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ON THE COMPLETENESS OF SOME L^p-SPACES OF OPERATOR-VALUED FUNCTIONS

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Abstract. In [3] there were studied Banach spaces of (equivalence classes of) functions Φ whose values are unbounded operators, in general, and which are p-integrable with respect to operator-valued measures having an operator density N with respect to some non-negative scalar measure μ. In the present short note it is shown that the values of all functions Φ are even bounded linear operators if and only if there is not any set A of positive finite measure μ such that the values of N on A have non-closed ranges. The result is used to answer a question raised by Górniak et al. [2].

1. For the reader's convenience we start with recalling the definition of some classes of L^p-spaces introduced in [3].

Let K be a non-trivial separable Hilbert space and H an infinite-dimensional separable Hilbert space over the field of complex numbers $\boldsymbol{C}$. The inner product and the norm in H are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. Let $\mathscr{B}$ be the Banach space of all bounded linear operators of H into K, and $\mathfrak{S}_\infty$ the subspace of all compact operators. For a bounded linear operator X, let $|X|$ and $\mathscr{R}(X)$ be the usual operator norm and the range of X, respectively. Let α be a symmetric gauge function (cf. [1], p. 96). It defines a norm $|\cdot|_\alpha$ on a certain linear subspace $\mathfrak{S}_\alpha$ of $\mathfrak{S}_\infty$, which becomes a Banach space under the norm $|\cdot|_\alpha$. The well-known Schatten classes are examples of such spaces (cf. [1], pp. 120–121). Note that in the case $K = \boldsymbol{C}$ the spaces $\mathfrak{S}_\alpha$ do not depend on the choice of α and $|\cdot|_\alpha = |\cdot|$. For more information about the spaces $\mathfrak{S}_\alpha$ see [1].

Let $(\Omega, \mathfrak{A}, \mu)$ be a positive measure space. A function $T: \Omega \to \mathscr{B}$ is called *measurable* if it is strongly (or, equivalently, weakly) measurable. Assertions concerning measurable functions are to be understood as assertions which are true for μ-almost all (abbreviated to "μ-a.a.") elements of the domain of definition, although we will not emphasize this each time.

Let $\mathscr{B}(H)$ be the Banach algebra of bounded linear operators in H and $N: \Omega \to \mathscr{B}(H)$ be a measurable function such that $N(\omega) \geqslant 0$ and $|N(\omega)| = 1$,

$\omega \in \Omega$. Here $N(\omega) \geqslant 0$ means $(N(\omega)x, x) \geqslant 0$ for all $x \in H$. Let

$$N(\omega) = \int_0^1 \lambda E(d\lambda; \omega)$$

be the spectral representation of $N(\omega)$, $\omega \in \Omega$. Let

$$P(\omega) := E((0, 1]; \omega)$$

and for $1 \leqslant p < \infty$

$$N(\omega)^{1/p} := \int_0^1 \lambda^{1/p} E(d\lambda; \omega), \qquad N(\omega)^{\# 1/p} := \int_0^1 \lambda^{-1/p} E(d\lambda \cap (0, 1]; \omega), \qquad \omega \in \Omega.$$

Moreover, let

$$P\colon \Omega \ni \omega \to P(\omega),$$

$$N^{1/p}\colon \Omega \ni \omega \to N(\omega)^{1/p}, \qquad N^{\# 1/p}\colon \Omega \ni \omega \to N(\omega)^{\# 1/p}.$$

Let $\mathscr{A}$ be the set of all (not necessarily densely defined and not necessarily bounded) linear operators from H to K. Let $1 \leqslant p < \infty$ and α be a symmetric gauge function. By $\mathscr{L}^p_\alpha(N\,d\mu)$ we denote the set of all functions $\Phi\colon \Omega \to \mathscr{A}$ with the following three properties:

$\Phi(\omega) N(\omega)^{1/p}$ exists and belongs to $\mathfrak{S}_\alpha$ for μ-a.a. $\omega \in \Omega$;

$\Phi N^{1/p}$ is measurable;

$\|\Phi\|_{p,\alpha} := \left(\int_\Omega |\Phi(\omega) N(\omega)^{1/p}|^p_\alpha \, \mu(d\omega)\right)^{1/p} < \infty$.

Two functions Φ, $\Psi \in \mathscr{L}^p_\alpha(N\,d\mu)$ are called *p-equivalent* if $\Phi N^{1/p} = \Psi N^{1/p}$. Let $L^p_\alpha(N\,d\mu)$ be the set of all p-equivalence classes of functions of $\mathscr{L}^p_\alpha(N\,d\mu)$. As usual, studying $L^p_\alpha(N\,d\mu)$ we work with representatives, i.e. with functions, instead of equivalence classes. The space $L^p_\alpha(N\,d\mu)$ is a Banach space under the norm $\|\cdot\|_{p,\alpha}$ (see [3], Theorem 7). Note that if $K = C$, the space $L^p_\alpha(N\,d\mu)$ does not depend on the choice of the symmetric gauge function α. In this case we will simply denote it by $L^p(N\,d\mu)$.

2. The following theorem answers the question under which conditions all elements of $L^p_\alpha(N\,d\mu)$ are not only $\mathscr{A}$-valued but even $\mathscr{B}$-valued, i.e. under which conditions for each p-equivalence class of $L^p_\alpha(N\,d\mu)$ there exists a $\mathscr{B}$-valued function belonging to this class.

THEOREM 1. *Let α be a symmetric gauge function and $1 \leqslant p < \infty$. The following two conditions are equivalent:*

(I) *All elements of $L^p_\alpha(N\,d\mu)$ are $\mathscr{B}$-valued.*

(II) *There does not exist a set $A \in \mathfrak{A}$ such that*

(i) $0 < \mu(A) < \infty$,

(ii) *$\mathscr{R}(N(\omega))$ is not closed for $\omega \in A$.*

Proof. The proof is divided into a number of steps.

Step 1. Let I be any subinterval of $[0, 1]$. Then the function $E(I; \cdot)$ is measurable and there exists a measurable function $x\colon \Omega \to H$ such that for $\omega \in \Omega$

(1) $x(\omega) \in \mathscr{R}(E(I;\omega))$,
(2) $\|x(\omega)\| = 1$ if $E(I;\omega) \neq 0$.
This result follows from [3], Lemma 1, and [4], Lemma 8.

Step 2. Let $C := \{\omega \in \Omega: \mathscr{R}(N(\omega))$ is not closed$\}$. Then C belongs to $\mathfrak{A}$. The range $\mathscr{R}(N(\omega))$ is not closed if and only if $P(\omega) \neq E((k^{-1}, 1]; \omega)$ for all k from the set of positive integers N. Since according to Step 1 the functions P and $E((k^{-1}, 1]; \cdot)$ are measurable, the result follows.

Step 3. (II) $\Rightarrow$ (I). Let $\Phi \in L^p_\alpha(N d\mu)$. In [3], the proof of Lemma 6, it was shown that there exists a function $T \in L^p_\alpha(P d\mu)$ such that $\Phi = TN^{\# 1/p}$. Note that the elements of $L^p_\alpha(P d\mu)$ are $\mathscr{B}$-valued and that T is equal to 0 outside a set of σ-finite measure μ. Since the closedness of $\mathscr{R}(N(\omega))$ is equivalent to the boundedness of $N(\omega)^{\# 1/p}$ and since the set C of Step 2 is measurable, we are done.

Step 4. Let A be a measurable set having the properties (i) and (ii). Then there exist a measurable subset $B \subseteq A$ and an increasing sequence $\{n_j\}_{j \in N} \subseteq N$ such that $\mu(B) > 0$ and $E((n_{j+1}^{-1}, n_j^{-1}]; \omega) \neq 0$ for all $\omega \in B$ and $j \in N$.

Choose a positive real number ε such that $\mu(A) - \varepsilon > 0$. Set $n_1 := 1$. For $n \in N$, $n > n_1$, let

$$A_n := \{\omega \in A: E((n^{-1}; n_1^{-1}]; \omega) \neq 0\}.$$

Since, for $\omega \in \Omega$, $\lim_{n \to \infty} E((0, n^{-1}], \omega) = 0$ with respect to the strong operator topology, we have

$$\bigcup_{n=n_1+1}^{\infty} A_n = A.$$

Choose $n_2 \in N$, $n_2 > n_1$, so large that

$$\mu\Big(\bigcup_{n=n_1+1}^{n_2} A_n\Big) > \mu(A) - \frac{\varepsilon}{2}$$

and set

$$B_1 := \bigcup_{n=n_1+1}^{n_2} A_n.$$

Assume that for a certain $k \in N$ we have already constructed an increasing sequence $\{n_j\}_{j=1}^{k+1} \subseteq N$ and a non-increasing sequence $\{B_j\}_{j=1}^{k} \subseteq \mathfrak{A}$ such that

$$\mu(B_j) > \mu(A) - \sum_{s=1}^{j} 2^{-s} \varepsilon$$

and

$$E((n_{j+1}^{-1}, n_j^{-1}]; \omega) \neq 0 \quad \text{for } \omega \in B_j,\ j = 1, \ldots, k.$$

Then using analogous arguments as in the construction of n_2 and B_1, we can find a positive integer $n_{k+2} > n_{k+1}$ and a measurable set $B_{k+1} \subseteq B_k$ such that

$$\mu(B_{k+1}) > \mu(B_k) - 2^{-(k+1)} \varepsilon$$

and

$$E\big((n_{k+2}^{-1}, n_{k+1}^{-1}]; \omega\big) \neq 0 \quad \text{for } \omega \in B_{k+1}.$$

If we set $B := \bigcap_{k=1}^{\infty} B_k$, we obtain

$$\mu(B) \geqslant \mu(A) - \varepsilon > 0$$

and

$$E\big((n_{j+1}^{-1}, n_j^{-1}]; \omega\big) \neq 0 \quad \text{for } \omega \in B \text{ and } j \in N.$$

Step 5. Assume that (II) is not true. Then there exist a measurable set B of positive finite measure μ and a bounded measurable function $\boldsymbol{x}$: $\Omega \to H$ such that the functional

$$x \to \big(N(\omega)^{\#1/p} x, \boldsymbol{x}(\omega)\big)$$

is an unbounded linear functional on $\mathscr{R}\big(N(\omega)\big)$ if $\omega \in B$, and

$$\boldsymbol{x}(\omega) = 0 \quad \text{if } \omega \in \Omega \backslash B.$$

If (II) is not true, there exists a measurable set A having properties (i) and (ii). Construct a set B and a sequence $\{n_j\}_{j \in N}$ as in Step 4. Choose a sequence $\{c_j\}_{j \in N}$ of positive real numbers such that

$$\sum_{j=1}^{\infty} c_j^2 < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} n_j^{1/p} c_j^2 = \infty.$$

By Step 1, for $j \in N$ there exists a measurable function $\boldsymbol{x}_j$: $\Omega \to H$ such that

$$\boldsymbol{x}_j(\omega) \in \mathscr{R}\Big(E\big((n_{j+1}^{-1}, n_j^{-1}]; \omega\big)\Big)$$

and

$$\|\boldsymbol{x}_j(\omega)\| = 1 \quad \text{if } E\big((n_{j+1}^{-1}, n_j^{-1}]; \omega\big) \neq 0.$$

Now set

$$\boldsymbol{x}(\omega) = 0 \quad \text{if } \omega \in \Omega \backslash B,$$

and

$$\boldsymbol{x}(\omega) = \sum_{j=1}^{\infty} c_j \boldsymbol{x}_j(\omega) \quad \text{if } \omega \in B.$$

Obviously, the function $\boldsymbol{x}$ is measurable. Fix $\omega \in B$ and set

$$y_k := \sum_{j=1}^{k} c_j \boldsymbol{x}_j(\omega), \quad k \in N.$$

The sequence $\{y_k\}_{k \in N}$ is bounded, since

$$\|y_k\|^2 = \sum_{j=1}^{k} c_j^2 < \sum_{j=1}^{\infty} c_j^2 < \infty.$$

But

$$\big(N(\omega)^{\#1/p} y_k, \boldsymbol{x}(\omega)\big) = \Big(N(\omega)^{\#1/p} \sum_{j=1}^{k} c_j \boldsymbol{x}_j(\omega), \sum_{j=1}^{\infty} c_j \boldsymbol{x}_j(\omega)\Big)$$

$$= \sum_{j=1}^{k} c_j^2 \big(N(\omega)^{\#1/p} \boldsymbol{x}_j(\omega), \boldsymbol{x}_j(\omega)\big) \geqslant \sum_{j=1}^{k} c_j^2 n_j^{1/p}$$

tends to ∞ if $k \to \infty$. Hence the linear functional $x \to (N(\omega)^{\# 1/p} x, \boldsymbol{x}(\omega))$ is unbounded if $\omega \in B$.

Step 6. (I) $\Rightarrow$ (II). Obviously, it is enough to prove the result if the dimension of K is 1. So we will assume $K = C$. If (II) does not hold, we can construct a set B and a function $\boldsymbol{x}$ as in Step 5. For $\omega \in B$ set

$$\Phi(\omega)x := (N(\omega)^{\# 1/p} x, \boldsymbol{x}(\omega)), \qquad x \in \mathscr{R}(N(\omega)^{1/p}),$$

and for $\omega \in \Omega \backslash B$ set $\Phi(\omega) = 0$. Since

$$\int_\Omega |\Phi(\omega) N(\omega)^{1/p}|^p \mu(d\omega) = \int_B \|\boldsymbol{x}(\omega)\|^p \mu(d\omega) < \infty,$$

the function Φ belongs to $L^p(N d\mu)$. But $\Phi(\omega) \notin \mathscr{B}$ if $\omega \in B$; hence (I) does not hold.

COROLLARY. *If μ is a σ-finite measure, then all elements of $L^p_\alpha(N d\mu)$ are $\mathscr{B}$-valued if and only if $\mathscr{R}(N(\omega))$ is closed for μ-a.a. $\omega \in \Omega$.*

3. In [2], pp. 108–109, Górniak et al. considered a σ-finite measure space $(\Omega, \mathfrak{A}, \nu)$, a measurable function $F'_\nu\colon \Omega \to \mathscr{B}(H)$ such that $F'_\nu(\omega) \geqslant 0$ for $\omega \in \Omega$, and the inner product space L^2_F of all (equivalence classes of) measurable functions $\boldsymbol{x}$ such that

$$\int_\Omega (F'_\nu(\omega)\boldsymbol{x}(\omega), \boldsymbol{x}(\omega))\, \nu(d\omega) < \infty.$$

They proved that the closedness of $\mathscr{R}(F'_\nu(\omega))$ for ν-a.a. $\omega \in \Omega$ is sufficient for the completeness of L^2_F and raised the question whether this condition is necessary for the completeness of L^2_F; cf. [2], Remark 5.6, and also [5], p. 211. Using the results of Section 1, we can answer this question in the affirmative.

THEOREM 2. *The space L^2_F is complete if and only if $\mathscr{R}(F'_\nu(\omega))$ is closed for ν-a.a. $\omega \in \Omega$.*

Proof. First note that we can assume that $F'_\nu(\omega) \neq 0$ for all $\omega \in \Omega$. Under this assumption set

$$N(\omega) := |F'_\nu(\omega)|^{-1} F'_\nu(\omega) \quad \text{and} \quad \mu(d\omega) := |F'_\nu(\omega)|\, \nu(d\omega), \qquad \omega \in \Omega.$$

The measure μ is σ-finite and $|N(\omega)| = 1$, $\omega \in \Omega$. Clearly, the space L^2_F does not change if we replace F'_ν by N and ν by μ. Let $K := C$ and consider the space $L^2(N d\mu)$. For $\boldsymbol{x} \in L^2_F$ define

$$\Phi(\omega)x := (x, \boldsymbol{x}(\omega)), \qquad x \in H,\ \omega \in \Omega.$$

It is not hard to see that the map $\boldsymbol{x} \to \Phi$ is an isometry of L^2_F onto the set of all $\mathscr{B}$-valued elements of $L^2(N d\mu)$. Thus L^2_F is complete if and only if all elements of $L^2(N d\mu)$ are $\mathscr{B}$-valued. Using the Corollary, we obtain the assertion.

REFERENCES

[1] I. Ts. Gokhberg and M. G. Kreĭn, *Introduction to the Theory of Linear Non-self-adjoint Operators in Hilbert Space* (in Russian), Nauka, Moscow 1965.

[2] J. Górniak, A. Makagon and A. Weron, *An explicit form of dilation theorems for semispectral measures*, in: *Prediction Theory and Harmonic Analysis. The Pesi Masani Volume*, V. Mandrekar and H. Salehi (Eds.), North-Holland Publishing Company, Amsterdam–New York–Oxford 1983, pp. 85–111.

[3] L. Klotz, *Some Banach spaces of measurable operator-valued functions*, Probab. Math. Statist. 12 (1991), pp. 85–97.

[4] — *Inclusion relations for some L^p-spaces of operator-valued functions*, Math. Nachr. 150 (1991), pp. 119–126.

[5] A. Makagon and H. Salehi, *Notes on infinite dimensional stationary sequences*, in: *Probability Theory on Vector Spaces IV. Proceedings of a Conference held in Łańcut, Poland, June 10–17, 1987*, Lecture Notes in Math. 1391, S. Cambanis and A. Weron (Eds.), Springer, Berlin–Heidelberg–New York 1989, pp. 200–238.

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