PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 18, Fasc. 2 (1998), pp. 247–270

ON APPROXIMATIONS TO GENERALIZED COX PROCESSES

BY

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Abstract. A refined estimate of the rate of convergence of one-dimensional distributions of nonrandomly centered generalized Cox processes to location mixtures of normal laws is presented. Asymptotic expansions for these distributions are constructed. Some estimates for the concentration functions of these distributions are proved.

1. Introduction. This paper* is a continuation of our works [1]–[3] and [7] in which we considered generalized Poisson and generalized doubly stochastic Poisson processes (called also *generalized Cox processes*). Some fragments of our research were described in [5]. Here we consider the accuracy of the approximation of one-dimensional distributions of generalized Cox processes by location mixtures of normal laws. We deal with nonrandomly centered generalized Cox processes since the very problem of construction of approximations assumes that the approximated random process should be nonrandomly centered. In [2] we gave necessary and sufficient conditions for the convergence of one-dimensional distributions of nonrandomly centered generalized Cox processes, proved some convergence rate estimate and formulated two theorems on asymptotic expansions. Here we sharpen the convergence rate estimate given in [2], prove and discuss the results on asymptotic expansions announced in [2] and present some estimates for the concentration functions of generalized Cox processes.

Let $N_1(t)$, $t \geqslant 0$, be a homogeneous Poisson process with unit intensity and let $\Lambda(t)$, $t \geqslant 0$, be a process independent of $N_1(t)$ and having the following properties: $\Lambda(0) = 0$, $\mathsf{P}(\Lambda(t) < \infty) = 1$ for any $t > 0$, the trajectories of $\Lambda(t)$ do not decrease and are right-continuous. A doubly stochastic Poisson process $N(t)$, called also a *Cox process*, is defined as the superposition of $N_1(t)$ and $\Lambda(t)$:

$$N(t) = N_1(\Lambda(t)), \quad t \geqslant 0.$$

* This research was supported by the Russian Foundation for Fundamental Research, Grants 96-01-01919, 97-01-00271, and by the Russian Humanitarian Scientific Foundation, Grant 97-02-02235.

In this case we shall say that the Cox process $N(t)$ is *controlled* by the process $\Lambda(t)$. The properties of Cox processes are described in rather full detail in [6].

Let $X_1, X_2, \ldots$ be identically distributed random variables (r.v.'s). Assume that for each $t \geqslant 0$ the r.v.'s $N(t), X_1, X_2, \ldots$ are independent. The process

$$S(t) = \sum_{j=1}^{N(t)} X_j, \quad t \geqslant 0, \tag{1.1}$$

will be called a *generalized Cox process* (for definiteness, we assume that $\sum_{j=1}^{0} = 0$). The processes of the form (1.1) play important roles in many practical problems. For example, if $\Lambda(t) \equiv \lambda t$ with $\lambda > 0$, then $S(t)$ turns into a classical generalized Poisson process which is widely used as a model of many real phenomena in physics, reliability theory, financial and actuarial mathematics, etc. Many applied problems which can be reduced to the analysis of special generalized Poisson processes are described in [3] and [5]. More general processes $S(t)$ of type (1.1) with random intensity $\Lambda'(t)$ are of course more adequate models for the processes of the payments of an insurance company or of the increments of stock prices where real intensity is essentially stochastic. It should be especially noted that the risk process, that is, the surplus of an insurance company, which plays the key role in actuarial mathematics is, by definition, a nonrandomly centered process (1.1) with $N(t)$ being the number of claims up to time t. In the classical definition of a risk process, $N(t)$ is a Poisson process. Therefore the results presented below concern the generalization of some of the classical asymptotical results of the risk theory to generalized Cox processes.

Throughout the paper, the symbols $\Rightarrow$, $\xrightarrow{P}$ and $\stackrel{d}{=}$ will denote weak convergence, convergence in probability and coincidence of distributions, respectively. The standard normal distribution function (d.f.) and its density will be denoted by Φ and ϕ, respectively.

For the sake of convenience, without loss of generality everywhere in what follows we will assume that $\mathsf{E}\Lambda(t) = t$, $t \geqslant 0$. This relation can be interpreted both as the proportionality of the mathematical expectation of the controlling process to time and (which is most important for asymptotic inference) as the parametrization of the controlling process $\Lambda(t)$ by its expectation. Since we consider one-dimensional distributions, we therefore will construct approximations to generalized Cox processes with infinitely increasing expectation of the controlling process. In addition to the above assumptions we will assume that there exists $\mathsf{D}X_1 = \sigma^2$, $0 < \sigma^2 < \infty$. Let us write $\mathsf{E}X_1 = a$. Then, as is easy to see, for $t \geqslant 0$ we have $\mathsf{E}S(t) = at$, $\mathsf{D}S(t) = \sigma^2 t + a^2 \mathsf{D}N(t)$, and hence $\mathsf{D}S(t) \neq \sigma^2 t$ if $a \neq 0$. Nevertheless, in the subsequent reasoning we will normalize $S(t)$ by $\sigma\sqrt{t}$ instead of $\sqrt{\mathsf{D}S(t)}$, thus formally not assuming the existence of the second moment of the controlling process $\Lambda(t)$. In [2] we proved the following result:

THEOREM 1. *Assume that* $\Lambda(t) \xrightarrow{P} \infty$ $(t \to \infty)$. *Then one-dimensional distributions of a nonrandomly centered and normalized generalized Cox process* (1.1) *weakly converge to the distribution of some r.v.* Z:

$$\frac{S(t)-at}{\sigma\sqrt{t}} \Rightarrow Z \quad (t \to \infty) \tag{1.2}$$

if and only if there exists an r.v. V *such that*

(i) $Z \stackrel{d}{=} \sqrt{(1+a^2/\sigma^2)}\, W + (a/\sigma) V$, *with* W *and* V *independent and* $\mathbf{P}(W < x) = \Phi(x)$, $x \in \boldsymbol{R}$;

(ii) $(\Lambda(t)-t)/\sqrt{t} \Rightarrow V$ $(t \to \infty)$.

From this theorem it follows, for example, that under conditions of Theorem 1 one-dimensional distributions of a nonrandomly centered and normalized generalized Cox process (1.1) are asymptotically normal:

$$\mathbf{P}\left(\frac{S(t)-at}{\sigma\sqrt{t}} < x\right) \Rightarrow \Phi\left(\frac{x}{\delta}\right) \quad (t \to \infty),$$

with some asymptotic variance $\delta^2 < \infty$, if and only if $\delta^2 \geqslant 1$ and

$$\mathbf{P}\left(\frac{\Lambda(t)-t}{\sqrt{t}} < x\right) \Rightarrow \Phi\left(\frac{ax}{\sigma\sqrt{\delta^2-1}}\right) \quad (t \to \infty).$$

2. Convergence rate estimates for nonrandomly centered generalized Cox processes. In this section we shall give some estimates for the rate of convergence in Theorem 1. It follows from Theorem 1 that the distribution of the r.v. $(S(t)-at)/(\sigma\sqrt{t})$ is close to the limit one if and only if the distribution of the r.v. $(\Lambda(t)-t)/\sqrt{t}$ is close to that of V or, which is in a certain sense the same, if the distribution of the r.v. $\Lambda(t)$ is close to that of $\sqrt{t}V+t$. However, in general, the latter r.v. can also take negative values while the controlling process of a Cox process has to be positive. Therefore, instead of $\sqrt{t}V+t$ we will deal with the "accompanying" process $\Lambda^*(t) = |\sqrt{t}V+t|$, which, as $t \to \infty$, behaves more and more like $\sqrt{t}V+t$, and hence like $\Lambda(t)$ (in [5] we considered the accompanying controlling process of the form $\Lambda^*(t) = \max\{0, \sqrt{t}V+t\}$).

Let $N^*(t)$ be a Cox process controlled by the process $\Lambda^*(t)$ and

$$S^*(t) = \sum_{j=0}^{N^*(t)} X_j. \tag{2.1}$$

By $F_t(x)$ and $F_t^*(x)$ we will denote the d.f.'s of the r.v.'s $(S(t)-at)/(\sigma\sqrt{t})$ and $(S^*(t)-at)/(\sigma\sqrt{t})$, respectively. Then from the identity

$$F_t(x) = (F_t(x)-F_t^*(x))+F_t^*(x) \tag{2.2}$$

it follows that given an appropriate estimate of the accuracy of the approximation of the distribution of the generalized Cox process $S(t)$ by the distribution of $S^*(t)$ and an estimate of the rate of convergence as $t \to \infty$ of the distributions of the process $S^*(t)$ to the limit one, we can obtain an appropriate estimate of the rate of convergence of the generalized Cox process $S(t)$.

First give an estimate for

$$\Delta(t) \equiv \sup_x |F_t(x) - F_t^*(x)|.$$

LEMMA 1. *Let $S_1(t)$ and $S_2(t)$ be two generalized Cox processes generated by one and the same sequence of r.v.'s $\{X_j\}_{j\geq 1}$ and controlled by processes $\Lambda_1(t)$ and $\Lambda_2(t)$, respectively. Then*

$$\sup_x |\mathbf{P}(S_1(t) < x) - \mathbf{P}(S_2(t) < x)| \leq \int_0^\infty \min\left(2, \frac{1}{\sqrt{\lambda}}\right) |\mathbf{P}(\Lambda_1(t) < \lambda) - \mathbf{P}(\Lambda_2(t) < \lambda)|\, d\lambda.$$

The proof of this statement can be found in [1].

From Lemma 1 it follows that

$$(2.3) \quad \Delta(t) \leq \int_0^\infty \min\left(2, \frac{1}{\sqrt{\lambda}}\right) |\mathbf{P}(\Lambda(t) < \lambda) - \mathbf{P}(|\sqrt{t}V + t| < \lambda)|\, d\lambda$$
$$\leq \int_0^\infty \min\left(2, \frac{1}{\sqrt{\lambda}}\right) |\mathbf{P}(\Lambda(t) < \lambda) - \mathbf{P}(\sqrt{t}V + t < \lambda)|\, d\lambda + \int_0^\infty \min\left(2, \frac{1}{\sqrt{\lambda}}\right) \mathbf{P}\left(V < -\frac{\lambda + t}{\sqrt{t}}\right) d\lambda$$
$$\leq \int_{-\sqrt{t}}^\infty \min\left(2\sqrt{t}, \frac{1}{\sqrt{u/\sqrt{t}+1}}\right) \left|\mathbf{P}\left(\frac{\Lambda(t) - t}{\sqrt{t}} < u\right) - \mathbf{P}(V < u)\right| d\lambda + \int_0^\infty \min\left(2, \frac{1}{\sqrt{\lambda}}\right) \mathbf{P}\left(V < -\frac{\lambda + t}{\sqrt{t}}\right) d\lambda \equiv \omega(t).$$

The first summand in $\omega(t)$ is the mean metric with the weight function

$$w(t, u) = \min\{2\sqrt{t}, (u/\sqrt{t}+1)^{-1/2}\}\, \mathbf{1}(u > -\sqrt{t}),$$

which characterizes the distance between the pre-limit and limit d.f.'s of the controlling process (here and in what follows by $\mathbf{1}(A)$ we denote the indicator function of a set A). It is easily seen that for $u < 0$ the derivative of the function $(u/\sqrt{t}+1)^{-1/2}$ with respect to t is negative, and hence, at these u, $w(t, u)$ does not exceed its value at the point t at which $2\sqrt{t} = (u/\sqrt{t}+1)^{-1/2}$. This value equals $\frac{1}{2}(|u| + \sqrt{u^2+1}) \leq |u| + 1$. It is obvious that with nonnegative u we have $w(t, u) \leqslant 1$ so that $w(t, u) \leqslant |u|+1$. Therefore the properties of the first summand in $\omega(t)$ are similar to those of well-studied difference pseudomoments (see [12], pp. 33 and 113–126). At the same time, the second summand in $\omega(t)$ does not depend on the pre-limit controlling process. It is easy to see that if, for example, V is the standard normal r.v., then

$$\int_0^\infty \min\left(2, \frac{1}{\sqrt{\lambda}}\right) \mathsf{P}\left(V < -\frac{\lambda+t}{\sqrt{t}}\right) d\lambda \leqslant \int_0^\infty \min\left(2, \frac{1}{\sqrt{\lambda}}\right) \frac{\sqrt{t}}{\lambda+t}\, \phi\left(\frac{\lambda+t}{\sqrt{t}}\right) d\lambda$$
$$\leqslant \exp\left\{-\frac{t}{2}\right\} \left(\frac{1}{2\sqrt{\pi t}}+\sqrt{\frac{t\pi}{8}}\right),$$

that is, in this case the second summand decreases exponentially as t grows.

Let us put

$$\beta_3 = \mathsf{E}|X_1|^3, \quad \mu_3 = \mathsf{E}|X_1-a|^3, \quad L_3 = C_0\beta_3/(\sigma^2+a^2)^{3/2},$$

where C_0 is an absolute constant in the Berry–Esseen inequality. It is well known that

$$0.4097 \approx \frac{\sqrt{10}+3}{6\sqrt{2\pi}} \leqslant C_0 \leqslant 0.7655.$$

LEMMA 2. *Assume that $\beta_3 < \infty$. Let N_λ be a Poisson r.v. with parameter $\lambda > 0$ independent of the sequence $\{X_j\}_{j\geqslant 1}$. Then*

$$\sup_x \left|\mathsf{P}\left(\frac{1}{\sigma\sqrt{\lambda}}\left(\sum_{j=1}^{N_\lambda} X_j - a\lambda\right) < x\right) - \Phi\left(\frac{x\sigma}{\sqrt{\sigma^2+a^2}}\right)\right| \leqslant \frac{L_3}{\sqrt{\lambda}}.$$

For the proof see [3] with the constant refined in [8] (¹).

Let us write

$$\varrho(t) = \sup_x \left|F_t(x) - \mathsf{E}\Phi\left(\frac{\sigma x - aV}{\sqrt{\sigma^2+a^2}}\right)\right|.$$

THEOREM 2. *Let $\beta_3 < \infty$, $\mathsf{E}|V| < \infty$. Then for all $t > 0$ such that $\mathsf{P}(V = -\sqrt{t}) = 0$ we have*

$$\varrho(t) \leqslant \frac{1}{\sqrt{t}} \inf_{0<q<1} \left[\frac{L_3}{\sqrt{1-q}} + Q(q)\mathsf{E}|V|\right] + \omega(t) \tag{2.4}$$

with $\omega(t)$ defined in (2.3) and

$$Q(q) = \max\left\{\frac{1}{q}, \frac{1}{\sqrt{2\pi e(1-q)}(1+\sqrt{1-q})}\right\}.$$

(¹) After the paper [8] had appeared its authors learned about the paper [10] published much earlier with exactly the same estimates which were independently obtained in [8].

Proof. Using (2.2) and (2.3) it suffices to estimate

$$\alpha(t) = \sup_x \left| F_t^*(x) - \mathsf{E}\,\Phi\left(\frac{\sigma x - aV}{\sqrt{\sigma^2+a^2}}\right)\right|.$$

With $\gamma > 0$, by N_γ we will denote a Poisson r.v. with parameter γ assuming it to be independent of the sequence $\{X_j\}_{j\geq 1}$. Then for any $\varepsilon \in (0, 1)$

$$\alpha(t) = \sup_x \left| \int_{-\infty}^{\infty} \left[\mathsf{P}\left(\frac{1}{\sigma\sqrt{t}}\left(\sum_{j=1}^{N_{|v\sqrt{t}+t|}} X_j - at\right) < x\right) - \Phi\left(\frac{\sigma x - av}{\sqrt{\sigma^2+a^2}}\right)\right] d\mathsf{P}(V<v)\right|$$

$$\leqslant \left(\int_{-\infty}^{-\varepsilon\sqrt{t}} + \int_{-\varepsilon\sqrt{t}}^{\infty}\right) \sup_x \left| \mathsf{P}\left(\frac{1}{\sigma\sqrt{t}}\left(\sum_{j=1}^{N_{|v\sqrt{t}+t|}} X_j - at\right) < x\right) - \Phi\left(\frac{\sigma x - av}{\sqrt{\sigma^2+a^2}}\right)\right| d\mathsf{P}(V<v)$$

$$\equiv I_1 + I_2.$$

By the Markov inequality we have

$$I_1 \leqslant \mathsf{P}(V < -\varepsilon\sqrt{t}) \leqslant \frac{\mathsf{E}\,|V|\,\mathbb{1}(V < -\varepsilon\sqrt{t})}{\varepsilon\sqrt{t}}. \tag{2.5}$$

At the same time we obtain

$$I_2 = \int_{-\varepsilon\sqrt{t}}^{\infty} \sup_x \left| \mathsf{P}\left(\frac{1}{\sigma\sqrt{v\sqrt{t}+t}}\left(\sum_{j=1}^{N_{v\sqrt{t}+t}} X_j - a(v\sqrt{t}+t)\right) < \frac{(\sigma x - av)\sqrt{t}}{\sigma\sqrt{v\sqrt{t}+t}}\right)\right.$$
$$\left. - \Phi\left(\frac{\sigma x - av}{\sqrt{a^2+\sigma^2}}\right)\right| d\mathsf{P}(V<v)$$

$$= \int_{-\varepsilon\sqrt{t}}^{\infty} \sup_y \left| \mathsf{P}\left(\frac{1}{\sigma\sqrt{v\sqrt{t}+t}}\left(\sum_{j=1}^{N_{v\sqrt{t}+t}} X_j - a(v\sqrt{t}+t)\right) < y\right)\right.$$
$$\left. - \Phi\left(\frac{\sigma y}{\sqrt{a^2+\sigma^2}}\sqrt{\frac{v}{\sqrt{t}}+1}\right)\right| d\mathsf{P}(V<v)$$

$$\leqslant \int_{-\varepsilon\sqrt{t}}^{\infty} \sup_y \left| \mathsf{P}\left(\frac{1}{\sigma\sqrt{v\sqrt{t}+t}}\left(\sum_{j=1}^{N_{v\sqrt{t}+t}} X_j - a(v\sqrt{t}+t)\right) < y\right)\right.$$
$$\left. - \Phi\left(\frac{\sigma y}{\sqrt{a^2+\sigma^2}}\right)\right| d\mathsf{P}(V<v)$$
$$+ \int_{-\varepsilon\sqrt{t}}^{\infty} \sup_y \left| \Phi(y) - \Phi\left(y\sqrt{\frac{v}{\sqrt{t}}+1}\right)\right| d\mathsf{P}(V<v) \equiv I_{21} + I_{22}.$$

Estimate I_{22} with the help of Lemma 6.3.2 in [9], according to which

$$|\Phi(x) - \Phi(px)| \leqslant x\phi(\min(1, p)x)\,|p-1| \quad \text{for } p > 0.$$

Then

$$I_{22} \leqslant \int_{-\varepsilon\sqrt{t}}^{0} \Big(\sup_{y>0} y\phi\Big(y\sqrt{\frac{v}{\sqrt{t}}+1}\Big)\Big) \Big|1-\sqrt{\frac{v}{\sqrt{t}}+1}\Big|\, d\mathbf{P}(V<v)$$

$$+\int_{0}^{\infty} (\sup_{y>0} y\phi(y)) \Big|1-\sqrt{\frac{v}{\sqrt{t}}+1}\Big|\, d\mathbf{P}(V<v) \equiv I_{221}+I_{222}.$$

Moreover,

$$I_{221} \leqslant \int_{-\varepsilon\sqrt{t}}^{0} [\sup_{y>0} y\phi(y\sqrt{1-\varepsilon})] \Big|1-\sqrt{\frac{v}{\sqrt{t}}+1}\Big|\, d\mathbf{P}(V<v)$$

$$= \frac{1}{\sqrt{2\pi e(1-\varepsilon)}} \int_{-\varepsilon\sqrt{t}}^{0} \frac{\big|\big(1-\sqrt{v/\sqrt{t}+1}\big)\big(1+\sqrt{v/\sqrt{t}+1}\big)\big|}{1+\sqrt{v/\sqrt{t}+1}}\, d\mathbf{P}(V<v)$$

$$\leqslant \frac{1}{\sqrt{2\pi et(1-\varepsilon)}(\sqrt{1-\varepsilon}+1)} \mathbf{E}|V|\mathbf{1}(-\varepsilon\sqrt{t}<V\leqslant 0),$$

$$I_{222} \leqslant \frac{1}{\sqrt{2\pi e}} \int_{0}^{\infty} \frac{\big|\big(1-\sqrt{v/\sqrt{t}+1}\big)\big(1+\sqrt{v/\sqrt{t}+1}\big)\big|}{1+\sqrt{v/\sqrt{t}+1}}\, d\mathbf{P}(V<v)$$

$$\leqslant \frac{1}{\sqrt{8\pi et}} \mathbf{E}|V|\mathbf{1}(V>0).$$

Therefore, with regard to (2.5) we obtain

$$\text{(2.6)} \quad I_1+I_{22} \leqslant \max\Big(\frac{1}{\varepsilon}, \frac{1}{\sqrt{2\pi e(1-\varepsilon)}(\sqrt{1-\varepsilon}+1)}, \frac{1}{\sqrt{8\pi e}}\Big) \frac{\mathbf{E}|V|}{\sqrt{t}} = Q(\varepsilon)\frac{\mathbf{E}|V|}{\sqrt{t}}$$

since the third term under the max sign is always less than the second one. Estimating I_{21} with the help of Lemma 2 we obtain

$$\text{(2.7)} \qquad I_{21} \leqslant L_3 \int_{-\varepsilon\sqrt{t}}^{\infty} \frac{d\mathbf{P}(V<v)}{\sqrt{v\sqrt{t}+t}} \leqslant \frac{L_3}{\sqrt{t(1-\varepsilon)}}.$$

By unifying (2.6) with (2.7), we arrive at the desired estimate. The theorem is proved.

COROLLARY 1. *Under the conditions of Theorem* 2,

$$\varrho(t) \leqslant \frac{1.32}{\sqrt{t}}\Big(\frac{\beta_3}{(a^2+\sigma^2)^{3/2}}+\mathbf{E}|V|\Big)+\omega(t).$$

To prove Corollary 1 it suffices to calculate the right-hand side of (2.4) with q being the solution of the equation $Q(q)\sqrt{1-q} = 0.7655$.

Remark 1. In many applied problems the data is represented by observations registered at equidistant time instants (time series). And if a seasonal component can be assumed pronounced in this series with a seasonality period t coinciding with the length of the time intervals between observations, then it is quite reasonable to assume from the very beginning that we deal not with $\Lambda(t)$ but with the "accompanying" controlling process $|\sqrt{t}V+t|$. This assumption results in $\omega(t)=0$.

Remark 2. The condition $\mathbf{P}(V=-\sqrt{t})=0$ guarantees the positiveness of the "accompanying" controlling process, and hence the correctness of the definition of all the Poisson r.v.'s considered in the proof of Theorem 2.

3. Asymptotic expansions for the distributions of generalized Cox processes. Relation (2.2) allows us to obtain an appropriate asymptotic expansion for $F_t(x)$ from the asymptotic expansion for $F_t^*(x)$ and an estimate for the difference $F_t(x)-F_t^*(x)$, e.g., given in Lemma 1.

We shall say that an r.v. Y satisfies the *Cramér condition* (see, e.g., [11], Section VI.3) if

$$\limsup_{|s|\to\infty}|\mathbf{E}\exp\{isY\}|<1. \tag{3.1}$$

The following statement is well known (see [11], Section VI.3). Let us put $S_n=X_1+\ldots+X_n$.

PROPOSITION. *Let in addition to the above assumptions the r.v.'s $\{X_j\}_{j\geqslant 1}$ satisfy the Cramér condition* (3.1) *and* $\mathbf{E}|X_1|^k<\infty$, *where $k\geqslant 3$ is integer. Then*

$$\sup_x\left|\mathbf{P}\left(\frac{S_n-an}{\sigma\sqrt{n}}<x\right)-\Phi(x)-\sum_{j=1}^{k-2}\tau^j Q_j(x)\right|=o(\tau^{k-2}),$$

where $\tau=n^{-1/2}$ and the functions $Q_j(x)$ are defined by

$$Q_j(x)=-\phi(x)\sum H_{j+2l-1}(x)\prod_{m=1}^{j}\frac{1}{k_m!}\left(\frac{\gamma_{m+2}}{(m+2)!\,\sigma^{m+2}}\right)^{k_m},\quad j=1,\ldots,k-2. \tag{3.2}$$

Here the summation is carried out over all nonnegative solutions $(k_1,\ldots,k_j)$ of the equation $k_1+2k_2+\ldots+jk_j=j$, $l=k_1+\ldots+k_j$; γ_{m+2} is the semi-invariant of order $m+2$ of the r.v. X_1, and $H_m(x)$ are the Chebyshev–Hermite polynomials of degree m, i.e.,

$$H_m(x)\,\phi(x)=(-1)^m\,\phi^{(m)}(x).$$

In particular, if we put $\alpha_l=\mathbf{E}X_1^l$, $l=1,2,\ldots$, then

$$Q_1(x)=-\phi(x)(x^2-1)\frac{\alpha_3}{6\sigma^3}, \tag{3.3}$$

$$Q_2(x)=-\phi(x)\left[(x^3-3x)\frac{\alpha_4-3\sigma^4}{24\sigma^4}+(x^5-10x^3+15x)\frac{\alpha_3^2}{72\sigma^6}\right]. \tag{3.4}$$

Now we shall give two analogs of this statement for generalized Cox processes. In the case $a = 0$ the asymptotic expansion for the d.f.'s of generalized Cox processes was constructed in [2]. Here we shall concentrate our attention on the situation $\mathsf{E} X_1 = a \neq 0$. In this case the limit distribution of nonrandomly centered generalized Cox processes is determined by Theorem 1 and is of the form

$$\mathsf{E}\Phi\left(\frac{\sigma x - aV}{\sqrt{\sigma^2+a^2}}\right),$$

where V is the limit r.v. for the standardized controlling process $(\Lambda(t)-t)/\sqrt{t}$. (As before, we assume that $\mathsf{E}\Lambda(t) \equiv t$.) At first we shall present a statement on the asymptotic expansion for the generalized Cox process with a rather peculiar structure of the limit r.v. V. However, as we shall see below, this situation turns out to be quite natural.

THEOREM 3. *Let the r.v. V have the form $V = V_0 - \mathsf{E}V_0$, where V_0 is a nonnegative r.v. satisfying the condition: there exist $\gamma > 0$ and a polynomial $P(h)$ such that for any $h \geqslant 0$*

$$\mathsf{E}\exp\{hV_0\} \leqslant P(h)\exp\{\gamma h^2\}. \tag{3.5}$$

Assume that $\mathsf{E}|X_1|^k < \infty$ for some integer $k \geqslant 3$ and that X_1 satisfies the Cramér condition (3.1). *Then for any $t \geqslant (\mathsf{E}V_0)^2$ we have*

$$\sup_x \left|\mathsf{P}\left(\frac{S(t)-at}{\sigma\sqrt{t}} < x\right) - \mathsf{E}\Phi\left(\frac{\sigma x - aV}{\sqrt{\sigma^2+a^2}}\right) - \sum_{j=1}^{k-2}\frac{w_j(x)}{t^{j/2}}\right| \leqslant \varepsilon(t),$$

where

$$\varepsilon(t) = \int_0^\infty \min\left(2, \frac{1}{\sqrt{\lambda}}\right)\left|\mathsf{P}(\Lambda(t)<\lambda) - \mathsf{P}(\Lambda_0(t)<\lambda)\right| d\lambda + o(t^{-(k-2)/2}),$$

$$\Lambda_0(t) = \sqrt{t}V + t,$$

$$w_j(x) = \sum_{\substack{l+m=j\\ l,m\geqslant 0}} \sum_{d=0}^{m} \int_{-\infty}^{x} \bar{P}_l(-D_y)P_{md}(-D_y)\chi_d(y)\,dy, \quad j = 0, \ldots, k-2,$$

$$\chi_d(y) = \frac{\sigma}{\sqrt{\sigma^2+a^2}}\mathsf{E}\left[V^d\phi\left(\frac{\sigma y - aV}{\sqrt{\sigma^2+a^2}}\right)\right], \quad d = 0, \ldots, k-2,$$

D_y is the operator of formal differentiation with respect to y, the polynomials $\bar{P}_l(\cdot)$ are defined by the relation

$$\bar{P}_l(it) = \sum \prod_{m=1}^{j} \frac{1}{k_m!}\left[\frac{(it)^{m+2}\alpha_{m+2}}{(m+2)!\,\sigma^{m+2}}\right]^{k_m}, \quad j = 1, \ldots, k-2,$$

in which the summation is carried over all nonnegative solutions $k_1, \ldots, k_j$ of the equation $k_1 + 2k_2 + \ldots + jk_j = j$, $\alpha_{m+2} = \mathsf{E}X_1^{m+2}$, $\bar{P}_0(x) \equiv 1$, the polynomials

$P_{md}(\cdot)$ *are defined by the formal equality*

$$\exp\left\{\sum_{l=1}^{k-2} v \frac{x^{l+1}\alpha_{l+1}}{(l+1)!\,\sigma^{l+1}} t^{-l/2}\right\} = \sum_{m=0}^{\infty} t^{-m/2} \sum_{d=0}^{m} v^d P_{md}(x). \tag{3.6}$$

Remark 3. It is easy to see that

$$P_{00}(x) \equiv 1, \quad P_{j0} \equiv 0, \quad j = 1, \ldots, k-2,$$

$$P_{11}(x) = \frac{x^2\alpha_2}{2\sigma^2}, \quad P_{21}(x) = \frac{x^3\alpha_3}{6\sigma^3}, \quad P_{22}(x) = \frac{x^4\alpha_2^2}{8\sigma^4},$$

$$P_{31}(x) = \frac{x^4\alpha_4}{24\sigma^4}, \quad P_{32}(x) = \frac{x^5\alpha_2\alpha_3}{12\sigma^5}, \quad P_{33}(x) = \frac{x^6\alpha_2^3}{48\sigma^6},$$

$$\bar{P}_1(x) = \frac{x^3\alpha_3}{6\sigma^3}, \quad \bar{P}_2(x) = \frac{x^4\alpha_4}{24\sigma^4} + \frac{x^6\alpha_3^2}{72\sigma^6}.$$

Remark 4. The first two functions w_1 and w_2 have the form

$$w_1(x) = -\int \phi\left(\frac{\sigma x - az}{\sqrt{a^2+\sigma^2}}\right)\left[\frac{\alpha_3}{6\sigma\alpha_2} H_2\left(\frac{\sigma x - az}{\sqrt{a^2+\sigma^2}}\right) + \frac{z}{2} H_1\left(\frac{\sigma x - az}{\sqrt{a^2+\sigma^2}}\right)\right] d\mathbf{P}(V < z),$$

$$w_2(x) = -\int \phi\left(\frac{\sigma x - az}{\sqrt{a^2+\sigma^2}}\right)\left[\frac{\alpha_4}{24\sigma^2\alpha_2} H_3\left(\frac{\sigma x - az}{\sqrt{a^2+\sigma^2}}\right) + \frac{\alpha_3}{72\sigma^4\alpha_2} H_5\left(\frac{\sigma x - az}{\sqrt{a^2+\sigma^2}}\right)\right.$$
$$+ z\left(\frac{\alpha_3}{12\sigma^3} H_4\left(\frac{\sigma x - az}{\sqrt{a^2+\sigma^2}}\right) + \frac{\alpha_3}{6\sigma\alpha_2} H_2\left(\frac{\sigma x - az}{\sqrt{a^2+\sigma^2}}\right)\right)$$
$$\left. + \frac{z^2\alpha_2^3}{8\sigma^2} H_3\left(\frac{\sigma x - az}{\sqrt{a^2+\sigma^2}}\right)\right] d\mathbf{P}(V < z),$$

where $H_m(\cdot)$ are the Chebyshev–Hermite polynomials.

Remark 5. Condition (3.5) holds if $V_0 = |\xi|$, where ξ is a bounded r.v. It also holds if ξ is a normal r.v. This condition does not hold for an exponential or Poisson r.v. V_0. The r.v. V_0 must have all moments.

Proof of Theorem 3. Using the relations (4.2), (4.4), (5.8)–(5.11) of the paper [1] to prove Theorem 3 it suffices to show that

$$J_1 \equiv \int_{|s| \leq \delta\sqrt{t}} \left|\frac{f_t^*(s) - h(s,t)}{s}\right| ds = o(t^{-(k-2)/2}), \tag{3.7}$$

$$J_2 \equiv \int_{\delta\sqrt{t} \leq |s| \leq At^{(k-2)/2}} \left|\frac{h(s,t)}{s}\right| ds = o(t^{-(k-2)/2}), \tag{3.8}$$

$$J_3 \equiv \int_{\delta\sqrt{t} \leq |s| \leq At^{(k-2)/2}} \left|\frac{f_t^*(s)}{s}\right| ds = o(t^{-(k-2)/2}), \tag{3.9}$$

where

$$f_t^*(s) = \mathbf{E}\exp\left\{is\left(\frac{S^*(t) - \alpha_1 t}{\sigma\sqrt{t}}\right)\right\},$$

(3.10) $$h(s,t)=\int e^{isx}\,d\Big(G(x)+\sum_{j=1}^{k-2} t^{-j/2} q_j(x)\Big)$$

$$=\sum_{j=0}^{k-2} t^{-j/2} \sum_{\substack{l+m=j\\ l,m\geqslant 0}} \sum_{d=0}^{m} V^d \exp\Big\{-\frac{s^2\alpha_2}{2\sigma^2}\Big\}\tilde{P}_l\Big(is\frac{\sqrt{\alpha_2}}{\sigma}\Big) P_{md}(is)\,\mathbf{E}\exp\Big\{\frac{is\alpha_1}{\sigma}V\Big\},$$

$$G(x)=\int_{-\infty}^{x}\Phi\Big(\frac{\sigma v-z\alpha_1}{\sqrt{\alpha_2}}\Big)\,d\mathbf{P}(V<v),$$

$\tilde{P}_0(x)\equiv 1$ and the polynomials $\tilde{P}_l(\cdot)$ are defined by formulas (3.3) with σ^2 and the semi-invariants γ_{m+2} replaced by α_2 and the moments α_{m+2}, respectively, and the polynomials $P_{md}(\cdot)$ are defined by relation (3.6).

Estimate the integrals J_i, $i=1,2,3$. Note that the characteristic function (ch.f.) $f_t^*(s)$ can be represented as

(3.11) $$f_t^*(s)=v^t\Big(\frac{s}{\sigma\sqrt{t}}\Big)\,\mathbf{E}\exp\Big\{V\sqrt{t}\Big(f_{X_1}\Big(\frac{s}{\sigma\sqrt{t}}\Big)-1\Big)\Big\},$$

where

$$v(s)=\exp\{f_{X_1}(s)-1-is\alpha_1\},\qquad f_{X_1}(s)=\mathbf{E}\exp\{isX_1\}.$$

Here $v(s)$ is the ch.f. of the r.v.

$$Y\equiv\sum_{i=1}^{N}X_i-\alpha_1,$$

where N is a Poisson r.v. with unit parameter independent of $X_1, X_2, \ldots$ so that $\mathbf{E}Y=0$, $\mathbf{D}Y=\alpha_2>0$, $\mathbf{E}|Y|^k<\infty$. Note also that the semi-invariants γ_j, $j=1,2,\ldots,k$, satisfy the relations

$$\gamma_1=0,\qquad \gamma_j=\alpha_j,\ j\geqslant 2.$$

To the function $v(s/(\sigma\sqrt{t}))^t$ we can apply Theorem 9.12 from [10] with n replaced by t. From this theorem it follows that for any $\varepsilon>0$ there exists a $\delta>0$ such that for $|s|\leqslant\delta\sqrt{t}$ we have

(3.12) $$\Big|v^t\Big(\frac{s}{\sigma\sqrt{t}}\Big)-\exp\Big\{-\frac{s^2\alpha_2}{2\sigma^2}\Big\}\Big(1+\sum_{j=1}^{k-2}\frac{\tilde{P}_j(is\sqrt{\alpha_2/\sigma^2})}{t^{j/2}}\Big)\Big|$$
$$\leqslant C\varepsilon t^{-(k-2)/2}(|s|^k+|s|^{3(k-2)})\exp\Big\{-\frac{s^2\alpha_2}{4\sigma^2}\Big\},\qquad C>0,$$

with polynomials $\tilde{P}(\cdot)$ of degrees $3j$ defined in (3.10).

Consider J_1. Let us put

(3.13) $$v(s,t)=\exp\Big\{\frac{-s^2\alpha_2}{2\sigma^2}\Big\}\Big(1+\sum_{j=1}^{k-2}t^{-j/2}\tilde{P}_j\Big(is\sqrt{\frac{\alpha_2}{\sigma^2}}\Big)\Big),$$

(3.14) $$f(s,t)=v(s,t)\,\mathbf{E}\exp\Big\{V\sqrt{t}\Big(f_{X_1}\Big(\frac{s}{\sigma\sqrt{t}}\Big)-1\Big)\Big\}.$$

Using the representation

$$f_{X_1}\left(\frac{s}{\sigma\sqrt{t}}\right) = 1+\frac{is\,\alpha_1}{\sigma\sqrt{t}}+r(s, t)$$

with

$$|r(s, t)| \leqslant \frac{s^2\alpha_2}{2\sigma^2 t}$$

and relations (3.11) and (3.12), for $|s| \leqslant \delta\sqrt{t}$ we obtain

$$(3.15)\quad |f_t^*(s)-f(s, t)| \leqslant \left|v^t\left(\frac{s}{\sigma\sqrt{t}}\right)-v(s, t)\right| \cdot \mathbf{E}\left|\exp\left\{V\sqrt{t}\left(f_{X_1}\left(\frac{s}{\sigma\sqrt{t}}\right)-1\right)\right\}\right|$$

$$\leqslant \left|v^t\left(\frac{s}{\sigma\sqrt{t}}\right)-v(s, t)\right| \cdot \mathbf{E}\exp\left\{|V|\,\frac{s^2\alpha_2}{2\sigma^2\sqrt{t}}\right\}$$

$$\leqslant C\varepsilon t^{-(k-2)/2}(|s|^k+|s|^{3(k-2)})\exp\left\{-\frac{s^2\alpha_2}{4\sigma^2}\right\}\mathbf{E}\exp\left\{|V|\frac{|s|\,\delta\alpha_2}{2\sigma^2}\right\}$$

$$\leqslant \varepsilon t^{-(k-2)/2} P_1(|s|)\exp\left\{\frac{|s|\,\delta\alpha_2}{2\sigma^2}\,\mathbf{E}|V_0|-\frac{s^2\alpha_2}{4\sigma^2}\left(1-\frac{\gamma\delta^2\alpha_2}{\sigma^2}\right)\right\},$$

where $P_1(\cdot)$ is some polynomial. The last inequality in (3.15) was obtained with the help of condition (3.5). Since δ is small enough, there is a function integrable with respect to s on the right-hand side of (3.15). Thus we have proved the inequality (see (3.7))

$$(3.16)\qquad J_1 \leqslant \bar{J}_1+o(t^{-(k-2)/2}),$$

where (see (3.13) and (3.14))

$$(3.17)\qquad \bar{J}_1 = \int\limits_{|s|\leqslant\delta\sqrt{t}}\left|\frac{f(s, t)-h(s)}{s}\right| ds.$$

Prove that $\bar{J}_1 = o(t^{-(k-2)/2})$. Let us write (see (3.13) and (3.2))

$$(3.18)\qquad f_1(s, t) = v(s, t)\,\mathbf{E}\exp\left\{V\sum_{j=1}^{k-1}\frac{(is)^j\alpha_j}{\sigma^j t^{(j-1)/2}}\right\},$$

$$(3.19)\qquad f_2(s, t) = v(s, t)\,\mathbf{E}\exp\left\{\frac{is\,\alpha_1}{\sigma}V\right\}\left[1+\sum_{l=1}^{k-2}\frac{V^l}{l!}\left(\sum_{j=2}^{k-1}\frac{(is)^j\alpha_j}{\sigma^j t^{(j-1)/2}}\right)^l\right],$$

$$(3.20)\qquad f_3(s, t) = v(s, t)\,\mathbf{E}\exp\left\{\frac{is\,\alpha_1}{\sigma}V\right\}\left[1+\sum_{j=1}^{k-2}t^{-j/2}\sum_{l=2}^{j}V^l P_{jl}(is)\right].$$

Then since

$$f_{X_1}\left(\frac{s}{\sigma\sqrt{t}}\right) = \sum_{j=1}^{k-1}\frac{(is)^j\alpha_j}{\sigma^j t^{j/2}}+r_k(s, t)$$

with

$$|r_k(s, t)| \leqslant \frac{|s|^k \alpha_k}{\sigma^k t^{k/2}},$$

by virtue of (3.15) we have

(3.21) $|f(s, t)-f_1(s, t)|$

$$\leqslant |v(s, t)| \mathbf{E} \left|\exp\left\{V \sum_{j=1}^{k-1} \frac{(is)^j \alpha_j}{\sigma^j t^{(j-1)/2}} + V r_k(s, t)\sqrt{t}\right\} - \exp\left\{V \sum_{j=1}^{k-1} \frac{(is)^j \alpha_j}{\sigma^j t^{(j-1)/2}}\right\}\right|$$

$$\leqslant |v(s, t)| \mathbf{E} |\exp\{\sqrt{t} V r_k(s, t)\} - 1|$$

$$\leqslant |v(s, t)| \sqrt{t} |r_k(s, t)| \mathbf{E} |V| \exp\{|V| \sqrt{t} |r_k(s, t)|\}$$

$$\leqslant |v(s, t)| \frac{|s|^k \alpha_k}{\sigma^k t^{(k-1)/2}} \mathbf{E} |V| \exp\left\{|V| \frac{|s|^k \alpha_k}{\sigma^k t^{(k-1)/2}}\right\}$$

$$\leqslant |v(s, t)| \frac{|s|^k \alpha_k}{\sigma^k t^{(k-1)/2}} (\mathbf{E} V^2)^{1/2} \left(\mathbf{E} \exp\left\{2|V| \frac{\delta^{k-1} |s| \alpha_k}{\sigma^k}\right\}\right)^{1/2}$$

$$\leqslant t^{-(k-1)/2} P_2(|s|) \exp\left\{\frac{\delta^{k-1} |s| \alpha_k}{\sigma^k} \mathbf{E} |V_0| - \frac{s^2 \alpha_2}{\sigma^2}\left(1 - \frac{\gamma \delta^{2(k-1)} \alpha_k^2}{\alpha_2 \sigma^{2k-2}}\right)\right\},$$

where $P_2(\cdot)$ is some polynomial and the last inequality is again due to condition (3.5). From this inequality it follows that for $\delta > 0$ small enough we have

$$\int_{|s| \leqslant \delta\sqrt{t}} \left|\frac{f(s, t)-f_1(s, t)}{s}\right| ds = o(t^{-(k-2)/2}). \tag{3.22}$$

Now consider the difference $f_1(s, t)-f_2(s, t)$ (see (3.18) and (3.19)). We have

$|f_1(s, t)-f_2(s, t)|$

$$\leqslant |v(s, t)| \mathbf{E} \left|\exp\left\{V \sum_{j=2}^{k-1} \frac{(is)^j \alpha_j}{\sigma^j t^{(j-2)/2}}\right\} - \sum_{l=0}^{k-2} \frac{V^l}{l!} \left(\sum_{j=2}^{k-1} \frac{(is)^j \alpha_j}{\sigma^j t^{(j-1)/2}}\right)^l\right|$$

$$\leqslant \frac{|v(s, t)|}{(k-1)!} \left(\sum_{j=2}^{k-1} \frac{|s|^j |\alpha_j|}{\sigma^j t^{(j-1)/2}}\right)^{k-1} \mathbf{E} |V|^{k-1} \exp\left\{|V| \sum_{j=2}^{k-1} \frac{|s|^j |\alpha_j|}{\sigma^j t^{(j-1)/2}}\right\}$$

$$\leqslant t^{-(k-1)/2} P_3(|s|) \exp\left\{-\frac{s^2 \alpha_2}{\sigma^2}\right\} (\mathbf{E} V^{2(k-1)})^{1/2} \left(\mathbf{E} \exp\left\{2|V| s \sum_{j=2}^{k-1} \frac{\delta^j |\alpha_j|}{\sigma^j}\right\}\right)^{1/2},$$

where $P_3(\cdot)$ is some polynomial. In the same way as (3.21), it follows that for $\delta > 0$ small enough we have

$$\int_{|s| \leqslant \delta\sqrt{t}} \left|\frac{f_1(s, t)-f_2(s, t)}{s}\right| ds = o(t^{-(k-2)/2}). \tag{3.23}$$

It is easy to see that

$$|f_2(s,t)-f_3(s,t)| \leqslant |v(s,t)|\, t^{-(k-1)/2} \sum_j \bar{R}_j(|s|)\, \mathsf{E}\, \tilde{P}_j(|V|), \tag{3.24}$$

where the sum over j contains a finite number of summands and $\bar{R}_j(\cdot)$ and $\tilde{P}_j(\cdot)$ are some polynomials. Hence it also follows that for $\delta > 0$ small enough we have

$$\int\limits_{|s| \leqslant \delta\sqrt{t}} \left| \frac{f_3(s,t)-f_2(s,t)}{s} \right| ds = o(t^{-(k-2)/2}). \tag{3.25}$$

Now consider the difference $f_3(s,t)-h(s,t)$ (see (3.10)). For this difference the inequality similar to (3.24) holds, and therefore

$$\int\limits_{|s| \leqslant \delta\sqrt{t}} \left| \frac{f_3(s,t)-h(s,t)}{s} \right| ds = o(t^{-(k-2)/2}). \tag{3.26}$$

Therefore from (3.22), (3.23), (3.25) and (3.26) it follows that $\bar{J}_1 = o(t^{-(k-2)/2})$ (see (3.17)), and hence by virtue of (3.16) we obtain $J_1 = o(t^{-(k-2)/2})$.

The relation $J_2 = o(t^{-(k-2)/2})$ (see (3.8)) follows directly from the form of the function $h(s,t)$ (see (3.10)).

Prove that $J_3 = o(t^{-(k-2)/2})$ (see (3.9)). Note that since the r.v. X_1 satisfies the Cramér condition, we have

$$\limsup_{|s|\to\infty} |f_{X_1}(s)| \leqslant q_1 < 1,$$

$$\limsup_{|s|\to\infty} |v(s)| \leqslant \limsup_{|s|\to\infty} \exp\{|f_{X_1}(s)|-1\} \leqslant q_2 < 1.$$

Therefore for $\delta\sqrt{t} \leqslant |s| \leqslant At^{(k-2)/2}$ we have

$$\left| v\left(\frac{s}{\sigma\sqrt{t}}\right) \right| \leqslant q_3 < 1$$

and (see (3.11))

$$\begin{aligned}
|f_t^*(s)| &\leqslant \left| v^t\left(\frac{s}{\sigma\sqrt{t}}\right) \right| \mathsf{E} \exp\{-V\sqrt{t}\} \left| \exp\left\{ V\sqrt{t} f_{X_1}\left(\frac{s}{\sigma\sqrt{t}}\right) \right\} \right| \\
&\leqslant q_3^t\, \mathsf{E} \exp\{-\sqrt{t}V\} \exp\left\{ \sqrt{t}V \left| f_{X_1}\left(\frac{s}{\sigma\sqrt{t}}\right) \right| \right\} \\
&\leqslant q_3^t\, \mathsf{E} \exp\{-\sqrt{t}V(1-q_4)\} \leqslant q_3^t \exp\{\sqrt{t}\,\mathsf{E}\,|V_0|\}, \qquad 0 < q_4 < 1.
\end{aligned}$$

Here the right-hand side tends to zero faster than any power of t. Hence it follows that $I_3 = o(t^{-(k-2)/2})$ (see (3.9)).

Now, the assertion of the theorem follows from (3.10) and the inversion formula for the Fourier transform. The theorem is proved.

Now consider the discrete-time case $t = n = 1, 2, \ldots$ and assume that the controlling process $\Lambda(n)$ has the form

$$\Lambda(n) = \sum_{i=1}^{n} Z_i, \tag{3.27}$$

where $\{Z_i\}$ are independent identically distributed r.v.'s, $Z_1 \geqslant 0$, $i \geqslant 1$. This representation occurs in the situation where $\Lambda(t)$ is a homogeneous stochastic process with independent increments, and the generalized Cox process is observed at equidistant time instants so that Z_i are the increments of the controlling process $\Lambda(t)$ on the time intervals between observations. We shall assume that $\mathbf{E} Z_1 = 1$ so that $\mathbf{E}\Lambda(n) = n$. Let us put

$$\nu_l = \mathbf{E}(Z_1 - 1)^l, \quad l = 1, 2, \ldots$$

Define the formal "semi-invariants" $æ_j$ by the equality

$$\log \mathbf{E} \exp\{(Z_1 - 1)(f_{X_1}(s) - 1)\} = \sum_{j=2}^{\infty} \frac{æ_j}{j!} (is)^j.$$

In particular,

$$æ_2 = \nu_2 \alpha_1^2, \quad æ_3 = 3\nu_2 \alpha_2 \alpha_1 + \nu_3 \alpha_1^3,$$

$$æ_4 = 3\nu_2 \alpha_2^2 + \nu_4 \alpha_1^4 + 6\alpha_1^2 \alpha_2 \nu_3 - 3\nu_2^2 \alpha_1^4.$$

THEOREM 4. *Assume that there exist $\gamma > 0$ and a polynomial $P(h)$ such that for any $h \geqslant 0$ the r.v. Z_1 satisfies the inequality*

$$\mathbf{E}\exp\{hZ_1\} \leqslant P(h)\exp\{\gamma h\}. \tag{3.28}$$

Let $\mathbf{E}|X_1|^k < \infty$ for some integer $k \geqslant 3$. Assume that X_1 satisfies the Cramér condition (3.1). *Then*

$$\sup_x \left| \mathbf{P}\left(\frac{S(n) - an}{\sigma\sqrt{n}}\right) - \Phi\left(\frac{\sigma x}{\sqrt{æ_2 + \alpha_2}}\right) - \sum_{j=1}^{k-2} \frac{v_j(x)}{n^{j/2}} \right| = o(n^{-(k-2)/2}),$$

where

$$v_j(x) = \frac{\sigma}{\sqrt{æ_2 + \alpha_2}} \sum_{\substack{l+m=j \\ l,m \geqslant 0}} \int_{-\infty}^{x} \bar{P}_l(-D_y) P_m^*(-D_y) \phi\left(\frac{\sigma y}{\sqrt{æ_2 + \alpha_2}}\right) dy,$$

$$j = 1, \ldots, k-2,$$

D_y is the operator of formal differentiation with respect to y, $P_0^*(x) \equiv \bar{P}_0(x) \equiv 1$, the polynomials $\bar{P}_l(\cdot)$ are defined in Theorem 3, *and $P_m^*(\cdot)$ are defined in the same way as $\bar{P}_l(\cdot)$ but with the moments α_{m+2} replaced with semi-invariants $æ_{m+2}$.*

Remark 6. We do not assume that the r.v.'s Z_i should satisfy the Cramér condition (3.1). Therefore they may be lattice.

Remark 7. Condition (3.28) is stronger than (3.5). This condition holds for any bounded r.v. Z_1, e.g., binomial, but it does not hold for exponential or Poisson r.v.'s. Note also that the r.v. Z_1 must have all moments.

Remark 8. The first functions $P_l^*(x)$ and $v_j(x)$ have the form

$$P_1^*(x)=\frac{x^3\,æ_3}{6\sigma^3},\qquad P_2^*(x)=\frac{x^4\,æ_4}{24\sigma^4}+\frac{x^6\,æ_3^2}{72\sigma^6},$$

$$v_1(x)=-\frac{æ_3+\alpha_3}{6\sigma(æ_2+\alpha_2)}H_2\left(\frac{\sigma x}{\sqrt{æ_2+\alpha_2}}\right)\phi\left(\frac{\sigma x}{\sqrt{æ_2+\alpha_2}}\right),$$

$$v_2(x)=-\frac{æ_4+\alpha_4}{24\sigma^2(æ_2+\alpha_2)}H_3\left(\frac{\sigma x}{\sqrt{æ_2+\alpha_2}}\right)\phi\left(\frac{\sigma x}{\sqrt{æ_2+\alpha_2}}\right)$$

$$-\frac{(æ_3+\alpha_3)^2}{72\sigma^4(æ_2+\alpha_2)}H_5\left(\frac{\sigma x}{\sqrt{æ_2+\alpha_2}}\right)\phi\left(\frac{\sigma x}{\sqrt{æ_2+\alpha_2}}\right),$$

where $H_m(\cdot)$ are Chebyshev–Hermite polynomials.

Remark 9. The functions $v_j(x)$ can be calculated by the formulas

$$v_j(x)=\frac{\sigma^2}{æ_2+\alpha_2}\bar{Q}_j\left(\frac{\sigma x}{\sqrt{æ_2+\alpha_2}}\right),\qquad j=1,\ldots,k-2,$$

where the functions $\bar{Q}_j(x)$ are defined by (3.2) with semi-invariants γ_{m+2} replaced by $æ_{m+2}+\alpha_{m+2}$. In particular,

$$\bar{Q}_1(x)=-\phi(x)H_2(x)\frac{æ_3+\alpha_3}{6\sigma^3},$$

$$\bar{Q}_2(x)=-\phi(x)\left[H_3(x)\frac{æ_4+\alpha_4}{24\sigma^4}+H_5(x)\frac{(æ_3+\alpha_3)^2}{72\sigma^6}\right].$$

Proof of Theorem 4. We will prove Theorem 4 using the same scheme as was used to prove Theorem 3. Here we have

$$f_n^*(s)\equiv \mathbf{E}\exp\left\{is\,\frac{S^*(n)-an}{\sigma\sqrt{n}}\right\} \tag{3.29}$$

$$=v^n\left(\frac{s}{\sigma\sqrt{n}}\right)\left[\mathbf{E}\exp\left\{(Z_1-1)\left(f_{X_1}\left(\frac{s}{\sigma\sqrt{n}}\right)-1\right)\right\}\right]^n.$$

For any $\varepsilon>0$ there exists a $\delta>0$ such that

$$\left|\mathbf{E}\exp\left\{(Z_1-1)\left(f_{X_1}\left(\frac{s}{\sigma\sqrt{n}}\right)-1\right)\right\}-\mathbf{E}\exp\left\{(Z_1-1)\sum_{l=1}^{k}\frac{(is)^l\alpha_l}{l!\,\sigma^l}\tau^l\right\}\right|$$

$$\leqslant \varepsilon\tau^k |s|^k \mathbf{E} \exp\left\{3|Z_1-1| \frac{|s|^k}{k!\,\sigma^k} \mathbf{E}\,|X_1|^k \tau^k\right\}$$

$$\leqslant \varepsilon\tau^k P_4(|s|) \exp\{\gamma_1 \delta^2 s^2 \tau^2\}, \quad |s| \leqslant \delta\sqrt{n},$$

where $\gamma_1 > 0$, and $P_4(\cdot)$ is some polynomial. The last inequality is due to condition (3.28).

Further, since the r.v. X_1 satisfies the Cramér condition and Z_1 is not degenerate at zero (because $\mathbf{E}Z_1 = 1$), we have

$$|f_n^*(s)| \leqslant \left(\mathbf{E}\left|\exp\left\{Z_1\left(f_{X_1}\left(\frac{s}{\sigma\sqrt{n}}\right)-1\right)\right\}\right|\right)^n$$

$$\leqslant (\mathbf{E}\exp\{-Z_1(1-q_5)\})^n \leqslant q_6^n, \quad |s| \geqslant \delta\sqrt{n},$$

with $q_5, q_6 \in (0, 1)$. Now the assertion of the theorem follows from the formulas similar to (3.7)–(3.9), (3.12). The theorem is proved.

The structure of the controlling process of the form (3.27) is typical in many applications. At the same time it helps us to understand better what goes on in the situation described by Theorem 3. Now we return to Theorem 3 and assume that $\Lambda(n)$ can be represented similarly to (3.27) but with not necessarily positive Z_i and consider some examples.

EXAMPLE 1. Let $\mathbf{E}Z_i = 0$ and $\mathbf{D}Z_i = \delta^2$. Set

$$\tilde{\Lambda}(n) = \frac{1}{\delta}|Z_1 + \ldots + Z_n| + n - \sqrt{\frac{2n}{\pi}}.$$

Then $\mathbf{E}\tilde{\Lambda}(n) \approx n$ and

$$\frac{\tilde{\Lambda}(n)-n}{\sqrt{n}} \Rightarrow |W| - \mathbf{E}|W| \quad (n \to \infty),$$

where W is a standard normal r.v.

EXAMPLE 2. Put $Z_0 = 0$. Then under the same conditions on $\{Z_i\}_{i\geqslant 1}$ as above, setting

$$\tilde{\Lambda}(n) = \frac{1}{\delta}\max_{0\leqslant i\leqslant n}(Z_0+Z_1+\ldots+Z_i)+n-\sqrt{\frac{2n}{\pi}},$$

we also obtain $\mathbf{E}\tilde{\Lambda}(n) \approx n$ and

$$\frac{\tilde{\Lambda}(n)-n}{\sqrt{n}} \Rightarrow |W| - \mathbf{E}|W| \quad (n \to \infty).$$

EXAMPLE 3. Under the conditions of Example 1 let us set

$$\tilde{\Lambda}(n) = \frac{1}{a^2\sqrt{n}}(Z_1+\ldots+Z_n)^2+n-\sqrt{n}.$$

Then $\mathsf{E}\tilde{\Lambda}(n) = n$ and

$$\frac{\tilde{\Lambda}(n)-n}{\sqrt{n}} \Rightarrow W^2 - \mathsf{E}\,W^2 \qquad (n \to \infty).$$

In all the three examples we observed the same structure of the limit r.v. as in Theorem 3 (the first two examples correspond to $V_0 = |W|$ while in the third example we have $V_0 = W^2$). These examples are nothing more than the illustrations of the invariance principle. So we can conclude that Theorem 3 corresponds to the situation similar to that in which $\Lambda(t)$ is a function of a Wiener process. As this is so, rather an unexpected structure of the r.v. V in Theorem 3 is due to the requirement that the controlling process should be positive.

Note that, in Examples 1 and 2, $\sqrt{2n/\pi}$ is the asymptotic expectation of the r.v.'s

$$\delta^{-1}|Z_1 + \ldots + Z_n| \quad \text{and} \quad \delta^{-1} \max_{0 \leqslant i \leqslant n} (Z_0 + Z_1 + \ldots + Z_n).$$

By subtracting exact expectations instead of their asymptotic values in the definitions of $\tilde{\Lambda}(n)$, we can provide the equality $\mathsf{E}\tilde{\Lambda}(n) \equiv n$.

Remark 10. By virtue of the representation (3.29) we can represent $S^*(n)$ as a sum of independent identically distributed r.v.'s and use well-known results to construct asymptotic expansions for its distribution (see, e.g., [11]). However, in this case the representation of the functions $v_j(x)$ turns out to be less convenient. By reducing the proof of Theorem 4 to that of Theorem 3 we express $v_j(x)$ in terms of the r.v.'s X_j and Z_j which are "atomic" for the problem under consideration.

4. Estimates for the concentration functions of generalized Cox processes. In this section we shall present some estimates for the concentration functions of one-dimensional distributions of generalized Cox processes $S(t)$. Recall that the *concentration function* of an r.v. Y is the function

$$Q_Y(l) = \sup_x \mathsf{P}(x \leqslant Y \leqslant x + l), \qquad l \geqslant 0,$$

see, e.g., [11], Section III.1.

It is well known that if $X_1, X_2, \ldots$ are independent and have identical nondegenerate distributions, then there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$(4.1) \qquad |f_{X_i}(s)| \leqslant 1 - \varepsilon s^2, \qquad |s| \leqslant \delta,\ i \geqslant 1,$$

where $f_{X_i}(s)$ is the ch.f. of the r.v. X_i (see, e.g., [11], Section I.2) and the concentration function of their partial sum $S_n = X_1 + \ldots + X_n$ satisfies the inequality

$$(4.2) \qquad Q_{S_n}(l) \leqslant C(\varepsilon, \delta)\frac{l+1}{\sqrt{n}}, \qquad l \geqslant 0,$$

with

$$C(\varepsilon, \delta) = \left(\frac{96}{95}\right)^2 \max(1, \delta^{-1}) \sqrt{\frac{\pi}{\varepsilon}}, \tag{4.3}$$

where ε and δ are the same as in (4.1).

Here we shall present three analogs of (4.2) for generalized Cox processes $S(t)$. Let us put

$$Q_t(l) = Q_{S(t)}(l), \quad l \geqslant 0.$$

First consider the most general case. Let V be the limit r.v. for the standardized controlling process $\Lambda(t)$:

$$\frac{\Lambda(t)-t}{\sqrt{t}} \Rightarrow V \quad (t \to \infty). \tag{4.4}$$

Set $\Lambda^*(t) = |t+\sqrt{t}V|$.

THEOREM 5. *Assume that* (4.4) *holds.*

I. *Let the r.v.* X_1 *be nondegenerate. Then*

$$Q_t(l) \leqslant \inf_{0<q<1} \left(C(\varepsilon, \delta) \frac{l+1}{\sqrt{(1-q)t}} + 2\left(\frac{96}{95}\right)^2 \max(1, \delta)\, \mathbf{P}(V < -q\sqrt{t})\right) + \omega(t).$$

II. *Let the r.v.* X_1 *be degenerate at a point* $\alpha \neq 0$. *Then*

$$Q_t(l) \leqslant \inf_{0<q<1} \left(C\left(\frac{11}{24}, 1\right) \frac{l|\alpha|^{-1}+1}{\sqrt{(1-q)t}} + 2\left(\frac{96}{95}\right)^2 (l|\alpha|^{-1}+1)\, \mathbf{P}(V < -q\sqrt{t})\right) + \omega(t).$$

Here $C(\varepsilon, \delta)$ *and* $\omega(t)$ *are defined in* (4.3) *and* (2.3), *respectively.*

Proof. I. Let $S^*(t)$ be the generalized Cox process generated by the sequence $X_1, X_2, \ldots$ and controlled by the process $\Lambda^*(t)$. Let us write $Q_t^*(l) = Q_{S^*(t)}(l)$, $l \geqslant 0$. Using the inequality (6.3) of [1] and the estimate (2.3) we have

$$Q_t(l) \leqslant Q_t^*(l) + \omega(t). \tag{4.5}$$

Apply Lemma 3 from [11], Section III.1, according to which for any r.v. Y and any $l \geqslant 0$, $a > 0$ we have the inequality

$$Q_Y(l) \leqslant \left(\frac{96}{95}\right)^2 \max(l, a^{-1}) \int_{-a}^{a} |f_Y(s)|\, ds, \tag{4.6}$$

where $f_Y(s)$ is the ch.f. of Y. It is easy to see that the ch.f. $f_t^*(s)$ of the r.v. $S^*(t)$ has the form

$$f_t^*(s) = \mathbf{E} \exp\{|\sqrt{t} + Vt|\,(f_{X_1}(s)-1)\}. \tag{4.7}$$

Let $q\in(0, 1)$. Since X_1 is nondegenerate, using (4.1) we therefore obtain

$$(4.8)\quad |f_t^*(s)| \leqslant \mathbf{E}\exp\{|\sqrt{t}V+t|(|f_{X_1}(s)|-1)\}$$
$$= \mathbf{E}\exp\{(\sqrt{t}V+t)(|f_{X_1}(s)|-1)\}\,\mathbf{1}(V\geqslant -\sqrt{t})$$
$$+\mathbf{E}\exp\{-(\sqrt{t}V+t)(|f_{X_1}(s)|-1)\}\,\mathbf{1}(V< -\sqrt{t})$$
$$\leqslant\Big(\int_{-\sqrt{t}}^{-q\sqrt{t}}+\int_{-q\sqrt{t}}^{\infty}\Big)\exp\{(\sqrt{t}v+t)(|f_{X_1}(s)|-1)\}\,d\mathbf{P}(V<v)+\mathbf{P}(V< -\sqrt{t})$$
$$\leqslant \exp\{-(1-q)t(1-|f_{X_1}(s)|)\}+\mathbf{P}(V< -q\sqrt{t})$$
$$\leqslant \exp\{-\varepsilon s^2(1-q)t\}+\mathbf{P}(V< -q\sqrt{t}),\qquad |s|\leqslant\delta.$$

If $l\geqslant 1$, then set $a=\delta/l$. Then applying (4.8) we obtain

$$(4.9)\quad Q_t^*(l)$$
$$\leqslant\Big(\frac{96}{95}\Big)^2 l\max(1,\delta^{-1})\Big(\int_{-\delta/l}^{\delta/l}\exp\{-\varepsilon s^2(1-q)t\}\,ds+\frac{2\delta}{l}\,\mathbf{P}(V< -\sqrt{t})\Big)$$
$$\leqslant l\sqrt{\frac{\varepsilon}{\pi}}C(\varepsilon,\delta)\int_{-\infty}^{\infty}\exp\{-\varepsilon(1-q)ts^2\}\,ds+2\Big(\frac{96}{95}\Big)^2\max(1,\delta)\,\mathbf{P}(V< -q\sqrt{t})$$
$$\leqslant C(\varepsilon,\delta)\frac{l+1}{\sqrt{(1-q)t}}+2\Big(\frac{96}{95}\Big)^2\max(1,\delta)\,\mathbf{P}(V< -q\sqrt{t}).$$

If $0\leqslant l<1$, then we set $a=\delta$ in (4.6) and again use (4.8) to obtain

$$(4.10)\quad Q_t^*(l)$$
$$\leqslant\Big(\frac{96}{95}\Big)^2\max(l,\delta^{-1})\Big(\int_{-\delta}^{\delta}\exp\{-\varepsilon(1-q)ts^2\}\,ds+2\delta\,\mathbf{P}(V< -\sqrt{t})\Big)$$
$$\leqslant (l+1)\sqrt{\frac{\varepsilon}{\pi}}C(\varepsilon,\delta)\int_{-\infty}^{\infty}\exp\{-\varepsilon(1-q)ts^2\}\,ds+2\Big(\frac{96}{95}\Big)^2\max(1,\delta)\,\mathbf{P}(V< -q\sqrt{t})$$
$$\leqslant C(\varepsilon,\delta)\frac{l+1}{\sqrt{(1-q)t}}+2\Big(\frac{96}{95}\Big)^2\max(1,\delta)\,\mathbf{P}(V< -q\sqrt{t}).$$

Now the first statement of the theorem follows from (4.5), (4.9) and (4.10).

To prove part II first assume that $\alpha=1$. Then we obviously have

$$f_t^*(s)=\mathbf{E}\exp\{-|t+\sqrt{t}V|(1-e^{is})\}.$$

Thus repeating the argument similar to that which was used to prove (4.8) we obtain the estimate

$$(4.11)\quad |f_t^*(s)|\leqslant\mathbf{E}\exp\{-|t+\sqrt{t}V|(1-\cos s)\}$$
$$\leqslant\exp\{-t(1-q)(1-\cos s)\}+\mathbf{P}(V< -q\sqrt{t}),\qquad q\in(0, 1).$$

But

(4.12) $$1-\cos s \geqslant \frac{11}{24} s^2, \quad |s| \leqslant 1.$$

Now, by putting $a=1$ in (4.6) and using (4.10) and (4.11), we obtain

(4.13) $$\begin{aligned} Q_t^*(l) &\leqslant \left(\frac{96}{95}\right)^2 \max(l, 1)\left(\int_{-1}^{1} \exp\{-t(1-q)(1-\cos s)\}\, ds + 2\mathbf{P}(V < -q\sqrt{t})\right) \\ &\leqslant \left(\frac{96}{95}\right)^2 (l+1)\left(\int_{-1}^{1} \exp\left\{-\frac{11}{24}(1-q)\, ts^2\right\} ds + 2\mathbf{P}(V < -q\sqrt{t})\right) \\ &\leqslant \left(\frac{96}{95}\right)^2 (l+1)\left(\int_{-\infty}^{\infty} \exp\left\{-\frac{11}{24}(1-q)\, ts^2\right\} ds + 2\mathbf{P}(V < -q\sqrt{t})\right) \\ &= C\left(\frac{11}{24}, 1\right)\frac{l+1}{\sqrt{(1-q)t}} + 2\left(\frac{96}{95}\right)^2 (l+1)\, \mathbf{P}(V < -q\sqrt{t}), \end{aligned}$$

and the desired result for the case $\alpha = 1$ follows from the relation (6.3) in [1] and (2.3). To prove part II for the case of an arbitrary $\alpha \neq 0$ note that, in (4.12), $Q_t^*(l) = Q_{N^*(t)}(l)$, where $N^*(t) \equiv N_1(\Lambda^*(t))$. Now the desired result follows from (4.13) and the obvious relation

$$Q_t^*(l) = Q_{N^*(t)}(l|\alpha|^{-1}),$$

which holds for an arbitrary $\alpha \neq 0$. The theorem is proved.

COROLLARY 2. *Let, in addition to the conditions of Theorem* 5, $\mathbf{E}|V| < \infty$.

I. *If the r.v.* X_1 *is nondegenerate, then*

$$Q_t(l) \leqslant \frac{1}{\sqrt{t}}\left(\sqrt{C(\varepsilon, \delta)(l+1)} + \frac{96}{95}\sqrt{2\max(1, \delta)\, \mathbf{E}|V|}\right)^2 + \omega(t).$$

II. *If the r.v.* X_1 *is degenerate at a point* $\alpha \neq 0$, *then*

$$Q_t(l) \leqslant \frac{l|\alpha|^{-1}+1}{\sqrt{t}}\left(\sqrt{C\left(\frac{11}{24}, 1\right)} + \frac{96}{95}\sqrt{2\mathbf{E}|V|}\right)^2 + \omega(t).$$

Proof. This statement easily follows from Theorem 5 with the help of the Markov inequality and the elementary fact: for any $c_1, c_2 > 0$

$$\inf_{0<q<1}\left(\frac{c_1}{1-q} + \frac{c_2}{q}\right) = (\sqrt{c_1} + \sqrt{c_2})^2.$$

Now we proceed to two special cases in which it is possible to simplify the estimates for the concentration functions of generalized Cox processes.

THEOREM 6. *Let* (4.4) *hold with* $V = V_0 - \mathsf{E}\, V_0$, *where* V_0 *is a nonnegative r.v.,* $\mathsf{E}\, V_0 < \infty$. *Then for* $t > (\mathsf{E}\, V_0)^2$ *the following statements hold*:

I. *If the r.v.* X_1 *is nondegenerate, then*

$$Q_t(l) \leqslant C(\varepsilon, \delta) \frac{l+1}{\sqrt{t - \sqrt{t}\, \mathsf{E}\, V_0}} + \Delta_0(t).$$

II. *If the r.v.* X_1 *is degenerate at a point* $\alpha \neq 0$, *then*

$$Q_t(l) \leqslant C\left(\frac{11}{24}, 1\right) \frac{l|\alpha|^{-1}+1}{\sqrt{t - \sqrt{t}\, \mathsf{E}\, V_0}} + \Delta_0(t).$$

Here $\Delta_0(t) = \sup_x |F_t(x) - F_t^0(x)|$, $F_t^0(x) = \mathsf{P}\left(S^0(t) < x\right)$ *and* $S^0(t)$ *is the generalized Cox process generated by the sequence* $X_1, X_2, \ldots$ *and controlled by the process* $\Lambda^0(t) = \sqrt{t}(V_0 - \mathsf{E}\, V_0) + t$.

The proof of this theorem follows the same scheme as that of Theorem 5. The only distinction is that instead of (4.8) we should use the following estimate for the ch.f. $f_t^0(s)$ of the r.v. $S^0(t)$:

$$|f_t^0(s)| \leqslant \mathsf{E} \exp\{-(\sqrt{t}V + t)(1 - |f_{X_1}(s)|)\} \leqslant \mathsf{E} \exp\{-(\sqrt{t}V + t)\varepsilon s^2\}$$
$$\leqslant \exp\{-(t - \sqrt{t}\, \mathsf{E}\, V_0)\varepsilon s^2\}, \quad |s| \leqslant \delta,$$

which holds for $t > (\mathsf{E}\, V_0)^2$.

Remark 11. To estimate $\Delta_0(t)$ one can use Lemma 1.

Finally let us consider the discrete-time case $t = n = 1, 2, \ldots$ and assume that

$$\Lambda(n) = \sum_{i=1}^{n} Z_i,$$

where Z_i are independent nonnegative identically distributed r.v.'s. Let us put $Q_n(l) = Q_{S(n)}(l)$.

THEOREM 7. *Assume that* $\text{æ} \equiv \mathsf{E}\, Z_1^{-1/2} < \infty$.

I. *If the r.v.* X_1 *is nondegenerate, then*

$$Q_n(l) \leqslant C(\varepsilon, \delta)\, \text{æ}\, \frac{l+1}{\sqrt{n}}.$$

II. *If the r.v.* X_1 *is degenerate at a point* $\alpha \neq 0$, *then*

$$Q_n(l) \leqslant C\left(\frac{11}{24}, 1\right) \text{æ}\, \frac{l|\alpha|^{-1}+1}{\sqrt{n}}.$$

Proof. In the case under consideration we have

$$f_n(s) \equiv \mathsf{E} \exp\{is\, S(n)\} = \left(\mathsf{E} \exp\{Z_1 (f_{X_1}(s) - 1)\}\right)^n,$$

and therefore if X_1 is nondegenerate, then, by (4.1) and Hölder's inequality, for $|s| \leqslant \delta$ we have

$$|f_n(s)| \leqslant (\mathsf{E} \exp\{-Z_1 \varepsilon s^2\})^n \leqslant \mathsf{E} \exp\{-n\varepsilon s^2 Z_1\}.$$

And if $\mathsf{P}(X_1 = 1) = 1$, then

$$f_n(s) = \left(\mathsf{E} \exp\{Z_1(e^{is} - 1)\}\right)^n,$$

so that by (4.10) for $|s| \leqslant 1$ we have

$$|f_n(s)| \leqslant \left(\mathsf{E} \exp\{Z_1(\cos s - 1)\}\right)^n$$

$$\leqslant \left(\mathsf{E} \exp\left\{-\frac{11}{24} s^2 Z_1\right\}\right)^n = \mathsf{E} \exp\left\{-\frac{11}{24} ns^2 Z_1\right\}.$$

Now it remains to use the inequalities similar to (4.6), (4.7), (4.8) and (4.13). The theorem is proved.

Remark 12. Condition (4.4) is actually used only as a hint on that $\omega(t)$ and $\Delta_0(t)$ in the above estimates should tend to zero as $t \to \infty$.

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Received on 7.10.1997