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A CHARACTERIZATION OF SIGN-SYMMETRIC LIOUVILLE-TYPE DISTRIBUTIONS

BY

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Abstract. Sign-symmetric Liouville-type distributions on n-dimensional space are characterized by certain $(n-1)$-dimensional distribution of quotients in a special form.

1. Introduction. In 1996 Gupta et al. [1] introduced a family of multivariate distributions which is an important generalization of many classes of distributions. It may be obtained by the following construction. For $\alpha, \beta > 0$ let $\mathscr{X}(\alpha, \beta)$ denote a distribution with probability density function

$$f(z) := \frac{\alpha}{2\Gamma(\beta/\alpha)} |z|^{\beta-1} \exp(-|z|^{\alpha}), \quad z \in \boldsymbol{R}.$$

When $Z_1, \ldots, Z_n$ are mutually independent, real-valued random variables and $Z_i \sim \mathscr{X}(\alpha_i, \beta_i)$ for some positive parameters α_i and β_i $(i = 1, \ldots, n)$, then the distribution of the vector

$$(1.1) \qquad (X_1, \ldots, X_n) := \left(\frac{Z_1 \cdot \Theta^{1/\alpha_1}}{(\sum_{j=1}^{n} |Z_j|^{\alpha_j})^{1/\alpha_1}}, \ldots, \frac{Z_n \cdot \Theta^{1/\alpha_n}}{(\sum_{j=1}^{n} |Z_j|^{\alpha_j})^{1/\alpha_n}} \right),$$

where Θ is a positive random variable independent of

$$(1.2) \qquad (U_1, \ldots, U_n) := \left(\frac{Z_1}{(\sum_{j=1}^{n} |Z_j|^{\alpha_j})^{1/\alpha_1}}, \ldots, \frac{Z_n}{(\sum_{j=1}^{n} |Z_j|^{\alpha_j})^{1/\alpha_n}} \right),$$

is called the *sign-symmetric Liouville-type distribution* and denoted by $\mathscr{SL}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$. (Besides, the distribution of the vector (1.2) is called the *sign-symmetric Dirichlet-type distribution* and denoted by $\mathscr{SD}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$.)

The problem of explaining relations between distributions of random vectors $X = (X_1, \ldots, X_n)$ and quotients $(X_1/X_n, \ldots, X_{n-1}/X_n)$ has a long history. In one of the recent investigations in this field Wesołowski [3] proved a theorem characterizing symmetrically invariant two-dimensional distributions by the Cauchy distribution of quotients. This result was generalized to finite-dimensional α-symmetrically invariant distributions by Szabłowski [2], who considered a certain Cauchy-like (so-called α-Cauchy) distribution of the quotients. On the other hand, Wesołowski [4] proved that a distribution of an α-spherical random vector is uniquely determined by a distribution of quotients. In this paper, methods adapted from [2] and [4] are applied to obtain a characterization of the sign-symmetric Liouville-type distribution that generalizes previous results.

We will use the following notation: If $\boldsymbol{x} = (x_1, \ldots, x_n)^T \in \boldsymbol{R}^n$, then

$$\|\boldsymbol{x}\|_\alpha := \Big(\sum_{i=1}^{n} |x_i|^\alpha\Big)^{1/\alpha}.$$

For $x \in \boldsymbol{R}$ and $q > 0$, $x^{\langle q \rangle} := \operatorname{sign}(x) \cdot |x|^q$. Throughout the paper $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are positive parameters and $p_i := \sum_{j=1}^{i} \beta_j/\alpha_j$ for $i = 1, \ldots, n$.

2. Characterization.

DEFINITION 1. Let $a_1, \ldots, a_n, b_1, \ldots, b_{n+1} > 0$. We say that a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ has a *distribution* $\mathscr{D}(a_1, \ldots, a_n; b_1, \ldots, b_{n+1})$ if its joint density function is

$$\text{(2.1)} \qquad \frac{\prod_{i=1}^{n} a_i}{2^n} \frac{\Gamma\big(\sum_{i=1}^{n+1} b_i\big)}{\sum_{i=1}^{n+1} \Gamma(b_i)} \prod_{i=1}^{n} |x_i|^{a_i b_i - 1} \Big(\sum_{i=1}^{n} |x_i|^{a_i} + 1\Big)^{-\sum_{i=1}^{n+1} b_i}.$$

The distribution $\mathscr{D}$ is a generalization of α-Cauchy distribution defined by Szabłowski [2]. More specifically, we have the following

Remark 1. A random vector $(X_1, \ldots, X_n) \sim \mathscr{D}(\alpha, \ldots, \alpha; 1/\alpha, \ldots, 1/\alpha)$ has the n-dimensional α-Cauchy distribution ($\alpha > 0$).

On the other hand, the distribution introduced by Definition 1 is a special case of the sign-symmetric Liouville distribution. Applying Proposition 3.2 from [1] we get

COROLLARY 1. *A random vector* $(X_1, \ldots, X_n) \sim \mathscr{D}(a_1, \ldots, a_n; b_1, \ldots, b_{n+1})$ *has the distribution* $\mathscr{SL}(a_1, \ldots, a_n; a_1 b_1, \ldots, a_n b_n; \Theta)$, *where* Θ *has the inverted beta-distribution* $\mathscr{IB}\big(\sum_{i=1}^{n} b_i, b_{n+1}, 1\big)$, *i.e. the probability density function of* Θ *is*

$$f(r) = \frac{1}{B\big(\sum_{i=1}^{n} b_i, b_{n+1}\big)} r^{\sum_{i=1}^{n} b_i - 1} \left(\frac{1}{r+1}\right)^{\sum_{i=1}^{n+1} b_i}, \qquad 0 < r < \infty,$$

B denoting the Euler beta-function.

The main result of this paper is

THEOREM 1. *A random vector* $\boldsymbol{X}$ *without an atom at* 0 *has the sign-symmetric Liouville-type distribution* $\mathscr{SL}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta)$ *if and only if the following three conditions hold*:

(i) $\boldsymbol{X} \sim -\boldsymbol{X}$ (*here the symbol* $\sim$ *denotes the equidistribution*);

(ii) *for some* $j \in \{1, \ldots, n\}$ *and for some* $\alpha > 0$ *the random vector*

$$Y_j := \left(\frac{X_1^{\langle \alpha_1/\alpha \rangle}}{X_j^{\langle \alpha_j/\alpha \rangle}}, \ldots, \frac{X_{j-1}^{\langle \alpha_{j-1}/\alpha \rangle}}{X_j^{\langle \alpha_j/\alpha \rangle}}, \frac{X_{j+1}^{\langle \alpha_{j+1}/\alpha \rangle}}{X_j^{\langle \alpha_j/\alpha \rangle}}, \ldots, \frac{X_n^{\langle \alpha_n/\alpha \rangle}}{X_j^{\langle \alpha_j/\alpha \rangle}}\right)$$

has the $(n-1)$*-dimensional distribution* $\mathscr{D}(\alpha, \ldots, \alpha;\ \beta_1/\alpha_1, \ldots, \beta_n/\alpha_n)$;

(iii) Y_j *and* $\sum_{i=1}^n |X_i|^{\alpha_i}$ *are independent.*

It is easily seen that sign-symmetric Liouville distributions contain α-symmetrically invariant distributions as a subclass. Hence and in view of Remark 1, Theorem 1 is an important generalization of Szabłowski's result [2].

3. Auxiliary results and proofs. We begin with three technical lemmas (an easy proof of the first one is left to the reader):

LEMMA 1. *If* $q > 0$, *then* $Z \sim \mathscr{L}(\alpha, \beta)$ *if and only if* $Z^{\langle q \rangle} \sim \mathscr{L}(\alpha/q, \beta/q)$.

LEMMA 2. *Let* $Z_1, \ldots, Z_n$ *be mutually independent, real-valued random variables and* $Z_i \sim \mathscr{L}(\alpha_i, \beta_i)$ *for some positive parameters* α_i *and* β_i $(i = 1, \ldots, n)$. *Fix* $j \in \{1, \ldots, n\}$ *and let* $q_i > 0$ *for* $i \neq j$. *Then the joint density function of the random vector*

$$\overset{\bowtie}{Z} = \left(\frac{Z_1^{\langle \alpha_1/q_1 \rangle}}{Z_j^{\langle \alpha_j/q_1 \rangle}}, \ldots, \frac{Z_{j-1}^{\langle \alpha_{j-1}/q_{j-1} \rangle}}{Z_j^{\langle \alpha_j/q_{j-1} \rangle}}, \frac{Z_{j+1}^{\langle \alpha_{j+1}/q_{j+1} \rangle}}{Z_j^{\langle \alpha_j/q_{j+1} \rangle}}, \ldots, \frac{Z_n^{\langle \alpha_n/q_n \rangle}}{Z_j^{\langle \alpha_j/q_n \rangle}}\right)$$

is

$$\int_{\boldsymbol{R}} \Big|\prod_{i \neq j} z^{\langle \alpha_j/q_i \rangle}\Big| \prod_{i \neq j} g_i(x_i z^{\langle \alpha_j/q_i \rangle}) \cdot \frac{\alpha_j}{2\Gamma(\beta_j/\alpha_j)} |z|^{\beta_j - 1} \exp(-|z|)^{\alpha_j}) \, dz, \tag{3.1}$$

where g_i *stands for the probability density function of* $Z_i^{\langle \alpha_i/q_i \rangle}$.

Proof. Let $f\colon \boldsymbol{R}^{n-1} \to \boldsymbol{R}$ be a bounded function. Since Z_i are independent, we see that

$$\mathbf{E} f(\overset{\bowtie}{Z}) = \int_{\boldsymbol{R}} \int_{\boldsymbol{R}^{n-2}} f\left(\frac{z_1}{z^{\langle \alpha_j/q_1 \rangle}}, \ldots, \frac{z_{j-1}}{z^{\langle \alpha_j/q_{j-1} \rangle}}, \frac{z_{j+1}}{z^{\langle \alpha_j/q_{j+1} \rangle}}, \ldots, \frac{z_n}{z^{\langle \alpha_j/q_n \rangle}}\right)$$

$$\times \prod_{i \neq j} g_i(z_i) \, dz_1 \ldots dz_{j-1} \, dz_{j+1} \ldots dz_n \cdot g(z) \, dz,$$

where g is the probability density function of Z_j. Let (with z fixed)

$$x_i := z_i / z^{\langle \alpha_j/q_i \rangle} \quad \text{for } i \neq j.$$

The Jacobian of this transformation is $\prod_{i \neq j} z^{\langle \alpha_j/q_i \rangle}$. Formula (3.1) is now easily seen. ■

LEMMA 3. *Let* $X = (X_1, \ldots, X_n) \sim \mathscr{SL}(\alpha_1, \ldots, \alpha_n;\ \beta_1, \ldots, \beta_n;\ \Theta)$. *Fix* $j \in \{1, \ldots, n\}$ *and let* $q_i > 0$ *for* $i \neq j$. *The random vector*

$$\overset{\bowtie}{X} = \left(\frac{X_1^{\langle \alpha_1/q_1 \rangle}}{X_j^{\langle \alpha_j/q_1 \rangle}}, \ldots, \frac{X_{j-1}^{\langle \alpha_{j-1}/q_{j-1} \rangle}}{X_j^{\langle \alpha_j/q_{j-1} \rangle}}, \frac{X_{j+1}^{\langle \alpha_{j+1}/q_{j+1} \rangle}}{X_j^{\langle \alpha_j/q_{j+1} \rangle}}, \ldots, \frac{X_n^{\langle \alpha_n/q_n \rangle}}{X_j^{\langle \alpha_j/q_n \rangle}}\right)$$

has the distribution $\mathscr{D}(q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_n;\ \beta_1/\alpha_1, \ldots, \beta_n/\alpha_n)$.

Proof. Since for every $k \in \{1, \ldots, j-1, j+1, \ldots, n\}$

$$\frac{X_k^{\langle \alpha_k/q_k \rangle}}{X_j^{\langle \alpha_j/q_k \rangle}} = \frac{Z_k^{\langle \alpha_k/q_k \rangle}}{Z_j^{\langle \alpha_j/q_k \rangle}},$$

the probability density function of $\overset{\bowtie}{X}$ is defined by (3.1). Consequently,

$$g(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

$$= \int_{\mathbf{R}} \Big|\prod_{i \neq j} z^{\langle \alpha_j/q_i \rangle}\Big| \prod_{i \neq j} \frac{q_i}{2\Gamma(\beta_i/\alpha_i)} |x_i z^{\langle \alpha_j/q_i \rangle}|^{(\beta_i/\alpha_i)q_i - 1} \exp(-|x_i z^{\langle \alpha_j/q_i \rangle}|^{q_i})$$

$$\times \frac{\alpha_j}{2\Gamma(\beta_j/\alpha_j)} |z|^{\beta_j - 1} \exp(-|z|^{\alpha_j})\, dz$$

$$= \frac{\alpha_j \prod_{i \neq j} q_i}{2^n \prod_{i=1}^n \Gamma(\beta_i/\alpha_i)} \prod_{i \neq j} |x_i|^{(\beta_i/\alpha_i)q_i - 1} \int_{\mathbf{R}} |z|^{\alpha_j p_n - 1} \exp\Big[-|z|^{\alpha_j}\Big(\sum_{i \neq j} |x_i|^{q_i} + 1\Big)\Big]\, dz.$$

Using symmetry and the integral formula $(\mu, \nu > 0)$

$$\int_0^\infty x^{v-1} \exp[-\mu x^p]\, dx = \frac{1}{|p|} \mu^{-v/p} \Gamma\Big(\frac{v}{p}\Big),$$

we get

$$\int_{\mathbf{R}} |z|^{\alpha_j p_n - 1} \exp\Big[-|z|^{\alpha_j}\Big(\sum_{i \neq j} |x_i|^{q_i} + 1\Big)\Big]\, dz = \frac{2}{\alpha_j}\Big(\sum_{i \neq j} |x_i|^{q_i} + 1\Big)^{-p_n} \Gamma(p_n).$$

Therefore

$$g(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

$$= \frac{\alpha_j \prod_{i \neq j} q_i}{2^n \prod_{i=1}^n \Gamma(\beta_i/\alpha_i)} \prod_{i \neq j} |x_i|^{(\beta_i/\alpha_i)q_i - 1} \frac{2}{\alpha_j}\Big(\sum_{i \neq j} |x_i|^{q_i} + 1\Big)^{-p_n} \Gamma(p_n)$$

$$= \frac{\prod_{i \neq j} q_i}{2^{n-1}} \frac{\Gamma(p_n)}{\prod_{i=1}^n \Gamma(\beta_i/\alpha_i)} \prod_{i \neq j} |x_i|^{(\beta_i/\alpha_i)q_i - 1} \Big(\sum_{i \neq j} |x_i|^{q_i} + 1\Big)^{-p_n}. \blacksquare$$

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. Let $\tilde{X} := (X_1^{\langle \alpha_1/\alpha \rangle}, \ldots, X_n^{\langle \alpha_n/\alpha \rangle})$ for some $\alpha > 0$. Then $\tilde{X} = U_\alpha \cdot \Theta^{1/\alpha}$, where

$$U_\alpha := \frac{(Z_1^{\langle \alpha_1/\alpha \rangle}, \ldots, Z_n^{\langle \alpha_n/\alpha \rangle})}{\|(Z_1^{\langle \alpha_1/\alpha \rangle}, \ldots, Z_n^{\langle \alpha_n/\alpha \rangle})\|_\alpha}$$

(U_α and Θ are independent). From Lemma 1 we get

$$U_\alpha \sim \mathscr{S}\mathscr{D}\Big(\alpha, \ldots, \alpha; \frac{\beta_1}{\alpha_1}\alpha, \ldots, \frac{\beta_n}{\alpha_n}\alpha\Big)$$

and

$$\tilde{X} \sim \mathscr{SL}\left(\alpha, \ldots, \alpha; \frac{\beta_1}{\alpha_1}\alpha, \ldots, \frac{\beta_n}{\alpha_n}\alpha; \Theta\right).$$

Furthermore,

$$X \sim \mathscr{SL}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; \Theta) \Leftrightarrow \tilde{X} \sim \mathscr{SL}\left(\alpha, \ldots, \alpha; \frac{\beta_1}{\alpha_1}\alpha, \ldots, \frac{\beta_n}{\alpha_n}\alpha; \Theta\right).$$

Wesołowski [4] showed the following fact. Let the distributions of random vectors

$$\left((-1)^{\varepsilon_1} Y_1, \ldots, (-1)^{\varepsilon_n} Y_n\right),$$

which have been constructed from a random vector $Y = (Y_1, \ldots, Y_n)$ concentrated on the unit α-sphere $S_\alpha := \{x \in \boldsymbol{R}^n \colon \|x\|_\alpha = 1\}$, coincide for any $(\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n$ and let a random vector $P = (P_1, \ldots, P_n)$ take the form $P = RY$ (for some positive random variable R independent of Y). Then the distribution of the vector P is uniquely determined by the distribution of the quotients $(P_1/P_j, \ldots, P_{j-1}/P_j, P_{j+1}/P_j, \ldots, P_n/P_j)$, $j = 1, \ldots, n$. Since $\tilde{X}$ obviously satisfies the assumptions of the above statement, applying Lemma 3 we obtain the result of Theorem 1. ∎

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