_{n+k-j} - \sum_{i=0}^{n} a_i\Big(\sum_{j=0}^{\infty} c_j Z_{n-i-j}\Big) \\ &= c_0 Z_{n+k} + c_1 Z_{n+k-1} + \ldots + c_{k-1} Z_{n+k} \\ &\quad + \sum_{m=0}^{n} (c_{k+m} - a_0 c_m - a_1 c_{m-1} - \ldots - a_m c_0) Z_{n-m} \\ &\quad + \sum_{m=n+1}^{\infty} (c_{k+m} - a_0 c_m - a_1 c_{m-1} - \ldots - a_n c_{m-n}) Z_{n-m}. \end{aligned} \tag{3.12}$$

We shall now verify that

$$c_{k+m} - a_0 c_m - a_1 c_{m-1} - \ldots - a_m c_0 = 0, \quad m = 0, 1, \ldots, n. \tag{3.13}$$

Indeed, by (3.2),

$$\begin{aligned} c_{k+m}-a_0 c_m-a_1 c_{m-1}-\dots-a_m c_0 \\ &= c_{k+m}+(c_0 h_k+c_1 h_{k-1}+\dots+c_{k-1} h_1) c_m \\ &\quad +(c_0 h_{k+1}+c_1 h_k+\dots+c_{k-1} h_2) c_{m-1} \\ &\quad \dots\dots\dots\dots\dots\dots \\ &\quad +(c_0 h_{k+m}+c_1 h_{k+m-1}+\dots+c_{k-1} h_{m+1}) c_0 \\ &= c_{k+m}-(c_k h_0) c_m-(c_k h_1+c_{k+1} h_0) c_{m-1}-\dots \\ &\quad \dots-(c_k h_m+c_{k+1} h_{m-1}+\dots+c_{k+m} h_0) c_0 \\ &= c_{k+m}-c_k(h_0 c_m+h_1 c_{m-1}+\dots+h_m c_0) \\ &\quad -c_{k+1}(h_0 c_{m-1}+\dots+h_{m-1} c_0) \\ &\quad \dots\dots\dots\dots\dots\dots \\ &\quad -c_{k+m}(h_0 c_0)=0. \end{aligned}$$

Hence, by (3.12) and (3.13), equality (3.8) holds with

$$r_n=\sum_{m=n+1}^{\infty}|c_{k+m}-a_0 c_m-a_1 c_{m-1}-\dots-a_n c_{m-n}|^{\alpha}. \tag{3.14}$$

Note that

$$\begin{aligned} -(a_0 c_m+a_1 c_{m-1}+\dots+a_n c_{m-n}) \\ &= (c_0 h_k+c_1 h_{k-1}+\dots+c_{k-1} h_1) c_m \\ &\quad +(c_0 h_{k+1}+c_1 h_k+\dots+c_{k-1} h_2) c_{m-1} \\ &\quad \dots\dots\dots\dots\dots\dots \\ &\quad +(c_0 h_{k+n}+c_1 h_{k+n-1}+\dots+c_{k-1} h_{n+1}) c_{m-n} \\ &= c_0(h_k c_m+h_{k+1} c_{m-1}+\dots+h_{k+m} c_{m-n}) \\ &\quad +c_1(h_{k-1} c_m+h_k c_{m-1}+\dots+h_{k+n-1} c_{m-n}) \\ &\quad \dots\dots\dots\dots\dots\dots \\ &\quad +c_{k-1}(h_1 c_m+h_2 c_{m-1}+\dots+h_{n+1} c_{m-n}) \\ &= \sum_{j=0}^{k-1} c_j\Big(\sum_{u=k-j}^{k-j+n} h_u c_{k-j+m-u}\Big) \\ &= -\sum_{j=0}^{k-1} c_j\Big(\sum_{u=0}^{k-j-1} h_u c_{k-j+m-u}+\sum_{u=k-j+n+1}^{k-j+m} h_u c_{k-j+m-u}\Big). \end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15), we get (3.9). ■

Notice that, by (1.12) and (3.8),

$$(3.16)\qquad 1-e_n \leqslant \Big(\sum_{j=0}^{k-1} |c_j|^\alpha\Big)^{-1} r_n,$$

where e_n is the efficiency of $\hat{X}_{n+k}$ defined in Section 1. It follows from (3.16) that in order to find an upper bound on $1-e_n$, it suffices to find an upper bound on r_n. The following lemmas are crucial steps in this direction.

LEMMA 3.1. *For $S_1(m)$ defined by* (3.10) *and any $d < 1-1/\alpha$, we have*

$$(3.17)\qquad \limsup_{n\to\infty} \frac{\sum_{m=n+1}^{\infty} |S_1(m)|^\alpha}{n^{(d-1)\alpha+1}} \leqslant K_1(k, \alpha, d),$$

where

$$(3.18)\quad K_1(k, \alpha, d) = \frac{2^\alpha}{|(d-1)\alpha+1|}\left|\frac{\Theta_q(1)}{\Phi_p(1)\Gamma(d)}\right|^\alpha \Big[1+k^\alpha \sum_{j=0}^{k-1} |c_j|^\alpha \Big(\sum_{u=0}^{k-j-1} |h_u|\Big)^\alpha\Big].$$

Proof. Since

$$|S_1(m)|^\alpha \leqslant 2^\alpha \Big[k^\alpha \sum_{j=0}^{k-1} |c_j|^\alpha \Big|\sum_{u=0}^{k-j-1} h_u c_{k-j+m-u}\Big|^\alpha + |c_{k+m}|^\alpha\Big],$$

we have

$$(3.19)\quad \sum_{m=n+1}^{\infty} |S_1(m)|^\alpha \leqslant 2^\alpha \Big[k^\alpha \sum_{j=0}^{k-1} |c_j|^\alpha \sum_{m=n+1}^{\infty} \Big|\sum_{k=0}^{k-j-1} h_u c_{k-j+m-u}\Big|^\alpha + \sum_{m=n+1}^{\infty} |c_{k+m}|^\alpha\Big]$$
$$=: 2^\alpha \Big[k^\alpha \sum_{j=0}^{k-1} |c_j|^\alpha R_1(n) + R_2(n)\Big].$$

In view of (3.19), to prove (3.17) and (3.18) it suffices to verify that

$$(3.20)\qquad \limsup_{n\to\infty} \frac{R_1(n)}{n^{(d-1)\alpha+1}} \leqslant \Big(\sum_{u=0}^{k-j-1} |h_u|\Big)^\alpha \frac{1}{|(d-1)\alpha+1|}\left|\frac{\Theta_q(1)}{\Phi_p(1)\Gamma(d)}\right|^\alpha$$

and

$$(3.21)\qquad \limsup_{n\to\infty} \frac{R_2(n)}{n^{(d-1)\alpha+1}} \leqslant \frac{1}{|(d-1)\alpha+1|}\left|\frac{\Theta_q(1)}{\Phi_p(1)\Gamma(d)}\right|^\alpha.$$

In order to find asymptotic upper bounds on $R_1(n)$ and $R_2(n)$, we use the relation

$$(3.22)\qquad \lim_{n\to\infty} \frac{c_n}{n^{\alpha-1}} = \frac{\Theta_q(1)}{\Phi_p(1)\Gamma(d)}, \qquad d < 1-\frac{1}{\alpha}$$

(see Kokoszka and Taqqu [17]). We verify below relation (3.20), the proof of (3.21) being the same. Notice that in the sum $\sum_{k=0}^{k-j-1} h_u c_{k-j+m-n}$ appearing in $R_1(n)$, $k-j+m-u \geqslant m+1$, and, consequently, by (3.22), for any fixed $\varepsilon > 0$ and sufficiently large n

$$R_1(n) \leqslant \Big(\sum_{n=0}^{k-j-1} |h_u|\Big)^{\alpha} \left(\left|\frac{\Theta_q(1)}{\Phi_p(1)\Gamma(d)}\right| + \varepsilon\right)^{\alpha} \int_n^{\infty} u^{(d-1)\alpha}\, du,$$

yielding (3.20). ∎

Lemma 3.2. *For $S_2(m)$ defined by* (3.11) *we have*

(a) *If $0 < d < 1-1/\alpha$, then*

$$\limsup_{n\to\infty} \frac{\sum_{m=n+1}^{\infty} |S_2(m)|^{\alpha}}{n^{-\alpha d}} \leqslant K_{21}(k, \alpha, d), \tag{3.23}$$

where

$$K_{21}(k, \alpha, d) = \left|\frac{k\Phi_p(1)}{\Theta_q(1)\Gamma(1-d)}\right|^{\alpha} \Big(\sum_{j=0}^{k-1} |c_j|^{\alpha}\Big)\Big(\sum_{j=0}^{\infty} |c_j|^{\alpha}\Big). \tag{3.24}$$

(b) *If $1/\alpha - 1 < d < 0$, then*

$$\limsup_{n\to\infty} \frac{\sum_{m=n+1}^{\infty} |S_2(m)|^{\alpha}}{n^{1-(d+1)\alpha}} \leqslant K_{22}(k, \alpha, d), \tag{3.25}$$

where

$$K_{22}(k, \alpha, d) = \frac{1}{(d+1)\alpha - 1}\left|\frac{k\Phi_p(1)}{\Theta_q(1)\Gamma(-d)}\right|^{\alpha} \Big(\sum_{j=0}^{k-1} |c_j|^{\alpha}\Big)\Big(\sum_{j=0}^{\infty} |c_j|\Big)^{\alpha}. \tag{3.26}$$

Proof. Recall that for any sequence $\{u_0, u_1, \ldots\}$ we defined

$$\boldsymbol{u}^{(j)} = \{0, \ldots, 0, u_j, u_{j+1}, \ldots\}.$$

Notice that for $d > 0$, $\boldsymbol{h} \in L^1$ and $\boldsymbol{c} \in L^{\alpha}$, and for $d < 0$, $\boldsymbol{h} \in L^{\alpha}$ and $\boldsymbol{c} \in L^1$. In either case

$$\|\boldsymbol{h}^{(j)} * \boldsymbol{c}\|_{\alpha}^{\alpha} = \sum_{v=j}^{\infty} \Big|\sum_{u=j}^{v} h_u c_{v-u}\Big|^{\alpha}. \tag{3.27}$$

Using equality (3.27) and changing the summation indices, it can be readily verified that

$$\sum_{m=n+1}^{\infty} |S_2(m)|^{\alpha} \leqslant k^{\alpha} \sum_{j=0}^{k-1} |c_j|^{\alpha}\, \|\boldsymbol{h}^{(k-j+n+1)} * \boldsymbol{c}\|_{\alpha}^{\alpha}. \tag{3.28}$$

We shall use in the sequel the following relation:

$$\lim_{n\to\infty} \frac{h_n}{n^{-d-1}} = \frac{\Phi_p(1)}{\Theta_q(1)\Gamma(-d)}, \quad |d| < 1 - \frac{1}{\alpha} \tag{3.29}$$

(see Kokoszka and Taqqu [17]).

Suppose $0 < d < 1-1/\alpha$. By the inequality of W. H. Young (inequality (2.3)) we obtain

$$\|h^{(k-j+n+1)} * c\|_\alpha \leqslant \|h^{(k-j+n+1)}\|_1 \|c\|_\alpha. \tag{3.30}$$

Relation (3.29) implies, for any fixed $\varepsilon > 0$ and sufficiently large n,

$$\|h^{(k-j+n+1)}\|_1 = \sum_{u=k-j+n+1}^{\infty} |h_u| \leqslant \sum_{u=n+1}^{\infty} |h_u| \tag{3.31}$$
$$\leqslant \left(\left|\frac{\Phi_p(1)}{\Theta_q(1)\Gamma(-d)}\right| + \varepsilon\right) \int_n^\infty u^{-d-1}\, du = \left(\left|\frac{\Phi_p(1)}{\Theta_q(1)\Gamma(-d)}\right| + \varepsilon\right) \frac{n^{-d}}{d}.$$

Hence, by (3.30) and (3.31),

$$\limsup_{n\to\infty} \frac{\|h^{(k-j+n+1)} * c\|_\alpha}{n^{-d}} \leqslant \left|\frac{\Phi_p(1)}{\Theta_q(1)\Gamma(1-d)}\right| \Big(\sum_{j=0}^{\infty} |c_j|^\alpha\Big)^{1/\alpha}. \tag{3.32}$$

Thus (3.23) follows from (3.28) and (3.32), completing the proof of part (a). The proof of part (b) is similar. Use (3.20), (3.29) and the inequality

$$\|h^{(h-j+n+1)} * c\|_\alpha \leqslant \|h^{(k-j+n+1)}\|_\alpha \|c\|_1. \ \blacksquare$$

Now we state our main theorem.

THEOREM 3.2. *Suppose $\{X_n\}$ is the invertible moving average solution of the FARIMA equations* (1.1) *defined in Theorems* 2.2 *and* 2.3. *Let $\hat{X}_{n+k}$ be the truncated predictor of X_{n+k} defined by* (3.7) *and* (3.2), *and let e_n be its efficiency defined by* (1.12). *Set*

$$K := 2^d \Big(\sum_{j=0}^{k-1} |c_j|^\alpha\Big)^{-1}$$

and

$$K_1 := K_1(k, d, \alpha), \qquad K_{21} := K_{21}(k, d, \alpha), \qquad K_{22} := K_{22}(k, d, \alpha)$$

(*see* (3.18), (3.24) *and* (3.26)).

(a) *If* $1/\alpha - 1 < d < 0$, *then*

$$\limsup_{n\to\infty} \frac{1-e_n}{n^{1-(d+1)\alpha}} \leqslant KK_{22}.$$

(b) *If* $0 < d < \frac{1}{2}(1-1/\alpha)$, *then*

$$\limsup_{n\to\infty} \frac{1-e_n}{n^{-d\alpha}} \leqslant KK_{21}.$$

(c) *If* $d = \frac{1}{2}(1-1/\alpha)$, *then*

$$\limsup_{n\to\infty} \frac{1-e_n}{\sqrt{n^{1-\alpha}}} \leqslant K(K_1 + K_{21}).$$

(d) *If* $\frac{1}{2}(1-1/\alpha) < d < 1-1/\alpha$, *then*

$$\limsup_{n\to\infty} \frac{1-e_n}{n^{(d-1)\alpha+1}} \leqslant KK_1.$$

Proof. Use inequality (3.16) and Lemmas 3.1 and 3.2. Note that $(d-1)\alpha+1 < 1-(d+1)\alpha$ if $d<0$, and for $d>0$, $(d-1)\alpha+1 < -\alpha d$ iff $d < \frac{1}{2}(1-1/\alpha)$. ∎

COROLLARY 3.1. *The predictor* $\hat{X}_{n+k}$ *defined in Theorem* 3.2 *is asymptotically efficient.*

By (3.10), we have $S_1(m) \equiv 0$ for the one-step predictor $\hat{X}_{n+1}$. This observation combined with Lemmas 3.1 and 3.2 yields

THEOREM 3.3. *Let* e_n *be the efficiency of the one-step predictor* $\hat{X}_{n+1}$ *for the FARIMA process defined in Theorem* 3.2.

(a) *If* $1/\alpha - 1 < d < 0$, *then*

$$\limsup_{n\to\infty} \frac{1-e_n}{n^{1-(d-1)\alpha}} \leqslant \frac{1}{(d+1)\alpha-1} \left|\frac{\Phi_p(1)}{\Theta_q(1)\Gamma(-d)}\right|^{\alpha} \Big(\sum_{j=0}^{\infty} |c_j|\Big)^{\alpha}.$$

(b) *If* $0 < d < 1-1/\alpha$, *then*

$$\limsup_{n\to\infty} \frac{1-e_n}{n^{-d\alpha}} \leqslant \left|\frac{\Phi_p(1)}{\Theta_q(1)\Gamma(1-d)}\right|^{\alpha} \sum_{j=0}^{\infty} |c_j|^{\alpha}.$$

The next theorem gives upper bounds on $1-e_n$ for the ARMA(p, q) time series defined by the equations

$$\Phi_p(B)X_n = \Theta_q(B)Z_n. \tag{3.33}$$

THEOREM 3.4. *Suppose the polynomials* Φ_p *and* Θ_q *in* (3.33) *have no roots in the closed unit disk* $\{z\colon |z| \leqslant 1\}$ *and no roots in common. Set*

$$r = \min\{|z|\colon \Theta_q(z) = 0\}, \qquad s = \min\{|z|\colon \Phi_p(z) = 0\}.$$

If $Q > \max(1/r, 1/s)$, *then for any* $\alpha \in (0, 2]$

$$\limsup_{n\to\infty} \frac{1-e_n}{Q^{\alpha n}} = 0, \tag{3.34}$$

where e_n *is defined by* (1.12).

Proof. Define coefficients c_j and h_j by

$$\sum_{j=0}^{\infty} c_j z^j = \frac{\Theta_q(z)}{\Phi_p(x)}, \qquad \sum_{j=0}^{\infty} h_j z^j = \frac{\Phi_p(z)}{\Theta_q(z)}, \qquad |z| \leqslant 1.$$

Notice that

$$\limsup_{j\to\infty} |c_j|^{1/j} = s, \qquad \limsup_{j\to\infty} |h_j|^{1/j} = r.$$

Observe that Proposition 3.1 and its proof remain valid for invertible ARMA processes, and so by (3.16) and (3.9)–(3.11) it suffices to show that

$$\limsup_{n\to\infty} Q^{-\alpha n} \sum_{m=n+1}^{\infty} |c_m|^\alpha = 0 \tag{3.35}$$

and

$$\limsup_{n\to\infty} Q^{-\alpha n} \sum_{m=n+1}^{\infty} \Big| \sum_{u=k-j+n+1}^{k-j+m} h_u c_{k-j+m-n}\Big|^\alpha = 0. \tag{3.36}$$

To verify (3.35), note that for any $q \in (1/s, Q)$ and sufficiently large m, $|c_m| < q^m$, and so

$$Q^{-\alpha n} \sum_{m=n+1}^{\infty} |c_m|^\alpha < \frac{q^\alpha}{1-q^\alpha}\left(\frac{q}{Q}\right)^{\alpha n}.$$

To verify (3.36), recall that the double sum in (3.36) is equal to $\|h^{(k-j+n+1)} * c\|_\alpha^\alpha$. Since

$$\|h^{(n)} * c\|_\alpha^\alpha \leqslant \|h^{(n)}\|_\alpha^\alpha \|c\|_{\alpha\wedge 1}^\alpha,$$

it is enough to show that

$$\limsup_{n\to\infty} Q^{-\alpha n} \sum_{m=n+1}^{\infty} |h_m|^\alpha = 0,$$

which is proved in the same way as (3.35). ∎

Remark. For the one-step predictor $\hat{X}_{n+1}$, equality (3.34) holds for any $Q > 1/r$.

The power function rates of convergence obtained in Theorems 3.2 and 3.3 are essentially slower than the exponential rates for ARMA processes. This reflects the fact that for FARIMA processes the future value X_{n+k} depends more strongly on the past (long range dependence) than is the case for ARMA processes. Consequently, a larger number of past observations have impact on the value of X_{n+k} and must be taken into account to obtain a desired accuracy of prediction.

In Kokoszka and Taqqu [18] and [17], a measure of dependence, the *codifference*, is used to describe quantitatively the dependence structure of infinite variance stable ARMA and FARIMA sequences. For ARMA processes the codifference decays exponentially to zero, whereas for FARIMA processes like a power function.

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