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SPECTRAL REPRESENTATION AND EXTRAPOLATION OF STATIONARY RANDOM PROCESSES ON LINEAR SPACES

BY

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Abstract. The paper deals with continuous Banach-space-valued stationary random processes on linear spaces. From von Waldenfels' [13] integral representation of positive definite functions on a linear space $\mathscr{L}$ we derive an analogue of Stone's theorem for a group of unitary operators over $\mathscr{L}$. It is used to obtain spectral representations of a general Banach-space-valued stationary random process over $\mathscr{L}$ and its covariance function. For the special class of Hilbert–Schmidt operator-valued stationary processes the explicit form of Kolmogorov's isomorphism theorem between temporal space and spectral space is established and with its aid there are studied some prediction problems. Our prediction results are similar to those proved in [5] for multivariate stationary processes on groups.

0. INTRODUCTION

There have been many papers that generalize the theory of stationary sequences and stationary processes on the real axis. In our paper by "stationary" we always mean "stationary in the wide sense". Generalizations are in two directions. On the one hand, there have been considered so-called Banach-space-valued processes, i.e. processes whose values are bounded linear operators from a Banach space into a Hilbert space. On the other hand, the parameter set has been generalized: The integers or the real axis have been replaced by locally compact abelian groups or even homogeneous spaces.

The present paper is devoted to the study of Banach-space-valued stationary processes X on a linear space $\mathscr{L}$ over $\boldsymbol{R}$. The processes are assumed to be stationary with respect to the additive group of $\mathscr{L}$. If we equip $\mathscr{L}$ with the discrete topology, then we consider X as a stationary process on a locally compact abelian group. But then we do not use the fact that $\mathscr{L}$ bears a linear structure. The question arises whether it is possible to regard this additional property of $\mathscr{L}$. It turned out that it is possible if X has an additional continuity property; see Definition 3.2 of the paper.

The starting point of our investigations is von Waldenfels' integral representation of positive definite functions on $\mathscr{L}$, which generalizes the well-

-known Bochner theorem; see [13], Satz 3. Using this result in Section 2 of our paper we obtain a spectral representation of groups of unitary operators on $\mathscr{L}$, i.e. an analogue of Stone's theorem. If we apply our theorem to the group of unitary operators which corresponds to X in a natural way, we immediately get a spectral representation for X and for the covariance function of X; see Theorem 3.3. Unfortunately, the spectral representation for general Banach-space-valued processes is not very useful in the prediction theory of such processes since up to now there has not been constructed a spectral space consisting of functions which is isometrically isomorphic to the Hilbert space spanned by the values of the process. Thus, in studying prediction problems we restrict our attention to processes of the class **CQS**, i.e. to continuous Hilbert–Schmidt operator-valued stationary processes on $\mathscr{L}$; see Definition 3.4.

For processes of the class **CQS** a spectral space can be constructed and Section 4 of the paper deals with the map of the space spanned by the values of the process onto the spectral space of the process.

The result of Section 4 is an analogue of Kolmogorov's isomorphism theorem and makes it possible to study problems of linear prediction as approximation problems in the spectral space. An isomorphism theorem of this type can be found in [9], Theorem 7.8; for some results in this direction see also [10]. But since in our paper the space spanned by the values of the process differs from that introduced in [9], we give complete proofs of our results.

Sections 5 and 6 deal with linear prediction problems of processes of the class **CQS** of the following type: Let $\mathscr{L}$ be the direct sum of some subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$ and assume that $\mathscr{L}_1$ has finite dimension. Let $\mathscr{S}_1$ be a subset of $\mathscr{L}_1$. Assume that the process is known on $\mathscr{S}_1+\mathscr{L}_2=\{x_1+x_2\colon x_1\in\mathscr{S}_1, x_2\in\mathscr{L}_2\}$. Construct the best linear prediction of an unknown value of the process on the basis of its values on $\mathscr{S}_1+\mathscr{L}_2$.

We show that such problems can be essentially considered as prediction problems for processes on $\mathscr{L}_1$. The method to obtain our results is a direct integral representation of the spectral space of the process. An analogous problem was studied for multivariate processes on groups in [5]. The present results are similar to those in [5].

In our paper we use the following notation. By N and C we denote the set of natural and complex numbers, respectively. For two Banach spaces $\mathscr{F}$ and $\mathscr{G}$, $\mathscr{B}(\mathscr{F}, \mathscr{G})$ denotes the Banach space of bounded linear operators from $\mathscr{F}$ to $\mathscr{G}$, equipped with the usual operator norm, and $\mathscr{B}_s(\mathscr{F}, \mathscr{G})$ denotes the space of bounded linear operators from $\mathscr{F}$ to $\mathscr{G}$, equipped with the strong operator topology. The norm in a Banach space $\mathscr{F}$ is denoted by $\|\cdot\|_{\mathscr{F}}$ and the inner product in a Hilbert space $\mathscr{H}$ by $(\cdot,\cdot)_{\mathscr{H}}$. If T is a linear operator, then T^* stands for its adjoint operator and $\mathscr{R}(T)$ for the range of T. If $\mathscr{X}$ is a topological space, then by $\mathfrak{B}(\mathscr{X})$ we denote the σ-algebra of Borel sets. Let $\mathscr{S}$ be a subset of a topological vector space $\mathscr{Y}$. Then $\overline{\text{lin}}_{\mathscr{Y}}\mathscr{S}$ denotes the closed linear hull of $\mathscr{S}$ in $\mathscr{Y}$.

1. PRELIMINARY FACTS

Throughout the paper, let $\mathscr{H}$ be a Hilbert space and $\mathscr{K}$ be a separable non-trivial Hilbert space over C. The (possibly finite) dimension of $\mathscr{K}$ is denoted by q. We set $N_q := \{1, 2, \ldots, q\}$ if $q < \infty$, and $N_q := N$ if $q = \infty$. We omit the symbol $\mathscr{K}$ in the notation of classes of operators acting on $\mathscr{K}$. Thus we write $\mathscr{B}$ for $\mathscr{B}(\mathscr{K}, \mathscr{K})$, and $\mathscr{B}_s$ instead of $\mathscr{B}_s(\mathscr{K}, \mathscr{K})$. The identity operator on $\mathscr{K}$ is denoted by I. Let $\mathscr{T}$ be the Banach space of trace class operators on $\mathscr{K}$ and $\mathscr{Q}(\mathscr{K}, \mathscr{H})$ be the Hilbert space of Hilbert–Schmidt operators from $\mathscr{K}$ to $\mathscr{H}$, i.e. $T \in \mathscr{Q}(\mathscr{K}, \mathscr{H})$ if and only if $T^* T \in \mathscr{T}$. We recall that the norm in $\mathscr{T}$ is defined by

$$\|T\|_{\mathscr{T}} := \operatorname{tr}(T^*T)^{1/2}, \qquad T \in \mathscr{T},$$

and the inner product in $\mathscr{Q}(\mathscr{K}, \mathscr{H})$ is defined by

$$(S, T)_{\mathscr{Q}(\mathscr{K},\mathscr{H})} := \operatorname{tr}(T^*S), \qquad T, S \in \mathscr{Q}(\mathscr{K}, \mathscr{H}).$$

Here the symbol $\operatorname{tr}T$ stands for the trace of an operator $T \in \mathscr{T}$, i.e.

$$\operatorname{tr} T := \sum_{i=1}^{q} (Te_j, e_j)_{\mathscr{K}},$$

where $\{e_j\}_{j \in N_q}$ denotes an orthonormal basis on $\mathscr{K}$. The symbol $T^{1/2}$ stands for the unique non-negative self-adjoint square root of a non-negative self-adjoint operator T. The properties of the spaces $\mathscr{Q}(\mathscr{K}, \mathscr{H})$ and $\mathscr{T}$ were studied in detail for example in [6]. In particular, the spaces $\mathscr{Q}(\mathscr{K}, \mathscr{H})$ and $\mathscr{T}$ are separable spaces (cf. [6], p. 119). Moreover, the following simple but useful fact holds true.

LEMMA 1.1. *There exists a countable dense subset $\mathscr{D}$ in the Hilbert space $\mathscr{Q}$ of Hilbert–Schmidt operators on $\mathscr{K}$. Moreover, the set $\mathscr{D}$ is dense in $\mathscr{B}_s$.*

Proof. The first assertion is clear from [6], p. 119. Now, let T be a bounded linear operator on $\mathscr{K}$ and $\{P_n\}_{n \in N}$ be a sequence of orthoprojectors on $\mathscr{K}$ with finite-dimensional range converging strongly to I if n tends to ∞. Then the sequence $\{TP_n\}_{n \in N}$ is a subset of $\mathscr{Q}$ and converges strongly to T if n tends to ∞. Hence $\mathscr{Q}$ is a dense subset of $\mathscr{B}_s$. Since the strong topology is weaker than the topology of the Hilbert space $\mathscr{Q}$, the set $\mathscr{D}$ is dense in the set of Hilbert–Schmidt operators on $\mathscr{K}$, equipped with the strong topology. It follows that $\mathscr{D}$ is dense in $\mathscr{B}_s$.

We remark that if $(\Omega, \mathfrak{A}, \mu)$ is a positive measure space and $(\Omega_1, \mathfrak{A}_1)$ is a measurable space, then all relations between $(\mathfrak{A}, \mathfrak{A}_1)$-measurable functions $f: \Omega \to \Omega_1$ are to be understood as relations which hold μ-almost everywhere. Moreover, in the notation of the integration we will often omit the integration variable. Let $\mathscr{X}$ be a subset of a topological vector space. A map $Z: \mathfrak{A} \to \mathscr{X}$ is called an *$\mathscr{X}$-valued measure* if Z is countably additive.

LEMMA 1.2. *Let* $M: \mathfrak{A} \to \mathcal{T}$ *be a* $\mathcal{B}_s$*-valued measure on the measurable space* $(\Omega, \mathfrak{A})$. *Then* M *is a* $\mathcal{T}$*-valued measure.*

Proof. We have to show that M is countably additive with respect to the $\|\cdot\|_{\mathcal{T}}$-topology if it is countably additive with respect to the strong topology. Let $\{P_n\}_{n\in N}$ be a sequence of finite-dimensional orthoprojectors on $\mathcal{K}$ tending to I in $\mathcal{B}_s$ if n tends to ∞. The mappings $M_n := P_n M P_n$, $n \in N$, are countably additive with respect to the strong topology. But since the values of M_n can be considered as operators on the finite-dimensional space $\mathcal{R}(P_n)$, the mappings M_n are countably additive with respect to the $\|\cdot\|_{\mathcal{T}}$-topology. On the other hand, for each $A \in \mathfrak{A}$ the sequence $\{M_n(A)\}_{n\in N}$ tends to $M(A)$ with respect to the $\|\cdot\|_{\mathcal{T}}$-topology (see [6], p. 119, Theorem 6.3). According to [4], Theorem IV.10.6, this yields the countable additivity of M with respect to the $\|\cdot\|_{\mathcal{T}}$-topology.

Let $\mathcal{T}^{\geqslant}$ be the set of non-negative self-adjoint operators in $\mathcal{T}$. Let M be a $\mathcal{T}^{\geqslant}$-valued measure on $(\Omega, \mathfrak{A})$. Set $\mu(A) := \operatorname{tr} M(A)$, $A \in \mathfrak{A}$. Then μ is a non--negative finite measure on $(\Omega, \mathfrak{A})$ and M is absolutely continuous with respect to μ. Since the Banach space $\mathcal{T}$ has the Radon–Nikodym property, the Radon–Nikodym derivative $dM/d\mu$ exists, and will be denoted by W. The function W is a $\mathcal{T}^{\geqslant}$-valued $(\mathfrak{A}, \mathfrak{B}(\mathcal{T}))$-measurable function such that $\|W(\omega)\|_{\mathcal{T}} = 1$ for μ-a.a. $\omega \in \Omega$. We remark that from Lemmas 1 and 5 in [8] it follows that $W^{1/2}$ is an $(\mathfrak{A}, \mathfrak{B}(\mathcal{Q}))$-measurable function.

Let $\mathcal{A}$ be the set of all (not necessarily densely defined and not necessarily bounded) linear operators on $\mathcal{K}$. Consider an $\mathcal{A}$-valued function Φ on Ω with the following properties:

(i) The function $\Phi W^{1/2}$ is defined and is a $\mathcal{Q}$-valued Bochner-measurable function on Ω.

(ii) The integral $\int \Phi W^{1/2} (\Phi W^{1/2})^* \, d\mu$ exists as a Bochner integral.

Two $\mathcal{A}$-valued functions Φ and Ψ with properties (i) and (ii) are called *equivalent* if $\Phi W^{1/2} = \Psi W^{1/2}$. Let $L^2(M)$ be the set of all equivalence classes of functions satisfying conditions (i) and (ii). As usual, we will work with representatives, i.e. with functions instead of equivalence classes.

THEOREM 1.3 ([9], Main Theorem I, 4.19). *The set* $L^2(M)$ *is a Hilbert space with inner product*

$$(\Phi, \Psi)_{L^2(M)} := \operatorname{tr} \int \Phi W^{1/2} (\Psi W^{1/2})^* \, d\mu$$
$$= \int \operatorname{tr} [\Phi W^{1/2} (\Psi W^{1/2})^*] \, d\mu, \qquad \Phi, \Psi \in L^2(M).$$

2. STONE'S THEOREM FOR A GROUP OF UNITARY OPERATORS OVER A LINEAR SPACE

Let $\mathcal{L}$ be a linear space over R and $(\mathcal{L}, +)$ be its additive group. We recall a theorem of von Waldenfels on the integral representation of positive definite functions on $(\mathcal{L}, +)$.

Let $\tilde{R}$ be the one-point compactification of R, i.e. $\tilde{R} := R \cup \{\infty\}$. A map $\varphi\colon \mathscr{L} \to \tilde{R}$ is called a *quasi-linear functional* on $\mathscr{L}$ if there exists a linear subspace $\mathscr{L}_1$ of $\mathscr{L}$ such that φ is a linear functional on $\mathscr{L}_1$ and the values of φ on $\mathscr{L} \backslash \mathscr{L}_1$ are equal to ∞. By $\langle \varphi, x \rangle$ we denote the value of a quasi-linear functional φ on the element x of $\mathscr{L}$ and by $\tilde{\mathscr{L}}$ the topological space of quasi-linear functionals on $\mathscr{L}$, equipped with the topology of pointwise convergence. We recall that $\tilde{\mathscr{L}}$ is compact.

THEOREM 2.1 ([13], Satz 3). *Let f be a positive definite C-valued function on $(\mathscr{L}, +)$ such that the map $R \ni \lambda \to f(\lambda x)$ is continuous at 0 for each $x \in \mathscr{L}$. Then there exists a unique non-negative finite regular Borel measure μ on $\mathfrak{B}(\tilde{\mathscr{L}})$ such that, for each $x \in \mathscr{L}$, we have*

$$\mu(\{\varphi \in \tilde{\mathscr{L}}\colon \langle \varphi, x \rangle = \infty\}) = 0 \quad \text{and} \quad f(x) = \int_{\tilde{\mathscr{L}}} e^{i\langle \varphi, x \rangle} \mu(d\varphi).$$

Let ν be a C-valued regular measure on $\mathfrak{B}(\tilde{\mathscr{L}})$ and $|\nu|$ be its variation. The uniqueness property of the measure μ in Theorem 2.1 yields the following general uniqueness result.

COROLLARY 2.2. *If ν is a C-valued bounded regular measure on $\mathfrak{B}(\tilde{\mathscr{L}})$ such that*

$$|\nu|(\{\varphi \in \tilde{\mathscr{L}}\colon \langle \varphi, x \rangle = \infty\}) = 0 \quad \text{and} \quad \int_{\tilde{\mathscr{L}}} e^{i\langle \varphi, x \rangle} \nu(d\varphi) = 0, \quad x \in \mathscr{L},$$

then ν is identically 0.

Proof. We write ν as a linear combination of four non-negative finite regular measures on $\mathfrak{B}(\tilde{\mathscr{L}})$, i.e. $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$. Let

$$f_j(x) := \int_{\tilde{\mathscr{L}}} e^{i\langle \varphi, x \rangle} \nu_j(d\varphi) = 0, \quad x \in \mathscr{L},\ j = 1, 2, 3, 4.$$

We have

$$f_1(x) - f_2(x) + i(f_3(x) - f_4(x)) = 0. \tag{2.1}$$

Because of $f_j(x)^* = f_j(-x)$ we get

$$f_1(x) - f_2(x) - i(f_3(x) - f_4(x)) = 0, \quad x \in \mathscr{L}.$$

Combining this with (2.1) we obtain $f_1 = f_2$ and $f_3 = f_4$, and by the uniqueness part of Theorem 2.1 it follows that $\nu_1 = \nu_2$ and $\nu_3 = \nu_4$, whence $\nu = 0$.

Let $\{U_x;\ x \in \mathscr{L}\}$ be a unitary representation of $(\mathscr{L}, +)$ on the Hilbert space $\mathscr{H}$, i.e. we have a family of unitary operators U_x acting on $\mathscr{H}$ such that $U_x U_y = U_{x+y}$ for all $x, y \in \mathscr{L}$. Assume further that for each $x \in \mathscr{L}$ the map $R \ni \lambda \to U_{\lambda x}$ is weakly continuous at 0, i.e.

$$\lim_{\lambda \to 0} (U_{\lambda x} u, v)_{\mathscr{H}} = (u, v)_{\mathscr{H}}, \quad u, v \in \mathscr{H}. \tag{2.2}$$

We recall that condition (2.2) is equivalent to the fact that for each $x \in \mathscr{L}$ the map $\boldsymbol{R} \ni \lambda \to U_{\lambda x}$ is continuous with respect to the strong operator topology.

Using Theorem 2.1 we can prove an analogue of Stone's integral formula for unitary representations of locally compact abelian groups.

THEOREM 2.3. *Let $\mathscr{L}$ be a linear space over $\boldsymbol{R}$ and $\{U_x;\ x \in \mathscr{L}\}$ be a unitary representation of the group $(\mathscr{L}, +)$ on the Hilbert space $\mathscr{H}$. Assume that $\{U_x;\ x \in \mathscr{L}\}$ has property (2.2). Then there exists a unique resolution of identity $\{E(B):\ B \in \mathfrak{B}(\tilde{\mathscr{L}})\}$ such that*

(i) *for arbitrary $u, v \in \mathscr{H}$ the measure $\mu_{u,v}(\cdot) := \big(E(\cdot)u, v\big)_{\mathscr{H}}$ is a regular measure on $\mathfrak{B}(\tilde{\mathscr{L}})$;*

(ii) *for each $x \in \mathscr{L}$ and all $u, v \in \mathscr{H}$ we have*

$$|\mu_{u,v}|(\{\varphi \in \tilde{\mathscr{L}}:\ \langle \varphi, x\rangle = \infty\}) = 0 \tag{2.3}$$

and

$$(U_x u, v)_{\mathscr{H}} = \int_{\tilde{\mathscr{L}}} e^{i\langle \varphi, x\rangle} \big(E(d\varphi)u, v\big)_{\mathscr{H}};$$

(iii) *a bounded linear operator on $\mathscr{H}$ commutes with every U_x, $x \in \mathscr{L}$, if and only if it commutes with every $E(B)$, $B \in \mathfrak{B}(\tilde{\mathscr{L}})$.*

Proof. Since the proof is similar to that of Theorem 2.2 in [11], we will sketch it only. The assumptions of the theorem, Theorem 2.1 and the polarization identity yield the existence and uniqueness of a $\boldsymbol{C}$-valued bounded regular measure $\mu_{u,v}$ on $\mathfrak{B}(\tilde{\mathscr{L}})$ such that for all $x \in \mathscr{L}$ the equality (2.3) holds and

$$(U_x u, v)_{\mathscr{H}} = \int_{\tilde{\mathscr{L}}} e^{i\langle \varphi, x\rangle} \mu_{u,v}(d\varphi), \qquad u, v \in \mathscr{H}.$$

Moreover, the measure $\mu_{u,u}$ is non-negative.

From the uniqueness of the measure $\mu_{u,v}$ it can be derived that for each $B \in \mathfrak{B}(\tilde{\mathscr{L}})$ the map $(u, v) \in \mathscr{H} \times \mathscr{H} \to \mu_{u,v}(B)$ is a semi-inner product on $\mathscr{H}$. Since the sesquilinear functional $\mu_{\cdot,\cdot}(B)$ is bounded, there exists a bounded non-negative self-adjoint operator $E(B)$ such that $\mu_{u,v}(B) = \big(E(B)u, v\big)_{\mathscr{H}}$, $u, v \in \mathscr{H}$, $B \in \mathfrak{B}(\tilde{\mathscr{L}})$. From the relations

$$\int_{\tilde{\mathscr{L}}} e^{i\langle \varphi, x\rangle} e^{i\langle \varphi, y\rangle} \big(E(d\varphi)u, v\big)_{\mathscr{H}} = (U_{x+y}u, v)_{\mathscr{H}} = \int_{\tilde{\mathscr{L}}} e^{i\langle \varphi, x\rangle} \big(E(d\varphi)U_y u, v\big)_{\mathscr{H}}$$

and from Corollary 2.2 it follows that

$$\int_{\tilde{\mathscr{L}}} e^{i\langle \varphi, y\rangle} \big(E((d\varphi) \cap B)u, v\big)_{\mathscr{H}} = \int_{B} e^{i\langle \varphi, y\rangle} \big(E(d\varphi)u, v\big)_{\mathscr{H}}$$

$$= \big(E(B)U_y u, v\big)_{\mathscr{H}}, \qquad x, y \in \mathscr{L},\ u, v \in \mathscr{H},\ B \in \mathfrak{B}(\tilde{\mathscr{L}}).$$

On the other hand, we have

$$\big(E(B)U_y u, v\big)_{\mathscr{H}} = \big(U_y u, E(B)v\big)_{\mathscr{H}} = \int_{\tilde{\mathscr{L}}} e^{i\langle \varphi, y\rangle} \big(E(d\varphi)u, E(B)v\big)_{\mathscr{H}}$$

and using Corollary 2.2 again we obtain

$$\big(E\big((\cdot)\cap B\big)u,\, v\big)_{\mathscr{H}} = \big(E(B)\,E(\cdot)\,u,\, v\big)_{\mathscr{H}}.$$

Hence $\big(E(B\cap B)u,\, v\big)_{\mathscr{H}} = \big(E(B)^2 u,\, v\big)_{\mathscr{H}}$, $u, v\in\mathscr{H}$, $B\in\mathfrak{B}(\tilde{\mathscr{L}})$. Thus $E(B) = E(B)^2$ is a projection operator and, consequently, $\{E(B)\colon B\in\mathfrak{B}(\tilde{\mathscr{L}})\}$ a resolution of identity. Finally, for a $T\in\mathscr{B}(\mathscr{H}, \mathscr{H})$ we have $(U_x Tu, v)_{\mathscr{H}} = (TU_x u, v)_{\mathscr{H}}$ if and only if

$$\int_{\tilde{\mathscr{L}}} e^{i\langle\varphi,x\rangle}\big(E(d\varphi)\,Tu,\, v\big)_{\mathscr{H}} = \int_{\tilde{\mathscr{L}}} e^{i\langle\varphi,x\rangle}\big(E(d\varphi)\,u,\, T^*v\big)_{\mathscr{H}}, \qquad x\in\mathscr{L},\ u,\ v\in\mathscr{H}.$$

This means that T commutes with every $E(B)$, $B\in\mathfrak{B}(\tilde{\mathscr{L}})$, by Corollary 2.2. The converse implication is obvious.

3. BANACH-SPACE-VALUED AND HILBERT–SCHMIDT OPERATOR-VALUED STATIONARY RANDOM PROCESSES ON LINEAR SPACES

Let $\mathscr{L}$ be a linear space over $\boldsymbol{R}$, $\mathscr{H}$ a Hilbert space, $\mathscr{F}$ a Banach space over $\boldsymbol{C}$ and $\mathscr{F}^*$ the Banach space of bounded linear functionals on $\mathscr{F}$.

Definition 3.1. A *Banach-space-valued stationary random process* $\boldsymbol{X}$ on $\mathscr{L}$ is a map $\boldsymbol{X}\colon \mathscr{L}\to\mathscr{B}(\mathscr{F}, \mathscr{H})$ such that the $\mathscr{B}(\mathscr{F}, \mathscr{F}^*)$-valued function $\tilde{\boldsymbol{K}}$ defined by $\tilde{\boldsymbol{K}}(x, y) := \boldsymbol{X}(y)^*\boldsymbol{X}(x)$ depends only on the difference $x-y$, i.e.

$$\tilde{\boldsymbol{K}}(x, y) = \tilde{\boldsymbol{K}}(x-y, 0) =: \boldsymbol{K}(x-y), \qquad x,\ y\in\mathscr{L}.$$

The function $\boldsymbol{K}$ is called the *covariance function* of the process $\boldsymbol{X}$. For some of its properties, in particular its positive definiteness, compare [12], Sections 1.1 and 1.2, and [1], Section 5.

For a subset $\mathscr{S}$ of $\mathscr{L}$ set $\boldsymbol{X}(\mathscr{S}) := \overline{\operatorname{lin}}_{\mathscr{H}}\{\boldsymbol{X}(x)\,a\colon x\in\mathscr{S}, a\in\mathscr{F}\}$. Particularly, the set $\boldsymbol{X}(\mathscr{L})$ is the subspace of $\mathscr{H}$ spanned by the values of $\boldsymbol{X}$. For $x\in\mathscr{L}$ define

$$U_x\boldsymbol{X}(y)\,a := \boldsymbol{X}(x+y)\,a, \qquad y\in\mathscr{L},\ a\in\mathscr{F}.$$

The map U_x can be extended to a unitary operator on $\boldsymbol{X}(\mathscr{L})$, which is also denoted by U_x. It is not difficult to see that the set $\{U_x\colon x\in\mathscr{L})$ is a unitary representation of $(\mathscr{L}, +)$ on $\boldsymbol{X}(\mathscr{L})$ with the property

$$\boldsymbol{X}(x) = U_x\boldsymbol{X}(0), \qquad x\in\mathscr{L}. \tag{3.1}$$

Definition 3.2. A Banach-space-valued stationary random process $\boldsymbol{X}$ on $\mathscr{L}$ is called *continuous* if for each $x\in\mathscr{L}$ the map $\boldsymbol{R}\ni\lambda\to\boldsymbol{X}(\lambda x)$ is weakly continuous at 0, i.e.

$$\lim_{\lambda\to 0}\big(\boldsymbol{X}(\lambda x)\,a,\, u\big)_{\mathscr{H}} = \big(\boldsymbol{X}(0)\,a,\, u\big)_{\mathscr{H}}, \qquad a\in\mathscr{F},\ u\in\mathscr{H}. \tag{3.2}$$

We recall that condition (3.2) is equivalent to several other continuity conditions. In what follows we only need the property that (3.2) is equivalent to the fact:

(3.3) $$\boldsymbol{R} \ni \lambda \to U_{\lambda x} \text{ is weakly continuous at } 0.$$

Now we derive spectral representations for continuous processes and their covariance functions from Theorem 2.3.

THEOREM 3.3. *Let X be a Banach-space-valued stationary random process on $\mathscr{L}$ and $\boldsymbol{K}$ its covariance function. Then there exists a unique $\mathscr{B}_s(\mathscr{F}, \mathscr{H})$-valued measure Z and a unique $B_s(\mathscr{F}, \mathscr{F}^*)$-valued measure F on $\mathfrak{B}(\tilde{\mathscr{L}})$ such that*

(3.4) $$(X(x)a, u)_{\mathscr{H}} = \int_{\tilde{\mathscr{L}}} e^{i\langle\varphi, x\rangle}(Z(d\varphi)a, u)_{\mathscr{H}}, \quad x \in \mathscr{L},\ a \in \mathscr{F},\ u \in \mathscr{H},$$

and

(3.5) $$\langle \boldsymbol{K}(x)a, b\rangle = \int_{\tilde{\mathscr{L}}} e^{i\langle\varphi, x\rangle}\langle F(d\varphi)a, b\rangle, \quad x \in \mathscr{L},\ a, b \in \mathscr{F}.$$

Moreover, for each $a \in \mathscr{F}$ the measure $\langle F(\cdot)a, a\rangle$ is non-negative.

Proof. Using (3.1) and (3.3) we conclude from Theorem 2.3 that there exist regular measures $\mu_{u,v}(\cdot) := (E(\cdot)u, v)_{\mathscr{H}}$ on $\mathfrak{B}(\tilde{\mathscr{L}})$. We set $Z(\cdot) := E(\cdot)X(0)$ and $F(\cdot) := X(0)^* E(\cdot)X(0)$ and see that (3.4) and (3.5) are satisfied. The uniqueness follows from Corollary 2.2.

The measures Z and F are called the *random* and the *spectral measures* of the process X, respectively.

For Banach-space-valued stationary random processes on locally compact abelian groups there have been proved several general prediction results (cf. [12] and [1]). Some of these results, for example the existence of the Wold decomposition of a process into its regular and singular parts, only depend on the fact that the underlying parameter set is an abelian group. Since $(\mathscr{L}, +)$ is also an abelian group, such results hold also for processes on $\mathscr{L}$; we omit the details.

In the following sections we will discuss special prediction problems for the class **CQS** of so-called continuous Hilbert–Schmidt operator-valued stationary random processes on $\mathscr{L}$.

DEFINITION 3.4. A *continuous Hilbert–Schmidt operator-valued stationary random process* X on $\mathscr{L}$ is a continuous Banach-space-valued stationary random process on $\mathscr{L}$ such that $\mathscr{F} = \mathscr{K}$ is a separable non-trivial Hilbert space and the values of X belong to $\mathscr{Q}(\mathscr{K}, \mathscr{H})$. The class of such processes is denoted by **CQS**.

Define τ by $\tau(B) := \operatorname{tr} F(B)$, $B \in \mathfrak{B}(\tilde{\mathscr{L}})$, and set $G := dF/d\tau$.

THEOREM 3.5. *The spectral measure F of a process X of the class* **CQS** *is a $\mathcal{T}^{\geqslant}$-valued measure. Further, for the $\mathcal{T}$-valued function K we have*

$$(3.6)\qquad (K(x)a, b)_{\mathscr{K}} = \int_{\mathscr{L}} e^{i\langle\varphi,x\rangle}(F(d\varphi)a, b)_{\mathscr{K}}$$
$$= \int_{\mathscr{L}} e^{i\langle\varphi,x\rangle}(G(\varphi)a, b)_{\mathscr{K}}\,\tau(d\varphi), \qquad x\in\mathscr{L},\ a, b\in\mathscr{K}.$$

Proof. The result follows immediately from Theorem 3.3 and Lemma 1.2.

From Theorem 3.5 it follows that for a spectral measure F of a process X of the class **CQS** one can construct the Hilbert space $L^2(F)$ as was done at the end of Section 1. It is called the *spectral space* of the process X. The method of attacking linear prediction problems for a process $X\in$ **CQS** is based on Kolmogorov's isomorphism theorem which enables us to study prediction problems for X as approximation problems in $L^2(F)$.

4. KOLMOGOROV'S ISOMORPHISM THEOREM

Let q be the dimension of the separable Hilbert space $\mathscr{K}$. Let $\mathscr{H}^q$ be the Hilbert space of all sequences $(u_j)_{j\in N_q} =:(u_j)$, $u_j\in\mathscr{H}$, with the property $\sum_{j=1}^{q}\|u_j\|_{\mathscr{H}}^2 < \infty$. We recall that the inner product in $\mathscr{H}^q$ is defined by

$$((u_j), (v_j))_{\mathscr{H}^q} := \sum_{j=1}^{q} (u_j, v_j)_{\mathscr{H}}, \qquad (u_j), (v_j)\in\mathscr{H}^q.$$

Let $\{e_j\}_{j\in N_q}$ be an orthonormal basis in $\mathscr{K}$. Then for $T\in\mathscr{Q}(\mathscr{K}, \mathscr{H})$ the sequence (Te_j) belongs to $\mathscr{H}^q$ because we have

$$\sum_{j=1}^{q} \|Te_j\|_{\mathscr{H}}^2 = \|T\|_{\mathscr{Q}(\mathscr{K},\mathscr{H})}^2 < \infty.$$

Let X be a process of the class **CQS** and F its spectral measure. For a subset $\mathscr{S}$ of $\mathscr{L}$ let $\mathscr{O}_{\mathscr{S}}(X)$ be the right operator linear hull of the operators $X(x)$, $x\in\mathscr{S}$, i.e. $\mathscr{O}_{\mathscr{S}}(X)$ is the set of operators of the form

$$\sum_{k=1}^{n} X(x_k)\,T_k, \qquad x_k\in\mathscr{S},\ T_k\in\mathscr{B},\ k = 1, 2, \ldots, n,\ n\in N.$$

Note that for an arbitrary $u\in\mathscr{K}$, $u \neq 0$, the set $\overline{\mathrm{lin}}_{\mathscr{H}}\{Tu\colon T\in\mathscr{O}_{\mathscr{S}}(X)\}$ coincides with $X(\mathscr{S})$. Set $X_q(\mathscr{S}) := \overline{\mathrm{lin}}_{\mathscr{H}^q}\{(Te_j)\colon T\in\mathscr{O}_{\mathscr{S}}(X)\}$.

LEMMA 4.1. *Let $X\in$* **CQS**. *Then for an arbitrary subset $\mathscr{S}$ of $\mathscr{L}$ the spaces $X(\mathscr{S})^q$ and $X_q(\mathscr{S})$ coincide.*

Proof. Since, for each $j\in N_q$, $\overline{\mathrm{lin}}_{\mathscr{H}}\{Te_j\colon T\in\mathscr{O}_{\mathscr{S}}(X)\}$ coincides with $X(\mathscr{S})$, we have $X_q(\mathscr{S}) \subseteq X(\mathscr{S})^q$. Now let (v_j) be an element of $X(\mathscr{S})^q$ that is orthogonal to $X_q(\mathscr{S})$. Then, in particular,

$$\sum_{j=1}^{q}(X(x)Te_j, v_j)_{\mathscr{H}}=0, \qquad x\in\mathscr{S},\ T\in\mathscr{B}. \tag{4.1}$$

Fix $j_0\in N_q$ and $x_0\in\mathscr{S}$ and choose the operator T such that $Te_j=0$ for $j\neq j_0$, and $Te_{j_0}=X(x_0)^*v_{j_0}$. Then (4.1) simplifies to

$$(X(x_0)^*v_{j_0}, X(x_0)^*v_{j_0})_{\mathscr{H}}=0,$$

and hence $X(x_0)^*v_{j_0}=0$. Since j_0 and x_0 may be arbitrarily chosen, it follows that for each $j\in N$, v_j is orthogonal to the range of all operators $X(x)$, $x\in\mathscr{S}$, and, consequently, $(v_j)_{j\in N_q}$ is orthogonal to $X(\mathscr{S})^q$. This proves the equality $X_q(\mathscr{S})=X(\mathscr{S})^q$.

Let $\mathcal{O}_{\mathscr{S}}$ be the operator linear hull of the functions $e^{-i\langle\cdot,x\rangle}I$, $x\in\mathscr{S}$, i.e. $\mathcal{O}_{\mathscr{S}}$ is the set of functions of the form

$$\sum_{k=1}^{n}\exp(-i\langle\cdot, x_k\rangle)T_k, \qquad x_k\in\mathscr{S},\ T_k\in\mathscr{B},\ k=1,2,\ldots,n,\ n\in N.$$

The closure of $\mathcal{O}_{\mathscr{S}}$ in $L^2(F)$ is denoted by $\sigma_{\mathscr{S}}(F)$. Let

$$T:=\sum_{k=1}^{n}X(x_k)T_k, \qquad x_k\in\mathscr{L},\ T_k\in\mathscr{B},\ k=1,2,\ldots,n,\ n\in N, \tag{4.2}$$

be an arbitrary element of $\mathcal{O}_{\mathscr{L}}(X)$. We set

$$V((Te_j)):=\sum_{k=1}^{n}\exp(-i\langle\cdot, x_k\rangle)T_k^*\in\mathcal{O}_{\mathscr{L}}. \tag{4.3}$$

THEOREM 4.2. *Let $X\in$ **CQS** and F be the spectral measure of X. There exists an antilinear isometric operator V from $X_q(\mathscr{L})$ onto $L^2(F)$ such that* (4.3) *holds. Moreover, for an arbitrary subset $\mathscr{S}$ of $\mathscr{L}$, the image of $X_q(\mathscr{S})$ under the map V coincides with $\sigma_{\mathscr{S}}(F)$.*

Proof. Let $T\in\mathcal{O}_{\mathscr{L}}(X)$ be of the form (4.2). Then by Theorem 3.5 we have

$$((Te_j)_{j\in N_q}, (Te_j)_{j\in N_q})_{\mathscr{H}^q}=\sum_{j=1}^{q}\Big(\sum_{k,l=1}^{n}T_l^*X(x_l)^*X(x_k)T_ke_j, e_j\Big)_{\mathscr{H}}$$

$$=\operatorname{tr}\int_{\mathscr{L}}\sum_{k,l=1}^{n}T_l^*\exp(i\langle\varphi, x_k-x_l\rangle)G(\varphi)T_k\,\tau(d\varphi)$$

$$=\operatorname{tr}\int_{\mathscr{L}}\sum_{l=1}^{n}\exp(-i\langle\varphi, x_l\rangle)T_l^*G(\varphi)^{1/2}\Big(\sum_{k=1}^{n}\exp(-i\langle\varphi, x_k\rangle)T_k^*G(\varphi)^{1/2}\Big)^*\tau(d\varphi)$$

$$=\Big(\sum_{k=1}^{n}\exp(-i\langle\varphi, x_k\rangle)T_k^*, \sum_{k=1}^{n}\exp(-i\langle\varphi, x_k\rangle)T_k^*\Big)_{L^2(F)}$$

$$=(V(Te_j), V(Te_j))_{L^2(F)}.$$

Hence the map V defined by (4.3) is isometric and can be extended to an isometric operator from $X_q(\mathscr{L})$ into $L^2(F)$. This extension will also be denoted by V. The antilinearity of V is clear and the relation $VX_q(\mathscr{S}) = \sigma_{\mathscr{S}}(F)$ is an immediate consequence of (4.3). It remains to show that the range of V coincides with $L^2(F)$. Let Ψ be an element of $L^2(F)$ orthogonal to $\sigma_{\mathscr{L}}(F)$. Choose a countable dense subset $\mathscr{D}$ of $\mathscr{Q}$ (cf. Lemma 1.1). We have

$$\operatorname{tr} \int_{\tilde{\mathscr{L}}} \exp(-i\langle\varphi, x\rangle) DG(\varphi)^{1/2} (\Psi(\varphi) G(\varphi)^{1/2})^* \tau(d\varphi) = 0, \qquad x \in \mathscr{L},\ D \in \mathscr{D}.$$

From Corollary 2.2 it follows that

$$\operatorname{tr}\big(DG(\varphi)^{1/2} (\Psi(\varphi) G(\varphi)^{1/2})^*\big) = 0 \tag{4.4}$$

for τ-a.a. $\varphi \in \tilde{\mathscr{L}}$ and all $D \in \mathscr{D}$. Fix $\varphi \in \tilde{\mathscr{L}}$ and choose $\{D_n\}_{n\in N} \subseteq \mathscr{D}$ which tends to $\Psi(\varphi) G(\varphi)$ in $\mathscr{Q}$ if n tends to ∞. From (4.4) we infer that

$$\operatorname{tr}\big(\Psi(\varphi) G(\varphi) (\Psi(\varphi) G(\varphi))^*\big) = 0,$$

and hence $\Psi(\varphi) G(\varphi) = 0$. This means that the kernel of $\Psi(\varphi) G(\varphi)^{1/2}$ contains the range, and hence the closure of the range of $G(\varphi)^{1/2}$. Since $\mathscr{K}$ is the orthogonal sum of the kernel of $G(\varphi)^{1/2}$ and the closure of the range of $G(\varphi)^{1/2}$, it follows that $\Psi(\varphi) G(\varphi)^{1/2} = 0$. Since φ was arbitrary, we obtain $\Psi = 0$ in $L^2(F)$, which proves the assertion.

5. SOME PREDICTION PROBLEMS AND THE SPACE $L^2(M)$

Let the linear space $\mathscr{L}$ be a direct sum of its subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$. Identifying an arbitrary $x \in \mathscr{L}$ of the form $x = x_1 + x_2$, $x_1 \in \mathscr{L}_1$, $x_2 \in \mathscr{L}_2$, with the ordered pair (x_1, x_2) we can and will consider $\mathscr{L}$ as the Cartesian product $\mathscr{L}_1 \times \mathscr{L}_2$. Let $\tilde{\mathscr{L}}^{(0)}$ be defined as

$$\tilde{\mathscr{L}}^{(0)} := \{\varphi \in \tilde{\mathscr{L}}\colon \langle\varphi, (x_1, 0)\rangle \neq \infty,\ x_1 \in \mathscr{L}_1\}.$$

For $\varphi \in \tilde{\mathscr{L}}^{(0)}$ define

$$\langle\varphi_1, x_1\rangle := \langle\varphi, (x_1, 0)\rangle, \qquad x_1 \in \mathscr{L}_1,$$
$$\langle\varphi_2, x_2\rangle := \langle\varphi, (0, x_2)\rangle, \qquad x_2 \in \mathscr{L}_2.$$

Then φ_2 is a quasi-linear functional on $\mathscr{L}_2$, i.e. belongs to $\tilde{\mathscr{L}}_2$, and φ_1 belongs to the linear space $\mathscr{L}_1'$ of linear functionals on $\mathscr{L}_1$. Moreover, we have

$$\langle\varphi, (x_1, x_2)\rangle = \langle\varphi_1, x_1\rangle + \langle\varphi_2, x_2\rangle, \qquad (x_1, x_2) \in \mathscr{L}. \tag{5.1}$$

Here we make the convention $\lambda + \infty = \infty$ for $\lambda \in \boldsymbol{R}$. Conversely, if $\varphi_1 \in \mathscr{L}_1'$ and $\varphi_2 \in \tilde{\mathscr{L}}_2$, we can define a quasi-linear functional $\varphi \in \tilde{\mathscr{L}}^{(0)}$ by (5.1). Thus, we can identify each $\varphi \in \tilde{\mathscr{L}}^{(0)}$ with the ordered pair (φ_1, φ_2) and the set $\tilde{\mathscr{L}}^{(0)}$ can be considered as the Cartesian product $\mathscr{L}_1' \times \tilde{\mathscr{L}}_2$.

From now on, we will assume that the space $\mathscr{L}_1$ has finite dimension d, i.e. $\mathscr{L}_1$ and $\mathscr{L}_1'$ can be indentified with the Euclidean space $\mathbf{R}^d$.

Let X be a process of class **CQS**. We deal with the following linear prediction problems:

(A) Let $\mathscr{S}_1$ be a subset of $\mathscr{L}_1$ and set $\mathscr{S} := \mathscr{S}_1 \times \mathscr{L}_2$. We ask the following question: when does the space $X(\mathscr{S})$ coincide with $X(\mathscr{L})$? If both spaces $X(\mathscr{S})$ and $X(\mathscr{L})$ are different we want to calculate the orthogonal projection of an element $X(x)u$, $x \in \mathscr{L}$, $u \in \mathscr{H}$, onto $X(\mathscr{S})$.

(B) Let $\mathscr{I}_1$ be an $(\mathscr{L}_1, +)$-invariant system of subsets of $\mathscr{L}_1$, i.e. if $x_1 \in \mathscr{L}_1$ and $\mathscr{S}_1 \in \mathscr{I}_1$, then the set $x_1 + \mathscr{S}_1$ belongs to $\mathscr{I}_1$. Let $\mathscr{I}$ be the system

$$\mathscr{I} := \mathscr{I}_1 \times \mathscr{L}_2 = \{\mathscr{S}_1 \times \mathscr{L}_2 \colon \mathscr{S}_1 \in \mathscr{I}_1\}.$$

The process X is called *$\mathscr{I}$-regular* if

$$\bigcap_{\mathscr{S} \in \mathscr{I}} X(\mathscr{S}) = 0.$$

It is called *$\mathscr{I}$-singular* if $X(\mathscr{S}) = X(\mathscr{L})$ for all $\mathscr{S} \in \mathscr{I}$. How to decide whether a given process X is $\mathscr{I}$-regular or $\mathscr{I}$-singular?

Let F be the spectral measure of X and $\tau := \operatorname{tr} F$. From the proofs of Theorems 3.3 and 2.3 it follows that $\tau(\{\varphi \in \tilde{\mathscr{L}} \colon \langle \varphi, x \rangle = \infty\}) = 0$ for each $x \in \mathscr{L}$. Since the space $\mathscr{L}_1$ is assumed to be finite dimensional, we obtain

$$\tau(\{\varphi \in \tilde{\mathscr{L}} \colon \langle \varphi, (x_1, 0) \rangle = \infty,\ x_1 \in \mathscr{L}\}) = 0,$$

and hence $\tau(\tilde{\mathscr{L}} \setminus \tilde{\mathscr{L}}^{(0)}) = 0$. Thus we can restrict our considerations to $\tilde{\mathscr{L}}^{(0)}$ without changing the spectral space $L^2(F)$ of X essentially.

Denote by $\mathfrak{B}^{(0)}$ the product of the σ-algebras $\mathfrak{B}(\mathscr{L}')$ and $\mathfrak{B}(\tilde{\mathscr{L}}_2)$:

$$\mathfrak{B}^{(0)} := \mathfrak{B}(\mathscr{L}') \otimes \mathfrak{B}(\tilde{\mathscr{L}}_2).$$

We have $\mathfrak{B}^{(0)} \subseteq \mathfrak{B}(\tilde{\mathscr{L}}^{(0)})$ and the inclusion can be proper in general. Let M be the restriction of F to $\mathfrak{B}^{(0)}$. Set $\mu := \operatorname{tr} M$ and $W := dM/d\mu$. Since for fixed $x \in \mathscr{L}$ the function

$$\tilde{\mathscr{L}}^{(0)} \ni \varphi \to \exp(i\langle \varphi, x \rangle) = \exp(i\langle \varphi_1, x_1 \rangle) \exp(i\langle \varphi_2, x_2 \rangle)$$

is $(\mathfrak{B}^{(0)}, \mathfrak{B}(\mathbf{R}))$-measurable, we obtain an isomorphism theorem analogous to Theorem 4.2 with $L^2(F)$ replaced by $L^2(M)$. Moreover, studying the behaviour of $X(\mathscr{S})$ as a subspace of $X(\mathscr{L})$ is equivalent to studying the behaviour of $X(\mathscr{S})^q$ as a subspace of $X(\mathscr{L})^q$. Thus, taking into consideration the results of Section 4 the prediction problems (A) and (B) can be formulated in an equivalent way as follows:

(A′) When does the space $\sigma_{\mathscr{S}}(M)$ coincide with $L^2(M)$ and how to calculate the orthogonal projection of a function $e^{-i\langle \cdot, x \rangle} I$, $x \in \mathscr{L}$, onto $\sigma_{\mathscr{S}}(M)$ if $\sigma_{\mathscr{S}}(M)$ does not coincide with $L^2(M)$?

(B′) How to decide whether $\bigcap_{\mathscr{S}\in\mathscr{J}}\sigma_{\mathscr{S}}(M)=0$ or $\sigma_{\mathscr{S}}(M)=L^2(M)$ for all $\mathscr{S}\in\mathscr{J}$?

In order to obtain some partial answers to the raised questions we write the space $L^2(M)$ as a direct integral. For the theory of direct integrals we refer to [3]. A short account of all results necessary for our purposes can be found in [5], where the direct integral method was used in the study of analogous problems on locally compact abelian groups.

6. DIRECT INTEGRAL REPRESENTATION OF $L^2(M)$ AND SOME PREDICTION RESULTS

We use the notation of the previous section. Let μ_1 and μ_2 be the first and the second marginal measures of μ, respectively, i.e.

$$\mu_1(B_1):=\mu(B_1\times\tilde{\mathscr{L}}_2),\qquad B_1\in\mathfrak{B}(\mathscr{L}'_1),$$

and

$$\mu_2(B_2):=\mu(\mathscr{L}'_1\times B_2),\qquad B_2\in\mathfrak{B}(\tilde{\mathscr{L}}_2).$$

LEMMA 6.1. *The measures μ_1 and μ_2 are regular measures on $\mathfrak{B}(\mathscr{L}'_1)$ and $\mathfrak{B}(\tilde{\mathscr{L}}_2)$, respectively.*

Proof. The regularity of μ_1 is clear since $\mathscr{L}'_1$ is a finite-dimensional space. To verify the regularity of μ_2 it is enough to show that for each $\varepsilon>0$ and each $B_2\in\mathfrak{B}(\tilde{\mathscr{L}}_2)$ there exists a compact set $K_2\subseteq B_2$ such that $\mu_2(B_2\backslash K_2)<\varepsilon$. But since the measure $\tau=\mathrm{tr}F$ is regular on $\mathfrak{B}(\tilde{\mathscr{L}})$, there exists a compact set $K\in\mathfrak{B}(\tilde{\mathscr{L}})$ such that $K\subseteq\mathscr{L}'_1\times B_2$ and $\tau\big((\mathscr{L}'_1\times B_2)\backslash K\big)<\varepsilon$ (cf. [2], Proposition 7.2.6). Since K is a subset of $\mathscr{L}'_1\times\tilde{\mathscr{L}}_2=\tilde{\mathscr{L}}^{(0)}$, it even belongs to $\mathfrak{B}(\tilde{\mathscr{L}}^{(0)})$. Let π_2 be the projection operator in $\tilde{\mathscr{L}}^{(0)}$ onto $\tilde{\mathscr{L}}_2$. From its continuity it follows that $K_2=\pi_2(K)$ is a compact subset of $\tilde{\mathscr{L}}_2$. Moreover, K_2 is a subset of B_2 and

$$\begin{aligned}\mu_2(B_2\backslash K_2)&=\mu\big(\mathscr{L}'_1\times(B_2\backslash K_2)\big)\\&=\mu\big((\mathscr{L}'_1\times B_2)\backslash(\mathscr{L}'_1\times K_2)\big)=\mu\big((\mathscr{L}'_1\times B_2)\backslash K\big)<\varepsilon.\end{aligned}$$

LEMMA 6.2. *There exists a function*

$$w\colon\mathfrak{B}(\mathscr{L}'_1)\times\tilde{\mathscr{L}}_2\ni(B_1,\varphi_2)\to w(B_1,\varphi_2)\in[0,\mu(\tilde{\mathscr{L}}^{(0)})]$$

with the following properties:

(i) *For each $B_1\in\mathfrak{B}(\mathscr{L}'_1)$ the function $\tilde{\mathscr{L}}_2\ni\varphi_2\to w(B_1,\varphi_2)$ is $\big(\mathfrak{B}(\tilde{\mathscr{L}}_2),\mathfrak{B}(R)\big)$-measurable.*

(ii) *For each $\varphi_2\in\tilde{\mathscr{L}}_2$ the function $\mathfrak{B}(\mathscr{L}'_1)\ni B_1\to w(B_1,\varphi_2)$ is a non-negative regular finite measure on $\mathfrak{B}(\mathscr{L}'_1)$.*

(iii) *For arbitrary $B_1\in\mathfrak{B}(\mathscr{L}'_1)$ and $B_2\in\mathfrak{B}(\tilde{\mathscr{L}}_2)$ we have*

$$\mu(B_1\times B_2)=\int_{B_2}w(B_1,\varphi_2)\,\mu_2(d\varphi_2).$$

Proof. The result is a consequence of the theorem in Section 21.2 of [7].

For μ_2-a.a. $\varphi_2 \in \tilde{\mathscr{L}}_2$ the function $W(\cdot, \varphi_2)$ is Bochner-integrable with respect to $w(\cdot, \varphi_2)$. Let M_{φ_2} be a $\mathscr{T}^{\geqslant}$-valued measure defined as

$$M_{\varphi_2}(B_1) := \int_{B_1} W(\varphi_1, \varphi_2) w(d\varphi_1, \varphi_2), \qquad B_1 \in \mathfrak{B}(\mathscr{L}'_1).$$

We wish to show that the Hilbert space $L^2(M)$ can be identified with the direct integral

$$\int_{\tilde{\mathscr{L}}_2}^{\oplus} L^2(M_{\varphi_2}) \mu_2(d\varphi_2) =: \mathscr{G}.$$

In order to do this we have to show first that the direct integral $\mathscr{G}$ can be constructed. This demands the construction of a so-called fundamental system of vector fields (cf. [3], p. 164).

Let $\mathscr{D}$ be a countable dense subset of $\mathscr{Q}$ (cf. Lemma 1.1). Let $\mathscr{D}_1$ be a countable dense subset of $\mathscr{L}_1$. It exists because $\mathscr{L}_1$ can be regarded as a finite-dimensional Euclidean space $\boldsymbol{R}^d$. For $x_1 \in \mathscr{L}_1$ and $T \in \mathscr{B}$ let $\mathscr{C}_{x_1,T}$ be a constant vector field on $\tilde{\mathscr{L}}_2$ defined by

$$\mathscr{C}_{x_1,T}(\varphi_2) := \exp(-i\langle\cdot, x_1\rangle) T, \qquad \varphi_2 \in \tilde{\mathscr{L}}_2, \tag{6.1}$$

i.e. the values of $\mathscr{C}_{x_1,T}$ are the functions

$$\mathscr{L}'_1 \ni \varphi_1 \to \exp(-i\langle\varphi_1, x_1\rangle) T \tag{6.2}$$

independent of φ_2.

Let $\mathscr{D}_{\mathscr{G}}$ be a countable set defined by $\mathscr{D}_{\mathscr{G}} = \{\mathscr{C}_{x_1,D}\colon x_1 \in \mathscr{D}_1, D \in \mathscr{D}\}$. To verify that $\mathscr{D}_{\mathscr{G}}$ forms a fundamental system of vector fields we have only to show that the functions of type (6.2) form a total set in each space $L^2(M_{\phi_2})$, $\varphi_2 \in \tilde{\mathscr{L}}_2$, if x_1 and D run through $\mathscr{D}_1$ and $\mathscr{D}$, respectively. But this can be done as at the end of the proof of Theorem 4.2 if we additionally take into consideration that the functions $\mathscr{L}_1 \ni x_1 \to \exp(-i\langle\varphi_1, x_1\rangle)$, $\varphi_1 \in \mathscr{L}'_1$, are continuous on $\mathscr{L}_1$. Thus the direct integral $\mathscr{G}$ is well defined.

Now we introduce a map j by assigning to $\Phi \in L^2(M)$ the vector field

$$j\Phi := \big(\Phi(\cdot, \varphi_2)\big)_{\varphi_2 \in \tilde{\mathscr{L}}_2}.$$

THEOREM 6.3. *The map j is a linear isometry of $L^2(F)$ onto the direct integral $\mathscr{G}$.*

Proof. For $\Phi \in L^2(M)$, we have

$$\int_{\tilde{\mathscr{L}}^{(0)}} \operatorname{tr}\left[\Phi(\varphi) W(\varphi)^{1/2} (\Phi(\varphi) W(\varphi)^{1/2})^*\right] \mu(d\varphi)$$

$$= \int_{\tilde{\mathscr{L}}_2} \int_{\mathscr{L}'_1} \operatorname{tr}\left[\Phi(\varphi) W(\varphi)^{1/2} (\Phi(\varphi) W(\varphi)^{1/2})^*\right] w(d\varphi_1, \varphi_2) \mu_2(d\varphi_2)$$

$$= \int_{\tilde{\mathscr{L}}_2} \|\Phi(\cdot, \varphi_2)\|^2_{L^2(M_{\varphi_2})} \mu_2(d\varphi_2),$$

and since the linearity of j is obvious, the map j is a linear isometry of $L^2(F)$ into $\mathcal{G}$. Now assume that the vector field $(\Psi(\cdot,\cdot\varphi_2))_{\varphi_2\in\tilde{\mathcal{L}}_2}\in\mathcal{G}$ is orthogonal to the range of j. Then, in particular,

$$\int_{\mathcal{L}_2}\exp(-i\langle\varphi_2, x_2\rangle)\int_{\mathcal{L}'_1}\exp(-i\langle\varphi_1, x_1\rangle)DW(\varphi)^{1/2}(\Psi(\varphi)W(\varphi)^{1/2})^* \times w(d\varphi_1, \varphi_2)\mu_2(d\varphi_2) = 0$$

for all x_1 of a countable dense subset $\mathcal{D}_1$ of $\mathcal{L}_1$ and all D of a countable dense subset $\mathcal{D}$ of $\mathcal{Q}$. Since according to Lemma 6.1 the measure μ_2 is regular, Corollary 2.2 implies that for μ_2-a.a. $\varphi_2\in\tilde{\mathcal{L}}_2$

$$\int_{\mathcal{L}'_1}\exp(-i\langle\varphi_1, x_1\rangle)DW(\varphi_1, \varphi_2)^{1/2}(\Psi(\varphi_1, \varphi_2)W(\varphi_1, \varphi_2)^{1/2})^* w(d\varphi_1, \varphi_2) = 0$$

for all $x_1\in\mathcal{D}_1$ and $D\in\mathcal{D}$. But since the functions of type (6.2) form a total set in $L^2(M_{\varphi_2})$ if x_1 runs through $\mathcal{D}_1$ and T runs through $\mathcal{D}$, it follows that $\Psi(\cdot, \varphi_2) = 0$ in $L^2(M_{\varphi_2})$ for μ_2-a.a. $\varphi_2\in\tilde{\mathcal{L}}_2$. Hence $\Psi = 0$ in $\mathcal{G}$.

Theorem 6.3 means that we can identify the space $L^2(M)$ and the direct integral $\mathcal{G}$. Let $\mathcal{S}_1$ be a subset of $\mathcal{L}_1$ and $\mathcal{S} = \mathcal{S}_1\times\mathcal{L}_2$. Let P be the orthoprojector in $L^2(M)$ onto $\sigma_{\mathcal{S}}(M)$. Let $\mathcal{O}_{\mathcal{S}_1}$ be the operator linear hull of functions

$$\mathcal{L}'_1\ni\varphi_1\to\exp(-i\langle\varphi_1, x_1\rangle)I,\qquad x_1\in\mathcal{S}_1.$$

Let $\sigma_{\mathcal{S}_1}(M_{\varphi_2})$ be the closure of $\mathcal{O}_{\mathcal{S}_1}$ in $L^2(M_{\varphi_2})$. Let $P(\varphi_2)$ be the orthoprojector in $L^2(M_{\varphi_2})$ onto $\sigma_{\mathcal{S}_1}(M_{\varphi_2})$. The following theorem is basic for establishing our prediction results.

THEOREM 6.4. *The operator P can be identified with the direct integral of operators*

$$\tilde{P} := \int_{\tilde{\mathcal{L}}_2}^{\oplus} P(\varphi_2)\mu_2(d\varphi_2).$$

Proof. The proof is divided into several steps.

Step 1. *The direct integral $\tilde{P} := \int_{\tilde{\mathcal{L}}_2}^{\oplus} P(\varphi_2)\mu_2(d\varphi_2)$ can be defined.*

According to [3], Proposition II.1.9, it suffices to show that there exists a countable subset of elements of $\mathcal{G}$ such that for μ_2-a.a. $\varphi_2\in\tilde{\mathcal{L}}_2$ the values at φ_2 of the elements of this subset form a total subset of $\sigma_{\mathcal{S}_1}(M_{\varphi_2})$. Let $\mathcal{D}_1$ be a countable dense subset of $\mathcal{S}_1$ and $\mathcal{D}$ a countable dense subset of $\mathcal{B}_s$ (cf. Lemma 1.1). Let $\mathcal{D}_{\mathcal{S}_1} := \{\mathcal{C}_{x_1,D}\colon x_1\in\mathcal{D}_1, D\in\mathcal{D}\}$, where $\mathcal{C}_{x_1,D}$ is defined by (6.1). Fix $\varphi_2\in\tilde{\mathcal{L}}_2$. Assume that $\Psi\in\sigma_{\mathcal{S}_1}(M_{\varphi_2})$ is such that

$$\int_{\mathcal{L}'_1}\exp(-i\langle\varphi_1, x_1\rangle)DW(\varphi_1, \varphi_2)^{1/2}(\Psi(\varphi_1)W(\varphi_1, \varphi_2)^{1/2})^* \times w(d\varphi_1, \varphi_2) = 0 \tag{6.3}$$

for all $x_1 \in \mathscr{D}_1$ and $D \in \mathscr{D}$. By a continuity argument, (6.3) is true for all $x_1 \in \mathscr{S}_1$ and $D \in \mathscr{D}$. For an arbitrary $T \in \mathscr{B}_s$ let $\{D_n\}_{n \in N} \subseteq \mathscr{D}$ be a sequence converging to T in $\mathscr{B}_{\mathscr{S}}$ if n tends to ∞. Since the operator norms of $\{D_n\}_{n \in N}$ are uniformly bounded, we easily obtain

$$(6.4) \qquad \left|\operatorname{tr}\left(\exp(-i\langle\varphi_1, x_1\rangle) D_n W(\varphi_1, \varphi_2)^{1/2} (\Psi(\varphi_1) W(\varphi_1, \varphi_2)^{1/2})^*\right)\right| \leqslant c\, \|\Psi\|_{L^2(M_{\varphi_2})}$$

with a constant c independent of x_1 and n. Moreover, according to [6], p. 119, we have

$$\lim_{n \to \infty} \|D_n W(\varphi_1, \varphi_2)^{1/2} - T W(\varphi_1, \varphi_2)^{1/2}\|_{\mathscr{2}} = 0, \qquad \varphi_1 \in \mathscr{L}'_1,$$

and hence

$$(6.5) \qquad \lim_{n \to \infty} \left|\operatorname{tr}\left(\exp(-i\langle\varphi_1, x_1\rangle) D_n W(\varphi_1, \varphi_2)^{1/2} (\Psi(\varphi_1) W(\varphi_1, \varphi_2)^{1/2})^*\right) - \operatorname{tr}\left(\exp(-i\langle\varphi_1, x_1\rangle) T W(\varphi_1, \varphi_2)^{1/2} (\Psi(\varphi_1) W(\varphi_1, \varphi_2)^{1/2})^*\right)\right| = 0, \qquad \varphi_1 \in \mathscr{L}_1.$$

Relations (6.4) and (6.5) show that the Lebesgue dominated convergence theorem can be applied. Consequently, (6.3) is true for all $x_1 \in \mathscr{L}_1$ and $T \in \mathscr{B}$. This shows that $\mathscr{D}_{\mathscr{S}_1}$ is total in $\sigma_{\mathscr{S}_1}(M_{\varphi_2})$, $\varphi_2 \in \tilde{\mathscr{L}}_2$.

Step 2. *The operator* $\tilde{P} := \int_{\tilde{\mathscr{L}}_2}^{\oplus} P(\varphi_2)\, \mu_2(d\varphi_2)$ *is an orthoprojector.*

This is an immediate consequence of the fact that the operators $P(\varphi_2)$, $\varphi_2 \in \tilde{\mathscr{L}}_2$, are orthoprojectors and of Proposition II.2.3 in [3].

Step 3. *The ranges of* P *and* $\tilde{P}$ *coincide.*

Let $\Psi \in L^2(M)$ be orthogonal to $\mathscr{R}(\tilde{P})$. Then

$$(6.6) \qquad 0 = \int_{\tilde{\mathscr{L}}_2} \exp(-i\langle\varphi_2, x_2\rangle) \int_{\mathscr{L}'_1} \exp(-i\langle\varphi_1, x_1\rangle) T W(\varphi)^{1/2} (\Psi(\varphi) W(\varphi)^{1/2})^* \times w(d\varphi_1, \varphi_2)\, \mu_2(d\varphi_2)$$
$$= \int_{\mathscr{L}^{(0)}} \exp(-i\langle\varphi, x\rangle) T W(\varphi)^{1/2} (\Psi(\varphi) W(\varphi)^{1/2})^* \mu(d\varphi)$$

for all $(x_1, x_2) \in \mathscr{S}$ and $T \in \mathscr{B}$. Thus Ψ is orthogonal to $\sigma_{\mathscr{S}}(M) = \mathscr{R}(P)$, and hence $\mathscr{R}(P) \subseteq \mathscr{R}(\tilde{P})$.

Let $\Psi \in \mathscr{R}(\tilde{P})$ be orthogonal to $\mathscr{R}(P)$. Then (6.6), Lemma 6.1, and Corollary 2.2 imply that for μ_2-a.a. $\varphi_2 \in \tilde{\mathscr{L}}_2$ we have

$$\int_{\mathscr{L}'_1} \exp(-i\langle\varphi_1, x_1\rangle) D W(\varphi_1, \varphi_2)^{1/2} (\Psi(\varphi_1, \varphi_2) W(\varphi_1, \varphi_2)^{1/2})^* w(d\varphi_1, \varphi_2) = 0$$

for all x_1 of a countable dense subset $\mathscr{D}_1$ of $\mathscr{S}_1$ and all D of a countable dense subset $\mathscr{D}$ of $\mathscr{B}_s$, and hence for all elements of $\mathscr{D}_{\mathscr{S}_1}$. But since $\mathscr{D}_{\mathscr{S}_1}$ is total in

$\sigma_{\mathscr{S}_1}(M_{\varphi_2})$, it follows that $\Psi(\cdot, \varphi_2) = 0$ in $L^2(M_{\varphi_2})$ for μ_2-a.a. $\varphi_2 \in \tilde{\mathscr{L}}_2$. Thus $\Psi = 0$ in $L^2(M)$ and $\mathscr{R}(P) = \mathscr{R}(\tilde{P})$.

Now we use Theorem 6.4 to establish some prediction results. The results of Theorem 6.5 are immediately clear from Theorem 6.4.

THEOREM 6.5. *Let $\mathscr{S}_1$ be a subset of $\mathscr{L}_1$ and $\mathscr{S} = \mathscr{S}_1 \times \mathscr{L}_2$. Then $\sigma_{\mathscr{S}}(M) = L^2(M)$ if and only if $\sigma_{\mathscr{S}_1}(M_{\varphi_2}) = L^2(M_{\varphi_2})$ for μ_2-a.a. $\varphi_2 \in \tilde{\mathscr{L}}_2$. If $\sigma_{\mathscr{S}}(M) \neq L^2(M)$, then the orthogonal projection of $\Psi \in L^2(M)$ onto $\sigma_{\mathscr{S}}(M)$ is equal to*

$$\int_{\mathscr{L}_2}^{\oplus} P(\varphi_2)\Psi(\cdot, \varphi_2)\mu_2(d\varphi_2),$$

where $P(\varphi_2)$ is the orthogonal projector onto $\sigma_{\mathscr{S}_1}(M_{\varphi_2})$ in $L^2(M_{\varphi_2})$, $\varphi_2 \in \tilde{\mathscr{L}}_2$. The distance from Ψ to $\sigma_{\mathscr{S}}(M)$ is equal to

$$\Big(\int_{\mathscr{L}_2}^{\oplus} \|\Psi(\cdot, \varphi_2) - P(\varphi_2)\Psi(\cdot, \varphi_2)\|^2_{L^2(M_{\varphi_2})}\mu_2(d\varphi_2)\Big)^{1/2}.$$

To obtain criteria for $\mathscr{J}$-singularity or $\mathscr{J}$-regularity we have to impose some restrictions on the system $\mathscr{J}$.

We will say that the system $\mathscr{J}_1$ of subsets of $\mathscr{L}_1$ is *countably generated* if there exists a countable subset $\mathscr{J}'_1$ such that for each $\mathscr{S}_1 \in \mathscr{J}_1$ there exist an $\mathscr{S}'_1 \in \mathscr{J}'_1$ and $x_1 \in \mathscr{L}_1$ such that $\mathscr{S}_1 = x_1 + \mathscr{S}'_1$.

THEOREM 6.6. *Let $\mathscr{J}_1$ be a countably generated $(\mathscr{L}_1, +)$-invariant system of subsets of $\mathscr{L}_1$ and $\mathscr{J} := \{\mathscr{S}_1 \times \mathscr{L}_2 \colon \mathscr{S}_1 \in \mathscr{J}_1\}$. Let X be a process of class* **CQS** *and let M be the restriction of its spectral measure to $\mathfrak{B}^{(0)}$. Then X is $\mathscr{J}$-singular if and only if there exists a set $B_2 \in \mathfrak{B}(\tilde{\mathscr{L}}_2)$ such that $\mu_2(B_2) = 0$ and $\sigma_{\mathscr{S}_1}(M_{\varphi_2}) = L^2(M_{\varphi_2})$ for all $\mathscr{S}_1 \in \mathscr{J}_1$ and all $\varphi_2 \in \tilde{\mathscr{L}}_2 \backslash B_2$.*

Proof. If the system $\mathscr{J}$ consists of countably many sets, then from Theorem 6.4 it follows that X is $\mathscr{J}$-singular if and only if there exists a set $B_2 \in \mathfrak{B}(\tilde{\mathscr{L}}_2)$ such that $\mu_2(B_2) = 0$ and $\sigma_{\mathscr{S}_1}(M_{\varphi_2}) = L^2(M_{\varphi_2})$ for all $\mathscr{S}_1 \in \mathscr{J}_1$ and all $\varphi_2 \in \tilde{\mathscr{L}}_2 \backslash B_2$. Now note that if $\mathscr{S}_1 = x_1 + \mathscr{S}'_1$, $x_1 \in \mathscr{L}_1$, $\mathscr{S}_1 \in \mathscr{J}_1$, $\mathscr{S}'_1 \in \mathscr{J}'_1$, then $\sigma_{\mathscr{S}_1}(M_{\varphi_2}) = L^2(M_{\varphi_2})$ if and only if $\sigma_{\mathscr{S}'_1}(M_{\varphi_2}) = L^2(M_{\varphi_2})$, $\varphi_2 \in \tilde{\mathscr{L}}_2$.

THEOREM 6.7. *Assume that the system $\mathscr{J}_1$ contains a countable subsystem $\{\mathscr{S}_1^{(j)} \colon j \in N\}$ such that*

$$\bigcap_{\mathscr{S} \in \mathscr{J}} \sigma_{\mathscr{S}}(M) = \bigcap_{j=1}^{\infty} \sigma_{\mathscr{S}^{(j)}}(M),$$

where $\mathscr{S}^{(j)}$ denotes the set $\mathscr{S}^{(j)} := \mathscr{S}_1^{(j)} \times \mathscr{L}_2$, $j \in N$. Then under the assumptions of Theorem 6.6 the process X is $\mathscr{J}$-regular if and only if

$$\bigcap_{j=1}^{\infty} \sigma_{\mathscr{S}_1^{(j)}}(M_{\varphi_2}) = \{0\} \quad \text{for } \mu_2\text{-a.a. } \varphi_2 \in \tilde{\mathscr{L}}_2.$$

Proof. If P_1 and P_2 are orthoprojectors in a Hilbert space, then the orthogonal projector onto $\mathscr{R}(P_1) \cap \mathscr{R}(P_2)$ is the strong limit of the sequence of operators $\{(P_1 P_2)^n\}_{n\in N}$. Using Propositions II.2.3 and II.2.4 of [3] we infer by induction that the orthogonal projector in $L^2(M)$ onto $\bigcap_{j=1}^{\infty} \sigma_{\mathscr{S}^{(j)}}(M)$ can be written as the direct integral of operators $Q = \int_{\tilde{\mathscr{L}}_2}^{\oplus} Q(\varphi_2)\mu_2(d\varphi_2)$, where $Q(\varphi_2)$ is the orthogonal projector in $L^2(M_{\varphi_2})$ onto $\bigcap_{j=1}^{\infty} \sigma_{\mathscr{S}_1^{(j)}}(M_{\varphi_2})$. It follows easily that

$$\bigcap_{\mathscr{S}\in\mathscr{J}} \sigma_{\mathscr{S}}(M) = \{0\} = \bigcap_{j=1}^{\infty} \sigma_{\mathscr{S}^{(j)}}(M)$$

if and only if

$$\bigcap_{j=1}^{\infty} \sigma_{\mathscr{S}_1^{(j)}}(M_{\varphi_2}) = \{0\} \quad \text{for } \mu_2\text{-a.a. } \varphi_2 \in \tilde{\mathscr{L}}_2.$$

Remark 6.8. Using Theorems 6.6 and 6.7 and prediction results for stationary processes on $\boldsymbol{R}$ one can obtain more concrete prediction results for processes of the class **CQS** if the space $\mathscr{L}_1$ is one-dimensional. We omit the details and refer to Section 5 of [5] where analogous results were obtained for multivariate processes on groups.

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