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CONDITIONING AND WEAK CONVERGENCE

BY

RYSZARD JAJTE* AND ADAM PASZKIEWICZ* (ŁÓDŹ)

Abstract. Several connections between the weak convergence of random variables, convergence of their distributions and conditioning have been described.

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1. Introduction. The Rademacher system (r_n) is a simple illuminating example of a sequence of functions having the same distributions and weakly converging to $0 = Er_n = E(r_n|r_1, \ldots, r_{n-1})$. One can expect more general relations between weak limits and conditional expectations for sequences of random variables. The analysis of such problems is the main goal of this paper. In Section 2 we discuss some connections between the weak and the almost everywhere convergence of sequences of functions. Roughly speaking, the weak convergence is not too far from the almost everywhere one, via conditioning or passing to a subsequence, and using martingale convergence theorems. In Section 3 we show that combining the weak convergence of random variables with the weak convergence of their distributions we get the convergence in measure and in L_1. Section 4 is devoted to a characterization of all distributions of weak limits of sequences of functions having distributions weakly converging to a given one.

2. Connections between weak and almost everywhere convergence. Let $(\Omega, \mathscr{F}, \mu)$ be a probability space and let $(\mathscr{A}_n)$ be a filtration in Ω, i.e. $(\mathscr{A}_n)$ is an increasing sequence of sub-σ-fields of $\mathscr{F}$ with $\sigma\left(\bigcup_{n\geqslant 1} A_n\right) = \mathscr{F}$. By $E^{\mathscr{A}_n}X$ we denote the conditional expectation of a random variable X with respect to $\mathscr{A}_n$.

For any uniformly integrable sequence (X_n) in $L_1(\Omega, \mathscr{F}, \mu)$, the sequence $Y_n = X_n - E^{\mathscr{A}_n}X_n$ is convergent in the $\sigma(L_1, L_\infty)$ topology ([2], see also Theorem 2.3 below). In particular, any uniformly integrable sequence (X_n) is weakly

* Institute of Mathematics, Łódź University. Work supported by KBN grant 2 P03A 023 15.

convergent in L_1 if only $E^{\mathscr{A}_n} X_n$ converges weakly in L_1. This is obviously true when $(E^{\mathscr{A}_n} X_n)$ is a martingale. In [3] a sequence $(X_n, \mathscr{A}_n)$ is called a *pseudo--martingale* when $(E^{\mathscr{A}_n} X_n)$ is a martingale. In fact, any weakly convergent sequence (X_n) is very close to a special pseudo-martingale. More precisely, we have the following

2.1. PROPOSITION. *Let $(\mathscr{A}_n)$ be an arbitrary filtration generated by finite partitions of Ω. Then, for any sequence (X_n) weakly convergent in L_1 to zero, there exists a sequence $\mathscr{B}_n$ of repetitions of $\mathscr{A}_m$'s i.e. $\mathscr{B}_n = \mathscr{A}_m$ for $n(m) \leqslant n < n(m+1)$, $n(1) < n(2) < \ldots$, such that $(X_n + \Delta_n, \mathscr{B}_n)$ is a pseudo-martingale for some sequence (Δ_n) converging to zero uniformly.*

Proof. Let $(A_1^{(m)}), \ldots, (A_{k(m)}^{(m)})$ be a partition of Ω generating $(\mathscr{A}_m)$. Take $n(m)$ such that

$$\Big|\int_\Omega X_n 1_{A_i^{(m)}}\Big| < \frac{\mu(A_i^{(m)})}{m} \quad \text{for } n > n(m),\ i = 1, 2, \ldots, k(m).$$

Obviously, we can assume that $n(1) < n(2) < \ldots$ It is enough to put $\Delta_n = -E^{\mathscr{B}_n} X_n$. ∎

2.2. COROLLARY. *If $X_n \to X$ weakly in L_1, then for any filtration $(\mathscr{A}_n)$ given by finite partitions with $\mathscr{A}_n \nearrow \mathscr{F}$ we have*

$$E^{\mathscr{B}_n} X_n \to X \text{ a.e. and in } L_1,$$

where $(\mathscr{B}_n)$ is a suitable sequence of repetitions of $\mathscr{A}_n$'s.

Proof. $X_n - X \to 0$ weakly. Thus we have $\|E^{\mathscr{B}_n}(X_n - X)\|_\infty \to 0$ and $E^{\mathscr{B}_n} X_n \to X$ a.e. and in L_1. ∎

For the sake of completeness we sketch the proof of the following theorem:

2.3. THEOREM (Jajte and Paszkiewicz [3]). *Let $(X_n, \mathscr{A}_n)$ be a pseudo-martingale with $\sigma\big(\bigcup_n \mathscr{A}_n\big) = \mathscr{F}$. Then (X_n) is weakly convergent to some random variable X in L_1 if and only if (X_n) is uniformly integrable. Then $X = \lim_n E^{\mathscr{A}_n} X_n$.*

Proof. Since any sequence weakly convergent in L_1 is uniformly integrable (so relatively compact in the $\sigma(L_1, L_\infty)$ topology [4]), it is enough to show that the uniform integrability of a pseudo-martingale (X_n) implies its weak convergence. Let (X_n) be uniformly integrable. We can assume that $\|X_n\|_1 \leqslant 1$. Take $g \in L_\infty(\Omega, \mathscr{F}, \mu)$, and let $\varepsilon > 0$. We fix an $h = \sum_{k=1}^N \lambda_k 1_{A_k}$ with $A_k \in \mathscr{F}$, $\|g - h\|_\infty < \varepsilon$. Since $\sigma\big(\bigcup_{n \geqslant 1} \mathscr{A}_n\big) = \mathscr{F}$, we find in the field $\bigcup_{n \geqslant 1} \mathscr{A}_n$, and consequently in some $\mathscr{A}_{n_0}$, the sets B_k satisfying $\mu(A_k \Delta B_k) < \delta$ with a δ such that $\mu(Z) < \delta$ implies

$$\int_Z |X_n - X| < \varepsilon / \|h\|_\infty .$$

Putting $h_1 = \sum_{k=1}^N \lambda_k 1_{B_k}$, we get

$$\begin{aligned}\Big|\int X_n g - \int X g\Big| &\leqslant \Big|\int (X_n - X)(g - h)\Big| + \Big|\int (X_n - X)(h - h_1)\Big| + \Big|\int (X_n - X) h_1\Big| \\ &\leqslant 2\varepsilon + \Big|\int (X_n - X) h_1\Big| < 3\varepsilon\end{aligned}$$

for n large enough, h_1 becomes $\mathscr{A}_{n_0}$-measurable, so

$$\int X_n h_1 = \int E^{\mathscr{A}_n}(X_n h_1) = \int h_1 E^{\mathscr{A}_n} X_n \to \int h_1 X. \quad \blacksquare$$

For a pseudo-martingale (X_n) in L_∞ we have the following

2.4. PROPOSITION. *If* $\sup_{n\geq 1} \|X_n\|_\infty < \infty$, *then* $X_n \to X$ *in* $\sigma(L_\infty, L_1)$ *topology, where* $X = \lim_n E^{\mathscr{A}_n} X_n$.

Proof. Writing $X_n = E^{\mathscr{A}_n} X_n + (1-E^{\mathscr{A}_n}) X_n$, we have for $g \in L_1$

$$\int X_n g = \int (E^{\mathscr{A}_n} X_n) g = \int X_n (1-E^{\mathscr{A}_n}) g \to \int Xg. \quad \blacksquare$$

In the last proposition the convergence in the $\sigma(L_\infty, L_1)$ topology cannot be replaced by the $\sigma(L_\infty, L_\infty^*)$ convergence.

2.5. EXAMPLE. Let $(\Omega, \mathscr{F}, \mu) = ([0, 1], \text{Borel } \lambda)$. Let φ be an extension to a continuous linear functional on L_∞ of the functional

$$f(X) = \lim_{\omega \to 0} X(\omega)$$

(for X's having this limit). Then, for

$$X_n(\omega) = 1_{[0,1/(2n)]}(\omega) - 1_{[1/(2n),1/n]}(\omega)$$

and σ-fields

$$\mathscr{A}_n = \{A \cup B;\ A \in \{\emptyset, [0, 1/n]\},\ B \in \text{Borel}[1/n, 1]\},$$

the sequence $(X_n, \mathscr{A}_n)$ is a pseudo-martingale. On the other hand, $\varphi(X_n) = 1$ and $X_n \to 0$ in the $\sigma(L_\infty, L_1)$ topology.

Remark. One can show that boundedness of a pseudo-martingale (X_n) in the L_1-norm does not imply its weak convergence in L_1. If (X_n) is L_p-bounded with $p > 1$, then it is uniformly integrable. Consequently, (X_n) converges weakly in L_p, since L_∞ is dense in L_q.

Now, we show that each weakly convergent sequence contains subsequences close to the martingale increments. The idea is analogous to that of Banach and Saks that each sequence weakly convergent to zero contains an "almost orthogonal" subsequence (see [5], Chapter 2, 38, and [6]).

2.6. THEOREM. *Let* $X_n \to X$ *weakly in* L_1. *For any sequence* (ε_k) *of positive numbers there exists a subsequence* $(X_{n(k)})$ *of* (X_n) *satisfying*

$$(*) \qquad \|X_{n(k)} - Y_k - X\|_1 < \varepsilon_k, \quad k = 1, 2, \ldots,$$

for some Y_k *being martingale increments, i.e.*

$$E(Y_k \mid 1, Y_1, \ldots, Y_{k-1}) = 0.$$

Proof. One can assume that $X = 0$. Let us take $n(1) = 1$ and let Y_1 be a simple function satisfying $\|X_1 - Y_1\| < \varepsilon_1$. Assume that $n(1) < \ldots < n(k)$, $Y_1, \ldots, Y_k$ have already been fixed in a way such that $(*)$

holds, and Y_k are simple functions. We can choose $n(k+1)$ large enough to have

$$\|E(X_{n(k+1)}|Y_1, \ldots, Y_k)\|_1 < \varepsilon_{k+1}/3.$$

Then we take a simple function Z satisfying

$$\|X_{n(k+1)}-Z\|_1 < \varepsilon_{k+1}/3,$$

and put $Y_{k+1} = Z-E(Z \mid Y_1, \ldots, Y_k)$. We have the estimation

$$\|X_{n(k+1)}-Y_{k+1}\|_1 \leqslant \|X_{n(k+1)}-Z\|_1+\|E(Z-X_{n(k+1)} \mid Y_1, \ldots, Y_k)\|_1 + \|E(X_{n(k+1)} \mid Y_1, \ldots, Y_k)\|_1 < \varepsilon_{k+1}.$$

The induction concludes the proof. ■

3. Weakly convergent sequences with weakly convergent distributions. Combining the weak convergence of random variables $X_n \to X$ with the weak convergence of their distributions $F_{X_n} \Rightarrow F_X$ one can easily obtain interesting observations. If $p > 1$, $X_n \to X$ weakly in L_p and $F_{X_n} \Rightarrow F_X$ (weakly), then $\|X_n - X\|_q \to 0$ for $1 < q < p$. Indeed, then the sequence $(|X_n|^q)$ is uniformly integrable. In particular, for any $\varepsilon > 0$ there exists a $c_0 > 0$ such that $c > c_0$ implies $\int_{(|X_n|^q > c)} |X_n|^q < \varepsilon$, which, together with the weak convergence of distribution, gives $\|X_n\|_q \to \|X\|_q$. Since $q > 1$, and $X_n \to X$ weakly in L_q, we get $X_n \to X$ in L_q (by the Radon theorem [4], Section 37).

The weak convergence $X_n \to X$ in L_1 of random variables satisfying $\|X_n\|_1 = \|X\|_1$ does not imply the convergence in L_1 (e.g. $X_n = 1+r_n \to 1$ weakly for the Rademacher system (r_n)). Anyway, we have the following result:

3.1. THEOREM. *Let $(X_n) \subset L_1$. Assume that $X_n \to X$ weakly in L_1 and $F_{X_n} \Rightarrow F_X$ (weakly). Then $X_n \to X$ in probability and, consequently, in L_1.*

Before starting the proof it is convenient to show the following lemma:

3.2. LEMMA. *Let $X_n \to X$ weakly in L_1 and $F_{X_n} \Rightarrow F_X$ weakly. Then, for any continuity point $a \in R$ of F_X, and $\varepsilon > 0$, we have*

$$\mu(X > a,\ X_n \leqslant a-\varepsilon) \to 0 \tag{1}$$

and, if $a-\varepsilon$ is a continuity point of F_X,

$$\mu(X \leqslant a-\varepsilon,\ X_n > a) \to 0 \quad \text{as } n \to \infty. \tag{2}$$

Proof. We prove only (1). In the sequel, we shall write $u_n = w_n + o(1)$ ($u_n \leqslant w_n + o(1)$, respectively) to say that $u_n = w_n + \delta_n$ ($u_n \leqslant w_n + \delta_n$, respectively) for some $\delta_n \to 0$.

First, let us observe that, for $a \in R$,

$$\int_{(X_n > a)} X_n \to \int_{(X > a)} X \quad \text{as } n \to \infty. \tag{3}$$

Indeed, take $\varepsilon > 0$. The sequence (X_n), being weakly convergent, is uniformly integrable [4], so

$$\Big|\int_{(X_n>c)} X_n\Big| \leqslant \int_{(|X_n|>c)} |X_n| < \varepsilon$$

for $c \geqslant c_0$ and all $n = 1, 2, \ldots$; we can additionally assume that $\big|\int_{(X>c)} X\big| < \varepsilon$ for $c \geqslant c_0$. Then, writing

$$\int_{(X_n>a)} X_n = \int_{(a<X_n\leqslant c_0)} X_n + \int_{(X_n>c_0)} X_n, \qquad \int_{(X>a)} X = \int_{(a<X\leqslant c_0)} X + \int_{(X>c_0)} X,$$

we have

$$\Big|\int_{(X_n>a)} X_n - \int_{(X>a)} X\Big| \leqslant \Big|\int_{(a<X_n\leqslant c_0)} X_n - \int_{(a<X\leqslant c_0)} X\Big| + 2\varepsilon.$$

Obviously, one can assume that c_0 is a continuity point of F_X. The weak convergence of measures $dF_{X_n} \to dF_X$ leads easily to the estimation

$$\Big|\int_{(a<X_n\leqslant c_0)} X_n - \int_{(a<X\leqslant c_0)} X\Big| = \Big|\int_{\mathbf{R}} u 1_{(a<u\leqslant c_0)}(u)\, dF_n(u) - \int_{\mathbf{R}} u 1_{(a<u\leqslant c_0)}(u)\, dF(u)\Big| < \varepsilon$$

for $n > n_0$, so finally we get (3).

The weak convergence of X_n to X and (3) imply, for $a \in \mathbf{R}$,

$$\int_{(X>a)} X_n = \int_{(X>a)} X + o(1) = \int_{(X_n>a)} X_n + o(1). \tag{4}$$

Moreover, for any continuity point a of F_X, we have

$$\mu(X > a,\, X_n \leqslant a) = \mu(X \leqslant a,\, X_n > a) + o(1) \tag{5}$$

(since $\mu(X > a) = \mu(X_n > a) + o(1)$). Subtracting $\int_{(X_n>a,X>a)} X_n$, we get from (4)

$$\int_{(X_n>a,X\leqslant a)} X_n = \int_{(X_n\leqslant a,X>a)} X_n + o(1). \tag{6}$$

Obviously, we have

$$\begin{aligned} \int_{(X_n\leqslant a,X>a)} X_n &= \int_{(X>a,a-\varepsilon<X_n\leqslant a)} X_n + \int_{(X>a,X_n\leqslant a-\varepsilon)} X_n \\ &\leqslant a\mu(X > a,\, a-\varepsilon < X_n \leqslant a) + (a-\varepsilon)\,\mu(X > a,\, X_n \leqslant a-\varepsilon). \end{aligned} \tag{7}$$

Moreover, by (5), (6) and (7) we have

$$\begin{aligned} a\mu(X > a,\, X_n \leqslant a) &= a\mu(X_n > a,\, X \leqslant a) + o(1) \\ &\leqslant \int_{(X_n>a,X\leqslant a)} X_n + o(1) = \int_{(X_n\leqslant a,X>a)} X_n + o(1) \\ &\leqslant a\mu(X > a,\, a-\varepsilon < X_n \leqslant a) + (a-\varepsilon)\,\mu(X > a,\, X_n \leqslant a-\varepsilon). \end{aligned}$$

Consequently,

$$a\mu(X > a,\, X_n \leqslant a-\varepsilon) \leqslant (a-\varepsilon)\,\mu(X > a,\, X_n \leqslant a-\varepsilon) + o(1),$$

which means

$$0 \leqslant -\varepsilon\mu(X > a,\, X_n \leqslant a-\varepsilon)+\delta_n$$

for some $\delta_n \to 0$, and gives immediately (1). In a similar way one can also prove (2). ■

3.3. Proof of Theorem 3.1. Let, for $\varepsilon > 0$, the numbers $a_0 < \ldots < a_m$ be fixed in a way such that F_X is continuous in each a_i; $a_i - a_{i-1} < \varepsilon$ for $i = 1, \ldots, m$; $F(a_0) < \varepsilon$, $F(a_m) > 1-\varepsilon$. Using Lemma 3.2 to X, X_n and once more to $-X$, $-X_n$ we obtain

$$\mu\big(a_{i-1} \leqslant X < a_i \wedge (X_n \geqslant a_i+\varepsilon \vee X_n < a_{i-1}-\varepsilon)\big) < \varepsilon/m \qquad \text{for } n > n(i).$$

Then $n > \max\big(n(i)\colon i = 1, \ldots, m\big)$ implies

$$\mu(|X-X_n| > 2\varepsilon) < 2\varepsilon+\mu(a_0 \leqslant X < a_n,\, |X-X_n| > 2\varepsilon) < 3\varepsilon.$$

The stochastic convergence is proved. The weak convergence of (X_n) implies its uniform integrability, so $X_n \to X$ in L_1. ■

3.4. Remark. If $X_n \to X$ weakly in L_p with $p > 1$ and $F_{X_n} \Rightarrow F_X$ (weakly), then, obviously, $X_n \to X$ weakly in L_1. Therefore, by Theorem 3.1, $X_n \to X$ in probability. If, additionally, the sequence $(|X_n|^p)$ is uniformly integrable, then $X_n \to X$ in L_p.

4. Distributions of weak limits. The following two theorems give complete characterization of distributions of weak limits. Roughly speaking, weak limits of sequences of random variables having distributions weakly converging to a given one can be described by a suitable conditioning. As usual, F_X and $p_X = dF_X$ denote a distribution function and a distribution law of X, respectively.

4.1. THEOREM. *Let (X, Y) be a random vector defined on a probability space, say $(M, \mathcal{M}, P)$, with X having finite expectation. Then there exists a probability space $(\Omega, \mathcal{F}, \mu)$ and random variables $\xi_n, \xi \in L_1(\Omega, \mathcal{F}, \mu)$ satisfying the conditions*

1° $\xi_n \to \xi$ *weakly in* $L_1(\Omega, \mathcal{F}, \mu)$,

2° $F_{\xi_n} = F_X$ *and* $F_\xi = F_{E(X|Y)}$.

4.2. THEOREM. *Let $\xi_n \to \xi$ weakly in $L_1(\Omega, \mathcal{F}, \mu)$, and $F_{\xi_n} \Rightarrow F$. Then there exist two random variables X, Y on a probability space $(M, \mathcal{M}, P)$ such that*

$$F = F_X, \qquad F_\xi = F_{E(X|Y)}.$$

Proof of Theorem 4.1. Let us put $\Omega = \boldsymbol{R}\times[0, 1]$, $F = \text{Borel}\,\Omega$, $\mu = dF_Y \times \lambda$, λ being the Lebesgue measure on $[0, 1]$. Let $F(y, \alpha)$, $y \in \boldsymbol{R}$, $\alpha \in \boldsymbol{R}$ be a regular conditional distribution for X given $(Y = y)$, i.e. for each y, $F(y, \cdot)$ is a distribution function and, for each α, $F(y, \alpha) = P(X < \alpha \mid Y = y)$ dF_Y-almost everywhere (see e.g. [1]). For $y \in \boldsymbol{R}$, we denote by $g(y, t)$ the rearrangement ("inverse") of distribution function $F(y, \alpha)$, i.e., for $0 \leqslant t \leqslant 1$, we put

$$g(y, t) = \inf\{\alpha;\ F(y, \alpha) \geqslant t\}.$$

Then, in particular, we have $g \in L_1(\Omega, F, \mu)$ and

$$\int_0^1 g(y, t)\,dt = E(X \mid Y = y)\ \ dF_Y\text{-a.e.}$$

Let us put, for $(y, x) \in \Omega$,

$$\xi(y, x) = E(X \mid Y = y), \qquad \xi_n(y, x) = g(y, \tau^n x),$$

where $\tau\colon [0, 1] \to [0, 1]$ is an arbitrary fixed measure preserving mixing transformation [2] (i.e. $\lambda(A \cap \tau^{-n} B) \to \lambda(A)\lambda(B)$ as $n \to \infty$ for each $A, B \in$ Borel $[0, 1]$). Then we have

$$F_{\xi_n}(\alpha) = \mu\{(y, x);\ g(y, \tau^n x) < \alpha\} = \mu\{(y, x);\ g(y, x) < \alpha\}$$

$$= \int_R P(X < \alpha \mid Y = y)\,dF_Y(y) = F_X(\alpha);$$

$$F_\xi(\alpha) = \mu\{(y, x);\ E(X \mid Y = y) < \alpha\} = F_{E(X|Y)}(\alpha).$$

By the Fubini theorem and the mixing property of τ, for any $\psi \in L_\infty(\Omega)$ we get

$$\int_\Omega \xi_n(y, x)\psi(y, x)\,dP = \int_\Omega g(y, \tau^n x)\psi(y, x)\,dP$$

$$= \int_R \Big(\int_0^1 g(y, \tau^n x)\psi(y, x)\,dx\Big)\,dF_Y(y) \to \int_R \Big(\int_0^1 g(y, x)\,dx \int_0^1 \psi(y, x)\,dx\Big)\,dF_Y(y)$$

$$= \int_R \Big(\int_0^1 E(X \mid Y = y)\psi(y, x)\,dx\Big)\,dF_Y(y) = \int_\Omega \xi(y, x)\psi(y, x)\,dP. \ \blacksquare$$

Proof of Theorem 4.2. Take a measurable space

$$(M, \mathcal{M}) = (R^2, \text{Borel } R^2).$$

Let P_n be a probability distribution in R^2 of the vector (ξ_n, ξ), $n = 1, 2, \ldots$ Taking a subsequence, if necessary, one can assume that $P_n \Rightarrow P$ for some probability measure P in R^2. For random variables $X(x, y) = x$ and $Y(x, y) = y$ on the probability space $(M, \mathcal{M}, P) = (R^2, \text{Borel } R^2, P)$, we have $F_X = F$. The equality

$$F_{E(X|Y)} = F_Y = F_\xi$$

is a consequence of the following lemma:

4.3. LEMMA. *Let $\xi_n \to \xi$ weakly in $L_1(\Omega, F, \mu)$ and the distributions on the plane $p_{(\xi_n, \xi)}$ converge weakly to p. Then for the coordinates $X(x, y) = x$ and $Y(x, y) = y$ on the probability space $(R^2$, Borel $R^2, p)$, we have $E(X \mid Y) = Y$.*

Proof. For any $\alpha < \beta$, we have

$$\int_{(Y\in(\alpha,\beta))} dp_{(\xi_n,\xi)} \to \int_{(Y\in(\alpha,\beta))} X\,dp$$

by the weak convergence of measures $p_{(\xi_n,\xi)} \Rightarrow p$ and the uniform integrability of ξ_n. On the other hand,

$$\int_{(Y\in(\alpha,\beta))} X\,dp_{(\xi_n,\xi)} = \int_{(\xi\in(\alpha,\beta))} \xi_n\,d\mu \to \int_{(\xi\in(\alpha,\beta))} \xi d\mu = \int_{(Y\in(\alpha,\beta))} Ydp.$$

Thus

$$\int_{(Y\in(\alpha,\beta))} X\,dp = \int_{(Y\in(\alpha,\beta))} Ydp,$$

which completes the proof. ■

4.4. COMMENTS. In the formulation of the last two theorems we used, just for clarity, two different probability spaces $(M, \mathcal{M}, P)$ and (Ω, F, μ). In fact, we can always assume that these spaces coincide. It in enough to consider separately the atoms and non-atomic parts of (Ω, F, μ) to show that:

1. If $F_{\xi_n} \Rightarrow F$ weakly for (ξ_n) on Ω, then there exists a random variable ξ on Ω such that $F = F_\xi$.

2. If $\xi_n \to \xi$ weakly in $L_1(\Omega)$ and $F_{\xi_n} \Rightarrow F$, then there exist X on Ω and a σ-field $\mathcal{A} \subset \mathcal{F}$ such that $F_X = F$ and $F_{E^{\mathcal{A}}X} = F_\xi$ (the idea of the universality of the interval [0, 1] as a probability space is used here if a non-atomic part of (Ω, F, μ) is non-trivial).

3. Obviously, the conditions $\xi_n \to \xi$ weakly in $L_1(\Omega)$ and $F_{\xi_n} \Rightarrow F$ do not imply the existence of a random variable X and a σ-field $\mathcal{A} \subset \mathcal{F}$ satisfying

$$\xi = E^{\mathcal{A}}X \quad \text{and} \quad F_X = F.$$

The sequence $\xi_n(\omega) = r_n(\omega)+\omega$ on $\Omega = [0, 1]$ is an example, where (r_n) is the Rademacher system.

Remark. The characterization of distributions of weak limits given in Theorems 4.1 and 4.2 is formulated for the case of the weak convergence in L_p with $p = 1$ just for simplicity. For a sequence (ξ_n) in L_p, the convergence in the $\sigma(L_p, L_q)$ topology, $1/p+1/q = 1$, $1 \leqslant p \leqslant \infty$, is equivalent to the $\sigma(L_p, L_\infty)$ convergence.

Note added in proof. The authors have recently been informed that Theorem 3.1 follows from the result of L. Pratelli (*Une caractérisation de la convergence dans L^1*, Sém. Prob. XXVI, Lecture Notes in Math. 1526, Springer, 1992; théorème (5.1), p. 66). For this reference, the authors thank P. Berti and P. Rigo.

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Institute of Mathematics, Łódź University
ul. Stefana Banacha 22, 90-238 Łódź, Poland
E-mail address: rjajte@math.uni.lodz.pl and adampasz@math.uni.lodz.pl

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