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COMPLETE EXACT LAWS

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Abstract. Consider independent and identically distributed random variables $\{X, X_n, n \geqslant 1\}$ with $xP\{X > x\} \sim a(\log x)^\alpha$, where $\alpha > -1$ and $P\{X < -x\} = o(P\{X > x\})$. Even though the mean does not exist, we establish Laws of Large Numbers of the form

$$\sum_{n=1}^{\infty} c_n P\left\{\left|\frac{\sum_{k=1}^{n} a_k X_k}{b_n} - L\right| > \varepsilon\right\} < \infty$$

for all $\varepsilon > 0$ and a particular nonsummable sequence $\{c_n, n \geqslant 1\}$, where $L \neq 0$.

Key words and phrases: Strong law of large numbers; weak law of large numbers; complete convergence.

Let $\{X, X_n, n \geqslant 1\}$ be independent and identically distributed random variables with

$$xP\{X > x\} \sim a(\log x)^\alpha, \quad \text{where } \alpha > -1 \text{ and } P\{X < -x\} = o(P\{X > x\}).$$

From Adler [1] we have Weak Laws of the form

$$\frac{\sum_{k=1}^{n} a_k X_k}{b_n} \xrightarrow{P} L,$$

where $a_k = k^a$ for all $a > -1$ and $L \neq 0$. (These limits were used to establish generalized one-sided Laws of the Iterated Logarithm.) Then in Adler [2] we established Strong Laws of the form

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_k X_k}{b_n} = L \text{ almost surely,}$$

where na_n was slowly varying at infinity and again $L \neq 0$.

The next question is whether we can extend almost sure convergence to complete convergence, i.e., $c_n = 1$ in (1). For our random variables the answer is a resounding 'no', but there is a similar result. We will show that

(1) $$\sum_{n=1}^{\infty} c_n P\left\{\left|\frac{\sum_{k=1}^{n} a_k X_k}{b_n} - L\right| > \varepsilon\right\} < \infty$$

for all $\varepsilon > 0$, where $c_n = (n \log n)^{-1}$ for the same nonzero L as in our Strong Laws.

Before we establish our result we need a few comments about notation. We define $\lg x = \log(\max\{e, x\})$ and $\lg_2 x = \lg(\lg x)$. Also, the constant C will denote a generic real number that is not necessarily the same in each appearance.

From Adler [2] we have

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} [(\lg k)^{b-\alpha-2}/k] X_k}{(\lg n)^b} = \frac{a}{(\alpha+1)b} \text{ almost surely,}$$

where both a and b are positive. So we set $a_n = (\lg n)^{b-\alpha-2}/n$, $b_n = (\lg n)^b$ and $L = a/((\alpha+1)b)$. As in our Strong Laws we partition our sum into the three terms:

(2) $$\begin{aligned} b_n^{-1} \sum_{k=1}^{n} a_k X_k &= b_n^{-1} \sum_{k=1}^{n} a_k \left[X_k I\left(|X_k| \leqslant k(\lg k)^{\alpha+2}\right) - EXI\left(|X| \leqslant k(\lg k)^{\alpha+2}\right)\right] \\ &+ b_n^{-1} \sum_{k=1}^{n} a_k X_k I\left(|X_k| > k(\lg k)^{\alpha+2}\right) \\ &+ b_n^{-1} \sum_{k=1}^{n} a_k EXI\left(|X| \leqslant k(\lg k)^{\alpha+2}\right). \end{aligned}$$

The last term converges to L by basic mathematics. Next we show that the first term converges to zero.

CLAIM. $\sum_{n=1}^{\infty} c_n P\left\{\left|\sum_{k=1}^{n} Y_k\right| > \varepsilon b_n\right\} < \infty$, *where*

$$Y_k = a_k \left[X_k I\left(|X_k| \leqslant k(\lg k)^{\alpha+2}\right) - EXI\left(|X| \leqslant k(\lg k)^{\alpha+2}\right)\right].$$

Proof. From Markov's inequality we get

$$\begin{aligned} P\left\{\left|\sum_{k=1}^{n} Y_k\right| > \varepsilon b_n\right\} &\leqslant \frac{C}{b_n^2} \sum_{k=1}^{n} EY_k^2 \\ &\leqslant \frac{C}{(\lg n)^{2b}} \sum_{k=1}^{n} \frac{(\lg k)^{2(b-\alpha-2)}}{k^2} EX^2 I\left(X \leqslant k(\lg k)^{\alpha+2}\right) \\ &\leqslant \frac{C}{(\lg n)^{2b}} \sum_{k=1}^{n} \frac{(\lg k)^{2(b-\alpha-2)}}{k^2} \int_1^{k(\lg k)^{\alpha+2}} (\lg x)^{\alpha} \, dx \end{aligned}$$

$$\leqslant \frac{C}{(\lg n)^{2b}} \sum_{k=1}^{n} \frac{(\lg k)^{2(b-\alpha-2)}}{k^2} k(\lg k)^{\alpha+2}[\lg k+(\alpha+2)\lg_2 k]^\alpha$$

$$\leqslant \frac{C}{(\lg n)^{2b}} \sum_{k=1}^{n} \frac{(\lg k)^{2b-\alpha-2}}{k}[\lg k]^\alpha = \frac{C}{(\lg n)^{2b}} \sum_{k=1}^{n} \frac{(\lg k)^{2b-2}}{k}.$$

Hence

$$P\{|\sum_{k=1}^{n} Y_k| > \varepsilon b_n\} \leqslant \begin{cases} C/(\lg n)^{2b} & \text{if } 0 < b < 1/2, \\ C\lg_2 n/\lg n & \text{if } b = 1/2, \\ C/\lg n & \text{if } b > 1/2, \end{cases}$$

whence

$$\sum_{n=1}^{\infty} c_n P\{|\sum_{k=1}^{n} Y_k| > \varepsilon b_n\} < \infty$$

since $c_n = (n\lg n)^{-1}$. ■

The real problem is the second term in (2). Even though the Borel–Cantelli lemma holds, this is not sufficient. We will use a result due to Hu et al. [3]. This result is quite optimal. Their theorem is:

THEOREM. *Let $\{(Y_{nk}, 1 \leqslant k \leqslant k_n), n \geqslant 1\}$ be an array of rowwise independent random variables and $\{c_n, n \geqslant 1\}$ a sequence of positive constants such that $\sum_{n=1}^{\infty} c_n = \infty$. Suppose that for all $\varepsilon > 0$ and some $\delta > 0$:*

(i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\{|Y_{nk}| > \varepsilon\} < \infty$,

(ii) *there exists $J \geqslant 2$ such that* $\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EY_{nk}^2 I(|Y_{nk}| \leqslant \delta)\right)^J < \infty$,

(iii) $\sum_{k=1}^{k_n} EY_{nk} I(|Y_{nk}| \leqslant \delta) \to 0$ *as* $n \to \infty$.

Then $\sum_{n=1}^{\infty} c_n P\{|\sum_{k=1}^{k_n} Y_{nk}| > \varepsilon\} < \infty$ *for all* $\varepsilon > 0$.

CLAIM. *(i) holds with*

$$c_n = (n\lg n)^{-1} \quad \text{and} \quad Y_{nk} = a_k X_k I(|X_k| > k(\lg k)^{\alpha+2})/b_n.$$

Proof. Let $0 < \varepsilon < 1$ and set γ_n to the greatest integer part of $n^{\varepsilon^{1/b}}$. We have

$$P\{|Y_{nk}| > \varepsilon b_n\} = P\{|X| I(|X| > k(\lg k)^{\alpha+2}) > \varepsilon k(\lg k)^{\alpha+2-b}(\lg n)^b\}$$

$$= P\{|X| > \max\{k(\lg k)^{\alpha+2}, \varepsilon k(\lg k)^{\alpha+2-b}(\lg n)^b\}\}.$$

Thus

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{n} P\{|Y_{nk}| > \varepsilon b_n\} = \sum_{n=1}^{\infty} c_n \Big[\sum_{k=1}^{\gamma_n} P\{|Y_{nk}| > \varepsilon b_n\} + \sum_{k=\gamma_n+1}^{n} P\{|Y_{nk}| > \varepsilon b_n\}\Big]$$

$$< C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{\gamma_n} \frac{[\lg(\varepsilon k(\lg k)^{\alpha+2-b}(\lg n)^b)]^\alpha}{\varepsilon k(\lg k)^{\alpha+2-b}(\lg n)^b}$$

$$+C\sum_{n=1}^{\infty} c_n \sum_{k=\gamma_n+1}^{n} \frac{\left[\lg\left(k(\lg k)^{\alpha+2}\right)\right]^{\alpha}}{k(\lg k)^{\alpha+2}}$$

$$= C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{\gamma_n} \frac{[\lg\varepsilon+\lg k+(\alpha+2-b)\lg_2 k+b\lg_2 n]^{\alpha}}{\varepsilon k(\lg k)^{\alpha+2-b}(\lg n)^b}$$

$$+C\sum_{n=1}^{\infty} c_n \sum_{k=\gamma_n+1}^{n} \frac{[\lg k+(\alpha+2)\lg_2 k]^{\alpha}}{k(\lg k)^{\alpha+2}}.$$

The second term is simpler to prove; it equals

$$C\sum_{n=1}^{\infty} c_n \sum_{k=\gamma_n+1}^{n} \frac{[\lg k+(\alpha+2)\lg_2 k]^{\alpha}}{k(\lg k)^{\alpha+2}} < C\sum_{n=1}^{\infty} c_n \sum_{k=\gamma_n+1}^{n} \frac{(\lg k)^{\alpha}}{k(\lg k)^{\alpha+2}}$$

$$= C\sum_{n=1}^{\infty} c_n \sum_{k=\gamma_n+1}^{n} \frac{1}{k(\lg k)^2} < C\sum_{n=1}^{\infty} c_n \int_{\gamma_n}^{\infty} \frac{dx}{x(\lg x)^2}$$

$$< C\sum_{n=1}^{\infty} \frac{c_n}{\lg n} = C\sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.$$

Let M be any integer larger than α; then the first term is

$$C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{\gamma_n} \frac{[\lg\varepsilon+\lg k+(\alpha+2)\lg_2 k+b(\lg_2 n-\lg_2 k)]^{\alpha}}{\varepsilon k(\lg k)^{\alpha+2-b}(\lg n)^b}$$

$$< C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{\gamma_n} \left\{ \frac{\sum_{j=0}^{M} [\lg\varepsilon+\lg k+(\alpha+2)\lg_2 k]^{\alpha-j}[\lg_2 n-\lg_2 k]^j}{k(\lg k)^{\alpha+2-b}(\lg n)^b} + \frac{y^{\alpha-M-1}[\lg_2 n-\lg_2 k]^{M+1}}{k(\lg k)^{\alpha+2-b}(\lg n)^b} \right\}$$

$$< C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{\gamma_n} \sum_{j=0}^{M} \frac{[\lg\varepsilon+\lg k+(\alpha+2)\lg_2 k]^{\alpha-j}(\lg_2 n)^j}{k(\lg k)^{\alpha+2-b}(\lg n)^b}$$

$$+C\sum_{n=1}^{\infty} c_n \sum_{k=1}^{\gamma_n} \frac{y^{\alpha-M-1}(\lg_2 n)^{M+1}}{k(\lg k)^{\alpha+2-b}(\lg n)^b},$$

where

$$\lg\varepsilon+\lg k+(\alpha+2)\lg_2 k < y < \lg\varepsilon+\lg k+(\alpha+2)\lg_2 k+b(\lg_2 n-\lg_2 k).$$

Since $y > C\lg k$, we have

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{\gamma_n} \frac{y^{\alpha-M-1}(\lg_2 n)^{M+1}}{k(\lg k)^{\alpha+2-b}(\lg n)^b} < C\sum_{n=1}^{\infty} \frac{c_n(\lg_2 n)^{M+1}}{(\lg n)^b} \sum_{k=1}^{\gamma_n} \frac{(\lg k)^{\alpha-M-1}}{k(\lg k)^{\alpha+2-b}}$$

$$\leqslant C\sum_{n=1}^{\infty} \frac{c_n(\lg_2 n)^{M+1}}{(\lg n)^b} \sum_{k=1}^{n} \frac{(\lg k)^{b-M-3}}{k} < C\sum_{n=1}^{\infty} \frac{c_n(\lg_2 n)^{M+1}}{(\lg n)^b} \sum_{k=1}^{n} \frac{(\lg k)^{b-3}}{k} < \infty$$

since

$$\sum_{k=1}^{n} \frac{(\lg k)^{b-3}}{k} \leqslant \begin{cases} C & \text{if } 0 < b < 2, \\ C \lg_2 n & \text{if } b = 2, \\ C(\lg n)^{b-2} & \text{if } b > 2. \end{cases}$$

Finally, we have

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{\gamma_n} \sum_{j=0}^{M} \frac{[\lg \varepsilon + \lg k + (\alpha+2)\lg_2 k]^{\alpha-j} (\lg_2 n)^j}{k(\lg k)^{\alpha+2-b} (\lg n)^b}$$

$$< C \sum_{n=1}^{\infty} c_n \sum_{k=1}^{\gamma_n} \sum_{j=0}^{M} \frac{(\lg k)^{\alpha-j} (\lg_2 n)^j}{k(\lg k)^{\alpha+2-b} (\lg n)^b} < C \sum_{n=1}^{\infty} \frac{c_n (\lg_2 n)^M}{(\lg n)^b} \sum_{k=1}^{n} \frac{(\lg k)^{b-2}}{k} < \infty$$

since

$$\sum_{k=1}^{n} \frac{(\lg k)^{b-2}}{k} \leqslant \begin{cases} C & \text{if } 0 < b < 1, \\ C \lg_2 n & \text{if } b = 1, \\ C(\lg n)^{b-1} & \text{if } b > 1, \end{cases}$$

which concludes the proof of part (i). ∎

CLAIM. (ii) *holds with* $J = 2$ *and* $\delta = 1$.

Proof. Again we select M as any integer larger than α. Thus

$$\sum_{k=1}^{n} EY_{nk}^2 I(|Y_{nk}| \leqslant 1)$$

$$= \sum_{k=1}^{n} \left\{ \frac{(\lg k)^{2(b-\alpha-2)} EX^2 I(X > k(\lg k)^{\alpha+2})}{k^2 (\lg n)^{2b}} \right.$$

$$\left. \times I\left((\lg k)^{b-\alpha-2} XI(X > k(\lg k)^{\alpha+2}) \leqslant k(\lg n)^b\right) \right\}$$

$$= \sum_{k=1}^{n} \frac{(\lg k)^{2(b-\alpha-2)}}{k^2 (\lg n)^{2b}} EX^2 I(X > k(\lg k)^{\alpha+2}) I(X \leqslant k(\lg k)^{\alpha+2-b} (\lg n)^b)$$

$$= \sum_{k=1}^{n} \frac{(\lg k)^{2(b-\alpha-2)}}{k^2 (\lg n)^{2b}} \int_{k(\lg k)^{\alpha+2}}^{k(\lg k)^{\alpha+2-b}(\lg n)^b} x^2 \, dF(x)$$

$$\leqslant C \sum_{k=1}^{n} \frac{(\lg k)^{2(b-\alpha-2)}}{k^2 (\lg n)^{2b}} \int_{1}^{k(\lg k)^{\alpha+2-b}(\lg n)^b} (\lg x)^\alpha \, dx$$

$$\leqslant C \sum_{k=1}^{n} \frac{(\lg k)^{2(b-\alpha-2)} k(\lg k)^{\alpha+2-b} (\lg n)^b [\lg(k(\lg k)^{\alpha+2-b}(\lg n)^b)]^\alpha}{k^2 (\lg n)^{2b}}$$

$$= C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2} [\lg k + (\alpha+2-b)\lg_2 k + b \lg_2 n]^\alpha}{k(\lg n)^b}$$

$$= C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2} [\lg k + (\alpha+2) \lg_2 k + b(\lg_2 n - \lg_2 k)]^{\alpha}}{k(\lg n)^b}$$

$$\leqslant C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^b} \Big[\sum_{j=0}^{M} [\lg k + (\alpha+2)\lg_2 k]^{\alpha-j} [\lg_2 n]^j + y^{\alpha-M-1} [\lg_2 n]^{M+1} \Big]$$

$$= C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^b} \sum_{j=0}^{M} [\lg k + (\alpha+2)\lg_2 k]^{\alpha-j} [\lg_2 n]^j$$

$$+ C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2} y^{\alpha-M-1} [\lg_2 n]^{M+1}}{k(\lg n)^b},$$

where $\lg k + (\alpha+2)\lg_2 k < y < \lg k + (\alpha+2)\lg_2 k + b(\lg_2 n - \lg_2 k)$.

For the first term

$$\sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^b} \sum_{j=0}^{M} [\lg k + (\alpha+2)\lg_2 k]^{\alpha-j} [\lg_2 n]^j$$

$$\leqslant C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^b} [\lg k]^{\alpha} [\lg_2 n]^M = \frac{C(\lg_2 n)^M}{(\lg n)^b} \sum_{k=1}^{n} \frac{(\lg k)^{b-2}}{k}.$$

Therefore the first term is less than

$$\begin{cases} C(\lg_2 n)^M/(\lg n)^b & \text{if } 0 < b < 1, \\ C(\lg_2 n)^{M+1}/\lg n & \text{if } b = 1, \\ C(\lg_2 n)^M/\lg n & \text{if } b > 1. \end{cases}$$

Working on the second term we have

$$\sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2} y^{\alpha-M-1} [\lg_2 n]^{M+1}}{k(\lg n)^b} \leqslant C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2} (\lg k)^{\alpha-M-1} [\lg_2 n]^{M+1}}{k(\lg n)^b}$$

$$= C \sum_{k=1}^{n} \frac{(\lg k)^{b-M-3} (\lg_2 n)^{M+1}}{k(\lg n)^b} \leqslant \frac{C(\lg_2 n)^{M+1}}{(\lg n)^b} \sum_{k=1}^{n} \frac{(\lg k)^{b-3}}{k}.$$

So this term is bounded by

$$\begin{cases} C(\lg_2 n)^{M+1}/(\lg n)^b & \text{if } 0 < b < 2, \\ C(\lg_2 n)^{M+2}/(\lg n)^2 & \text{if } b = 2, \\ C(\lg_2 n)^{M+1}/(\lg n)^2 & \text{if } b > 2. \end{cases}$$

Thus for all $b > 0$

$$\sum_{n=1}^{\infty} c_n \Big(\sum_{k=1}^{n} E Y_{nk}^2 I(|Y_{nk}| \leqslant 1) \Big)^2 < \infty, \quad \text{where } c_n = (n \lg n)^{-1}. \ \blacksquare$$

CLAIM. (iii) *holds, where once again* $\delta = 1$.

Proof. Let M be any integer larger than α. Thus

$$\sum_{k=1}^{n} EY_{nk} I(|Y_{nk}| \leqslant 1)$$

$$= \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2} EXI(X > k(\lg k)^{\alpha+2}) I\big((\lg k)^{b-\alpha-2} XI(X > k(\lg k)^{\alpha+2}) \leqslant k(\lg n)^{b}\big)}{k(\lg n)^{b}}$$

$$= \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^{b}} EXI(X > k(\lg k)^{\alpha+2}) I(X \leqslant k(\lg k)^{\alpha+2-b}(\lg n)^{b})$$

$$\leqslant C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^{b}} \int_{k(\lg k)^{\alpha+2}}^{k(\lg k)^{\alpha+2-b}(\lg n)^{b}} \frac{(\lg x)^{\alpha} dx}{x}$$

$$= C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^{b}} \big[[\lg k + b\lg_2 n + (\alpha+2-b)\lg_2 k]^{\alpha+1} - [\lg k + (\alpha+2)\lg_2 k]^{\alpha+1}\big]$$

$$= C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^{b}} \big[[\lg k + (\alpha+2)\lg_2 k + b(\lg_2 n - \lg_2 k)]^{\alpha+1} - [\lg k + (\alpha+2)\lg_2 k]^{\alpha+1}\big]$$

$$\leqslant C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^{b}} \Big[\sum_{j=1}^{M} [\lg k + (\alpha+2)\lg_2 k]^{\alpha+1-j} (\lg_2 n)^{j} + y^{\alpha-M}(\lg_2 n)^{M+1}\Big],$$

where $\lg k + (\alpha+2)\lg_2 k < y < \lg k + (\alpha+2)\lg_2 k + b(\lg_2 n - \lg_2 k)$.

The first term goes to zero since

$$\sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^{b}} \sum_{j=1}^{M} [\lg k + (\alpha+2)\lg_2 k]^{\alpha+1-j} (\lg_2 n)^{j}$$

$$\leqslant C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^{b}} [\lg k + (\alpha+2)\lg_2 k]^{\alpha} (\lg_2 n)^{M}$$

$$\leqslant C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^{b}} [\lg k]^{\alpha} (\lg_2 n)^{M} = C \sum_{k=1}^{n} \frac{(\lg k)^{b-2} (\lg_2 n)^{M}}{k(\lg n)^{b}}$$

$$= \frac{C(\lg_2 n)^{M}}{(\lg n)^{b}} \sum_{k=1}^{n} \frac{(\lg k)^{b-2}}{k} \to 0$$

for all $b > 0$.

Finally, we have

$$\sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^{b}} y^{\alpha-M} (\lg_2 n)^{M+1} \leqslant C \sum_{k=1}^{n} \frac{(\lg k)^{b-\alpha-2}}{k(\lg n)^{b}} [\lg k]^{\alpha-M} (\lg_2 n)^{M+1}$$

$$= \frac{C(\lg_2 n)^{M+1}}{(\lg n)^{b}} \sum_{k=1}^{n} \frac{(\lg k)^{b-M-2}}{k} \to 0$$

for all $b > 0$, which concludes the proof of part (iii), and hence our claim that

$$\sum_{n=1}^{\infty} (n \lg n)^{-1} P\left\{\left|\frac{\sum_{k=1}^{n} [(\lg k)^{b-\alpha-2}/k]\, X_k}{(\lg n)^b} - \frac{a}{(\alpha+1)\, b}\right| > \varepsilon\right\} < \infty$$

for all $\varepsilon > 0$. ▪

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