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ASYMPTOTICS OF THE SUPREMUM OF SCALED BROWNIAN MOTION

BY

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Abstract. We consider the problem of estimating the tail of the distribution of the supremum of scaled Brownian motion $B(f(t))$ processes with linear drift.

Using the *local time* technique we obtain asymptotics and bounds of

$$P\left(\sup_{t\geqslant t_0}\left(B(f(t))-t\right) > u\right),$$

which are expressed in terms of the expected value of the *local time* of $B(f(t))-t$ processes at level u.

As an application we obtain upper bounds for the tail of distribution of the supremum for some Gaussian processes with stationary increments.

Key words and phrases: Brownian motion, exponential bound, fractional Brownian motion, Gaussian process, local time, scaled Brownian motion.

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1. Introduction. We consider the problem of estimating the tail of the distribution of the supremum for scaled Brownian motion $B(f(t))$ with linear drift:

$$P\left(\sup_{t\geqslant t_0}\left(B(f(t))-t\right) > u\right), \tag{1.1}$$

where $B(t)$ is a standard Brownian motion and $f(t)$ is the scale function.

It is well known that if $f(t) = t$, then $P\left(\sup_{t\geqslant 0}\left(B(t)-t\right) > u\right) = e^{-2u}$. Dębicki et al. [5] considered (1.1) for the scale function $f(t) = t^{2H}$, where $H \in (0, 1)$. Using *local time* technique they obtained the upper and lower bound of (1.1)

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and proved the asymptotic

(1.2)
$$\lim_{u\to\infty} \frac{P\left(\sup_{t\geqslant 0}(B(t^{2H})-t) > u\right)}{l_{t^{2H},0}(u)} = 1,$$

where $l_{t^{2H},0}(u) = E\mathscr{L}\left(u; \{B(s^{2H})-s;\ s \geqslant 0\}\right)$ is the expected value of *local time* of the process $B(t^{2H})-t$ at level u.

The aim of this paper is to generalize this result and to prove that the tail of the distribution of the supremum for scaled Brownian motion $B(f(t))$ with linear drift behaves as the corresponding expected value of *local time* for a much wider class of scale functions f.

The paper is organized as follows. In Section 2 we introduce the idea of *local time* for scaled Brownian motion and present Lemma 2.2, which is of crucial interest for the method of proof. In Section 3 we obtain upper and lower bounds of (1.1), which are expressed in terms of the expected value of local time of $B(f(t))-t$. Section 4 deals with the asymptotic of (1.1), which is given in Theorem 4.1. We give some applications of obtained results in Section 5, where the class of Gaussian processes with stationary increments is considered.

The distribution of (1.1) is related to the distribution of the supremum of Brownian motion with appropriate nonlinear drift. In particular, if $f(t)$ is strictly increasing and $f(0) = 0$, then

(1.3)
$$P\left(\sup_{t\geqslant 0}\left(B(f(t))-t\right) > u\right) = P\left(\sup_{t\geqslant 0}\left(B(t)-g(t)\right) > u\right),$$

where $g(t)$ is the inverse function to the function $f(t)$. In Section 6, using (1.3) and geometric approach, we find exponential bounds of (1.1).

2. Local times for SBM processes. In this section we introduce the concept of *local times* for *scaled Brownian motion* (SBM) with linear drift. Namely, we consider $\{B(f(t))-t;\ t \geqslant 0\}$, where $B(t)$ is a standard Brownian motion. We assume that the scale function $f(t)$ satisfies:

B1: $f(t) \in C^1([0, \infty))$ is a strictly increasing function of t.

B2: There exists $\varepsilon_0 > 0$ and t_0 such that, for all $t \geqslant t_0$, the function
$$\frac{x^{2-\varepsilon_0}}{f(t+x)-f(t)}$$
is an increasing function of $x \geqslant 0$.

Dębicki et al. [5] considered SBM processes with the scale function of the form $f(t) = t^{2H}$ for $H \in (0, 1)$. Note that this class of scale functions belongs to the family of functions satisfying **B1** and **B2**.

The following function will play the crucial role in the sequel:
$$l_{f,t_0}(u) = \frac{1}{\sqrt{2\pi}} \int_{t_0}^{\infty} \frac{1}{\sqrt{f(t)}} \exp\left(-\frac{(t+u)^2}{2f(t)}\right) dt.$$

Let

$$\mathscr{L}(u;\{X(t);\, t \geqslant t_0\}) = \mathscr{L}_{t_0}(u) = \lim_{\delta\to 0} \frac{1}{2\delta} \int_{t_0}^{\infty} \mathbf{1}\,[X(t) \in (u-\delta,\, u+\delta)]\, dt$$

be the *local time* of the process $\{X(t);\, t \geqslant t_0\}$ at level u, where the limit exists almost surely. The existence of $\mathscr{L}(u;\{X(t);\, t \geqslant t_0\})$ for $X(t) = B(f(t))-t$ can be proved following papers of Geman and Horowitz [11], Berman [3] (see also Dębicki et al. [5]). Moreover, there exists an explicit formula for the expected value of $\mathscr{L}_{t_0}(u)$.

LEMMA 2.1. $E\mathscr{L}_{t_0}(u) = l_{f,t_0}(u)$.

Remark 2.1. Note that by **B2** we have $E\mathscr{L}(u;\{B(f(s))-s;\, s \geqslant t_0\}) < \infty$.

Let

$$\tau(u) = \inf\{t \geqslant t_0 \colon B(f(t))-t = u\}$$

be the hitting time of level u by $B(f(t))-t$ after time t_0 and $\nu_u(\cdot)$ be its distribution (note that ν_u is defective). By $\nu_u^{\circ}(\cdot)$ we denote the conditional distribution of $\tau(u)$ under the condition that $\tau(u) < \infty$. Consider jointly $(\mathscr{L}_{t_0}(u),\, \tau(u))$ under the condition that $\tau(u) < \infty$. Let $E(\mathscr{L}_{t_0}(u) \mid \tau(u) = t)$ denote a version of the conditional probability of $E(\mathscr{L}_{t_0}(u) \mid \tau(u))$.

The following lemma plays an important role in the sequel:

LEMMA 2.2. *If* **B1** *and* **B2** *hold, then for all* $u \geqslant 0$, $t \geqslant t_0$,

$$1/2 \leqslant E(\mathscr{L}_{t_0}(u) \mid \tau(u) = t) \leqslant 1/\varepsilon_0,$$

where ε_0 *is defined in* **B2**.

Before proving Lemma 2.2 we need the following result:

LEMMA 2.3. *If the function* $f(t)$ *satisfies conditions* **B1** *and* **B2**, *then*:

(i) *for each* $t \geqslant t_0$ *the function*

$$y_t(z) = \frac{z}{\sqrt{f(z+t)-f(t)}}$$

is strictly increasing from 0 *to* ∞;

(ii) *it follows that*

$$1 \leqslant f_t^*(z) = \frac{f(z+t)-f(t)}{f(z+t)-f(t)-\frac{1}{2}zf'(t+z)} \leqslant \frac{2}{\varepsilon_0} \tag{2.1}$$

for all $z \geqslant 0$, $t \geqslant t_0$ *and* ε_0 *defined by* **B2**.

Proof. (i) is a consequence of assumption **B2** and can be verified by standard calculations. We show only that $y_t(0) = 0$. To see this notice that

$$\lim_{z\to 0} y_t^2(z) = \lim_{z\to 0} \frac{z^2}{f(z+t)-f(t)} = \lim_{z\to 0} \frac{2z}{f'(t)} = 0, \tag{2.2}$$

where (2.2) follows from the l'Hôspital theorem.

To prove (ii) note that from assumption **B1** we have

$$f_t^*(z) = \frac{f(z+t)-f(t)}{f(z+t)-f(t)-\frac{1}{2}zf'(t+z)} \geqslant 1 \tag{2.3}$$

for all $z \geqslant 0$, $t \geqslant t_0$. So we have only to show that

$$f_t^*(z) = \frac{f(z+t)-f(t)}{f(z+t)-f(t)-\frac{1}{2}zf'(t+z)} \leqslant \frac{2}{\varepsilon_0}.$$

From assumption **B2** we obtain for all $z \geqslant 0$, $t \geqslant t_0$

$$\begin{aligned} 0 &\leqslant \frac{d}{dz}\frac{z^{2-\varepsilon_0}}{f(z+t)-f(t)} \qquad (2.4)\\ &= \big((2-\varepsilon_0)\big(f(z+t)-f(t)\big)-zf'(z+t)\big)\frac{z^{1-\varepsilon_0}}{\big(f(z+t)-f(t)\big)^2}, \end{aligned}$$

which implies

$$\frac{zf'(z+t)}{f(z+t)-f(t)} \leqslant 2-\varepsilon_0 \tag{2.5}$$

for all $z \geqslant 0$, $t \geqslant t_0$. Hence

$$\begin{aligned} \frac{1}{f_t^*(z)} &= \frac{f(z+t)-f(t)-\frac{1}{2}zf'(t+z)}{f(z+t)-f(t)} = 1-\frac{1}{2}\frac{zf'(z+t)}{f(z+t)-f(t)}\\ &\geqslant 1-\frac{1}{2}(2-\varepsilon_0) = \frac{\varepsilon_0}{2}, \qquad (2.6) \end{aligned}$$

where inequality (2.6) follows from (2.5). This completes the proof. ∎

Now we are prepared to prove Lemma 2.2.

Proof of Lemma 2.2. Let $t \geqslant t_0$ be given. Since the process $B\big(f(t)\big)-t$ has independent increments, we obtain

$$\begin{aligned} E\big(\mathscr{L}_{t_0}(u) \mid \tau(u) = t\big) &= E\Big(\mathscr{L}\big(0;\{B\big(f(t+s)-f(t)\big)-s;\ s \geqslant 0\}\big)\Big) \qquad (2.7)\\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{f(t+x)-f(t)}}\\ &\quad \times \exp\left(-\frac{x^2}{2\left[f(x+t)-f(t)\right]}\right)dx. \end{aligned}$$

In (2.7) we substitute

$$y = \frac{x}{\sqrt{f(x+t)-f(t)}}.$$

Note that from Lemma 2.3 it follows that the function

$$y_t(x) = \frac{x}{\sqrt{f(x+t)-f(t)}}$$

is strictly increasing from 0 to ∞. Hence there exists an inverse $z_t(y)$. We have

$$\begin{aligned}(2.8)\quad E\big(\mathscr{L}_{t_0}(u) \mid \tau(u)=t\big) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{f(t+x)-f(t)}} \\ &\quad\times \exp\left(-\frac{x^2}{2[f(x+t)-f(t)]}\right)dx \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{f(x+t)-f(t)}{f(x+t)-f(t)-\frac{1}{2}xf'(x+t)}\exp\left(-\frac{y^2}{2}\right)dy \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty f_t^*(z_t(y))\exp\left(-\frac{y^2}{2}\right)dy.\end{aligned}$$

Now from Lemma 2.3 we obtain

$$\begin{aligned}E\big(\mathscr{L}_{t_0}(u) \mid \tau(u)=t\big) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty f_t^*(z_t(y))\exp\left(-\frac{y^2}{2}\right)dy \\ &\leqslant \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{2}{\varepsilon_0}\exp\left(-\frac{y^2}{2}\right)dy = \frac{1}{\varepsilon_0}\end{aligned}$$

and

$$\begin{aligned}E\big(\mathscr{L}_{t_0}(u) \mid \tau(u)=t\big) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty f_t^*(z_t(y))\exp\left(-\frac{y^2}{2}\right)dy \\ &\geqslant \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left(-\frac{y^2}{2}\right)dy = \frac{1}{2}.\end{aligned}$$

This completes the proof. ∎

3. Bounds of distribution of the supremum of SBM. In this section we apply Lemma 2.2 to derive bounds of (1.1).

THEOREM 3.1. *If the function $f(t)$ satisfies* **B1** *and* **B2**, *then for each $u>0$*

$$(3.1)\qquad 2 \geqslant \frac{P\big(\sup_{t\geqslant t_0}\big(B(f(t))-t\big)>u\big)}{l_{f,t_0}(u)} \geqslant \varepsilon_0,$$

where ε_0 is defined in **B2**.

Proof. The idea is to consider

$$\text{(3.2)}\qquad \frac{l_{f,t_0}(u)}{P\left(\sup_{t\geqslant t_0}\{B(f(t))-ct\}>u\right)}=\frac{E\mathscr{L}_{t_0}(u)}{P\left(\sup_{t\geqslant t_0}\{B(f(t))-ct\}>u\right)}$$
$$=\frac{\int_{t_0}^{\infty}E\left(\mathscr{L}_{t_0}(u)\mid\tau(u)=t\right)\mu_u(dt)}{P\left(\tau(u)<\infty\right)}=\int_{t_0}^{\infty}E\left(\mathscr{L}_{t_0}(u)\mid\tau(u)=t\right)\mu_u^{\circ}(dt).$$

We obtain (3.1) inserting the result of Lemma 2.2 into (3.2). ∎

Dębicki et al. [5] used the *local time* technique to obtain bounds of (1.1) in the case of $f(t)=t^{2H}$ for $H\in(0,1)$. The following corollary extends the result obtained in [5].

COROLLARY 3.1. *Let $f(t)\in C^2([0,\infty))$ be increasing and such that there exists $1>\varepsilon>0$ such that for all $t>t_0$*

$$\text{(3.3)}\qquad f'(t)/t>f''(t)/(1-\varepsilon).$$

Then for each $u>0$

$$2\geqslant\frac{P\left(\sup_{t\geqslant t_0}(B(f(t))-t)>u\right)}{l_{f,t_0}(u)}\geqslant\varepsilon.$$

Proof. Let ε be such that (3.3) holds. By Theorem 3.1 it suffices to prove that the function $f(t)$ satisfies condition **B2**. In fact, we show that for each $x>0$, $t>t_0$

$$\frac{d}{dx}\frac{x^{2-\varepsilon}}{f(t+x)-f(t)}\geqslant 0.$$

We have

$$\text{(3.4)}\qquad \frac{d}{dx}\frac{x^{2-\varepsilon}}{f(t+x)-f(t)}=\frac{(2-\varepsilon)x^{1-\varepsilon}\left(f(t+x)-f(t)\right)-x^{2-\varepsilon}f'(t+x)}{\left(f(t+x)-f(t)\right)^2}.$$

Set

$$F_t(x)=(2-\varepsilon)\left(f(t+x)-f(t)\right)-xf'(t+x).$$

The idea is to prove that $F_t(x)\geqslant 0$ for all $x>0$, $t>t_0$.

Let $t>t_0$ be given. We have

$$\text{(3.5)}\qquad F_t(0)=0,$$

$$\text{(3.6)}\qquad \frac{d}{dx}F_t(x)=(1-\varepsilon)f'(t+x)-xf''(t+x).$$

Notice that

- if $f''(t+x)<0$, then, by (3.6),

$$\frac{d}{dx}F_t(x)\geqslant 0;$$

- if $f''(t+x) \geqslant 0$, then, by (3.3),

$$\frac{d}{dx} F_t(x) = (1-\varepsilon) f'(t+x) - x f''(t+x) \geqslant t f''(t+x) \geqslant 0.$$

Hence $F_t(x) \geqslant 0$ for all $x > 0$, $t > t_0$, which completes the proof. ∎

Remark 3.1. Note that if $f(t) \in C^2([0, \infty))$, $f(t)$ is increasing and concave, then Corollary 3.1 holds, which follows from the fact that $f''(x) \leqslant 0$ in this case.

4. Asymptotic of distribution of the supremum. Thorem 3.1 gives an upper and lower bound of (1.1). In this section we give conditions under which the exact asymptotic of (1.1) can be expressed in terms of the expected value of the *local time.*

We assume that $f(t)$ satisfies **B1**, **B2** and moreover that

B3: There exists a function $\Upsilon(t) \to \infty$ such that

$$\frac{\Upsilon(t)}{\sqrt{f(t+\Upsilon(t)) - f(t)}} \to \infty \quad \text{for } t \to \infty \tag{4.1}$$

and

$$f'(t+x)/f'(t) \to 1 \tag{4.2}$$

uniformly for $x \in [0, \Upsilon(t)]$ when $t \to \infty$.

Note that the class of scale functions of the form $f(t) = t^{2H}$, $H \in (0, 1)$, satisfies **B1**–**B3** for $\Upsilon(t) = t^{(H+1)/2}$. In Section 5 we give other examples of scale functions that satisfy **B1**–**B3**.

The main result of this section is given in the following theorem:

THEOREM 4.1. *If the function $f(t)$ satisfies* **B1**–**B3**, *then*

$$\lim_{u \to \infty} \frac{P\left(\sup_{t \geqslant t_0}\left(B(f(t)) - t\right) > u\right)}{l_{f,t_0}(u)} = 1. \tag{4.3}$$

Proof. Following the proof of Lemma 2.2 we have

$$\frac{l_{f,t_0}(u)}{P\left(\sup_{t \geqslant t_0}\{B(f(t)) - ct\} > u\right)} = \frac{E\mathscr{L}_{t_0}(u)}{P\left(\sup_{t \geqslant t_0}\{B(f(t)) - ct\} > u\right)}$$
$$= \frac{\int_{t_0}^{\infty} E\left(\mathscr{L}_{t_0}(u) \mid \tau(u) = t\right) \mu_u(dt)}{P\left(\tau(u) < \infty\right)} = \int_0^{\infty} E\left(\mathscr{L}_{t_0}(u) \mid \tau(u) = t\right) \mu_u^{\circ}(dt),$$

where

$$E\left(\mathscr{L}_{t_0}(u) \mid \tau(u) = t\right) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f_t^*(z_t(y)) \exp\left(-\frac{y^2}{2}\right) dy \tag{4.4}$$

does not depend on u ($f_t^*(x)$ was defined by (2.1)). Since for $u \to \infty$ the mass of the measure $\mu_u^\circ(dt)$ is moving to $+\infty$, it suffices to prove that

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty f_t^*(z_t(y)) \exp\left(-\frac{y^2}{2}\right) dy \to 1 \quad \text{as } t \to \infty. \tag{4.5}$$

Note that from the proof of Lemma 2.2 and (4.1) we obtain

$$z_t^{-1}(\Upsilon(t)) = \frac{\Upsilon(t)}{\sqrt{f(t+\Upsilon(t)) - f(t)}} \to \infty,$$

which implies that for any $y \geqslant 0$ there exists t^* such that $\Upsilon(t) > z_t(y)$ for $t > t^*$. Hence to prove (4.5) it suffices to show that

$$f_t^*(x) \to 2 \tag{4.6}$$

uniformly for $x \in [0, \Upsilon(t)]$ for $t \to \infty$. This is a consequence of the following chain of transformations:

$$\begin{aligned} \frac{1}{f_t^*(x)} &= 1 - \frac{1}{2} \frac{x f'(t+x)}{f(t+x) - f(t)} \\ &= 1 - \frac{1}{2} \frac{f'(t+x)}{f'(t+\varrho(t, x))} && (4.7) \\ &= 1 - \frac{1}{2} \frac{f'(t+x)}{f'(t)} \frac{f'(t)}{f'(t+\varrho(t, x))} && (4.8) \\ &\to 1 - \frac{1}{2} = \frac{1}{2}, && (4.9) \end{aligned}$$

where (4.7) follows from Taylor's formula with $\varrho(t, x) \in [0, x] \subset [0, \Upsilon(t)]$, and (4.8) is a consequence of assumption **B3**. This completes the proof. ∎

Now we give two corollaries that will be of special interest in next sections.

COROLLARY 4.1. *Let $f(t) \in C^2([0, \infty))$ be increasing and convex. If there exists $1 > \varepsilon > 0$ such that for all $t > t_0$*

$$f'(t)/t \geqslant f''(t)/(1-\varepsilon), \tag{4.10}$$

then

$$\lim_{u \to \infty} \frac{P\left(\sup_{t \geqslant t_0} (B(f(t)) - t) > u\right)}{l_{f,t_0}(u)} = 1.$$

Proof. By Corollary 3.1 it suffices to prove that assumption **B3** holds for the function $f(t)$. Let $\Upsilon(t) = t^{1-\varepsilon/2}$.

First we show (4.2). Note that $f'(t)$ is increasing, so it suffices to prove that

$$f'(t+\Upsilon(t))/f'(t) \to 1 \quad \text{as } t \to \infty. \tag{4.11}$$

To show (4.11) notice that

$$\frac{f'(t+\Upsilon(t))}{f'(t)}=\frac{f'(t+t^{1-1/\varepsilon})}{f'(t)}\geqslant 1 \tag{4.12}$$

and

$$\begin{aligned}
f'(t+t^{1-1/\varepsilon})-f'(t) &= \int_t^{t+t^{1-1/\varepsilon}} f''(s)\,ds \\
&\leqslant \int_t^{t+t^{1-1/\varepsilon}} (1-\varepsilon)\frac{f'(s)}{s}\,ds \qquad (4.13)\\
&\leqslant \int_t^{t+t^{1-1/\varepsilon}} \frac{f'(s)}{s}\,ds \\
&= \log(t+t^{1-1/\varepsilon})\,f'(t+t^{1-1/\varepsilon})-\log(t)\,f'(t) \qquad (4.14)\\
&\quad -\int_t^{t+t^{1-1/\varepsilon}} \log(s)\,f''(s)\,ds \\
&\leqslant \log(t+t^{1-1/\varepsilon})\,f'(t+t^{1-1/\varepsilon})-\log(t)\,f'(t) \\
&\quad -\log(t)\int_t^{t+t^{1-1/\varepsilon}} f''(s)\,ds \\
&= f'(t+t^{1-1/\varepsilon})\log\left(\frac{t+t^{1-1/\varepsilon}}{t}\right), \qquad (4.15)
\end{aligned}$$

where (4.13) follows from (4.10), and (4.14) is a consequence of integration by parts. Hence we get

$$\frac{f'(t+t^{1-1/\varepsilon})}{f'(t)}\leqslant\frac{1}{1-\log\big((t+t^{1-1/\varepsilon})/t\big)}\to 1 \qquad \text{as } t\to\infty,$$

which gives (4.11).

To prove (4.1) notice that for $t\to\infty$

$$\begin{aligned}
\frac{\Upsilon(t)}{\sqrt{f(t+\Upsilon(t))-f(t)}} &\geqslant \frac{\Upsilon(t)}{\sqrt{f(t+\Upsilon(t))}} \\
&\geqslant \frac{\Upsilon(t)}{\sqrt{f(2t)}} \qquad (4.16)\\
&= \frac{t^{1-\varepsilon/2}}{\sqrt{f(2t)}}\to\infty, \qquad (4.17)
\end{aligned}$$

where inequality (4.16) follows from (4.2). Equality (4.17) is a consequence of assumption **B2**. This completes the proof. ∎

COROLLARY 4.2. *If* $f(t) \in C^2([0, \infty))$ *is increasing, concave and*

$$|f''(x)/f'(x)| \searrow 0 \quad \text{as } x \to \infty, \tag{4.18}$$

then

$$\lim_{u \to \infty} \frac{P\left(\sup_{t \geqslant t_0} \left(B(f(t)) - t\right) > u\right)}{l_{f,t_0}(u)} = 1.$$

Proof. By Remark 3.1 and Corollary 3.1 assumptions **B1** and **B2** are satisfied for f. To prove that **B3** holds we take

$$\Upsilon(t) = \sqrt{|f'(x)/f''(x)|}. \tag{4.19}$$

To prove (4.1) we consider

$$\frac{\Upsilon(t)}{\sqrt{f\left(t+\Upsilon(t)\right) - f(t)}} \geqslant \sqrt{\frac{\Upsilon(t)}{f'(t)}} \geqslant \sqrt{\frac{\Upsilon(t)}{f'(0)}} \to \infty. \tag{4.20}$$

Notice that $f'(t)$ is decreasing (because $f(t)$ is concave), so it suffices to show that

$$\frac{f'\left(t+\Upsilon(t)\right)}{f'(t)} \to 1 \quad \text{as } t \to \infty. \tag{4.21}$$

From Taylor's expansion we obtain

$$\frac{f'\left(t+\Upsilon(t)\right)}{f'(t)} = \frac{f'(t) + \Upsilon(t) f''\left(t+\varrho(t)\right)}{f'(t)} = 1 + \frac{\Upsilon(t) f''\left(t+\varrho(t)\right)}{f'(t)}, \tag{4.22}$$

where $\varrho(t) \in [0, \Upsilon(t)]$. Hence it suffices to prove that

$$\frac{\Upsilon(t) f''\left(t+\varrho(t)\right)}{f'(t)} \to 0 \quad \text{as } t \to \infty.$$

But from (4.19) we have

$$\frac{\Upsilon(t) f''\left(t+\varrho(t)\right)}{f'(t)} = \left|\sqrt{\left|\frac{f'(t)}{f''(t)}\right|} \frac{f''\left(t+\varrho(t)\right)}{f'(t)}\right|$$

$$\leqslant \sqrt{\left|\frac{f'(t)}{f''(t)}\right|} \left|\frac{f''\left(t+\varrho(t)\right)}{f'\left(t+\varrho(t)\right)}\right| \tag{4.23}$$

$$\leqslant \sqrt{\left|\frac{f'(t)}{f''(t)}\right|} \left|\frac{f''(t)}{f'(t)}\right| = \sqrt{\left|\frac{f''(t)}{f'(t)}\right|} \to 0 \quad \text{as } t \to \infty. \tag{4.24}$$

This completes the proof. ∎

5. Application to Gaussian integrated processes. Dębicki et al. [5] considered the asymptotic of

$$P\big(\sup_{t\geqslant 0} B_H(t)-t > u\big), \tag{5.1}$$

where $B_H(t)$ is a fractional Brownian motion with the Hurst parameter H. For such a class of Gaussian processes with stationary increments, under the condition that $H\in(\frac{1}{2}, 1)$, Dębicki et al. [5] obtained an asymptotical upper bound

$$\limsup_{u\to\infty} \frac{P\left(\sup_{t\geqslant 0} B_H(t)-t > u\right)}{l_{\sigma^2_{B_H},0}(u)} \leqslant 1, \tag{5.2}$$

where $\sigma^2_{B_H}(t) = D^2\big(B_H(t)\big)$.

In this section we obtain an analogous result for another class of Gaussian processes with stationary increments which are important in Gaussian fluid models. Namely, we consider $\eta(t) = \int_0^t Z(s)\,ds$, where $Z(t)$ is a stationary centered Gaussian process with the covariance function $R(t) = \mathbf{Cov}\big(Z(t), Z(0)\big)$. We assume that $R(t) > 0$ and is continuous. Let

$$\sigma^2_\eta(t) = D^2\big(\eta(t)\big) = 2\int_0^t ds\int_0^s R(v)\,dv.$$

Note that $\sigma^2_\eta(t)\in C^2[0, \infty)$ is increasing and convex.

PROPOSITION 5.1. *If $Z(t)$ is a stationary centered Gaussian process with a positive and continuous covariance function $R(t) = \mathbf{Cov}\big(Z(t), Z(0)\big)$ such that $tR(t)\to 0$ as $t\to\infty$, then*

$$\limsup_{u\to\infty} \frac{P\left(\sup_{t\geqslant 0} \eta(t)-t > u\right)}{l_{\sigma^2_\eta,0}(u)} \leqslant 1. \tag{5.3}$$

Proof. Since the variance function $\sigma^2_\eta(t)$ of $\eta(t)$ is convex and $\eta(t)$ has stationary increments, we have the following relation between the covariance function $r_\eta(s, t)$ of $\eta(t)$ and the covariance function $r_{B(\sigma^2)}(s, t)$ of $B\big(\sigma^2(t)\big)$:

$$r_\eta(s, t) \leqslant r_{B(\sigma^2)}(s, t) \quad \text{for each } s, t\geqslant 0.$$

Moreover, $r_\eta(t, t) = r_{B(\sigma^2_\eta)}(t, t)$ for each $t\geqslant 0$. Thus from the Slepian theorem (see Piterbarg [13]) we have

$$P\big(\sup_{t\geqslant 0}\eta(t)-t > u\big) \leqslant P\big(\sup_{t\geqslant 0} B\big(\sigma^2(t)\big)-t > u\big). \tag{5.4}$$

Since for each $T > 0$

$$P\big(\sup_{t\geqslant 0} B\big(\sigma^2(t)\big)-t > u\big) \approx P\big(\sup_{t\geqslant T} B\big(\sigma^2(t)\big)-t > u\big) \quad \text{as } u\to\infty,$$

to obtain (5.3) it suffices to verify assumption (4.10) of Corollary 4.1, that is to prove that for $\varepsilon = 1/2$ there exists a point t_0 such that for $t > t_0$

$$\dot{\sigma}_\eta^2(t)/\ddot{\sigma}_\eta^2(t) \geqslant t/(1-\tfrac{1}{2}).$$

This follows from the fact that

$$\dot{\sigma}_\eta^2(t) = 2\int_0^t R(v)\,dv, \qquad \ddot{\sigma}_\eta^2(t)\,t = 2R(t)\,t \to 0.$$

Hence the proof is completed. ∎

Proposition 5.1 covers mainly the case of short range dependence structure of $\eta(t)$. Another important class of Gaussian integrated processes, that are considered in the theory of fluid models, are processes which have long range dependence structure. Recently Dębicki and Palmowski [6] analyzed processes $\eta(t) = \int_0^t Z(s)\,ds$, where $Z(t)$ is a stationary centered Gaussian process with covariance function $R(t) = t^{2H-2}L(t)$ for $H \in (\frac{1}{2}, 1)$, where $L(t)$ is slowly varying at ∞.

The following proposition deals with such a class of processes.

PROPOSITION 5.2. *If $Z(t)$ is a stationary centered Gaussian process with positive and continuous covariance function $R(t) = \mathbf{Cov}(Z(t), Z(0))$ such that $R(t) = t^{2H-2}L(t)$ as $t \to \infty$ and $H \in (\frac{1}{2}, 1)$, then*

$$\limsup_{u\to\infty} \frac{\boldsymbol{P}\left(\sup_{t\geqslant 0}\eta(t)-t > u\right)}{l_{\sigma_\eta^2,0}(u)} \leqslant 1. \tag{5.5}$$

Proof. The proof goes analogously to the proof of Proposition 5.1. Namely, it suffices to verify assumption (4.10) of Corollary 4.1. Since $R(t) = t^{2H-2}L(t)$, from Karamata's theorem (see [4]) we obtain

$$\dot{\sigma}_\eta^2(t) = \frac{2}{2H-1}t^{2H-1}L(t), \qquad \sigma_\eta^2(t) = \frac{2}{(2H-1)2H}t^{2H}L(t)$$

for $t \to \infty$. Consequently, there exists a point t_0 such that for $0 < \varepsilon < 2-2H$ and $t > t_0$

$$\dot{\sigma}_\eta^2(t)/\ddot{\sigma}_\eta^2(t) \geqslant t/(1-\varepsilon),$$

which completes the proof. ∎

6. A geometric approach. The distribution of $\boldsymbol{P}\left(\sup_{t\geqslant 0}B(f(t))-t > u\right)$ is related to the distribution of the supremum of Brownian motion with an appropriate nonlinear drift. In particular, if $f(t)$ is strictly increasing and $f(0) = 0$, then

$$\boldsymbol{P}\left(\sup_{t\geqslant 0}\left(B(f(t))-t\right) > u\right) = \boldsymbol{P}\left(\sup_{t\geqslant 0}\left(B(t)-g(t)\right) > u\right), \tag{6.1}$$

where $g(t)$ is the inverse function to the function $f(t)$. In this section, using (6.1), we find an exponential upper bounds of (1.1).

For a given continuous positive function $f(t)$ define

$$M_f(u) = \min\left\{\frac{(u+t)^2}{2f(t)}:\ t \geqslant 0\right\}.$$

This function plays the main role in the sequel.

THEOREM 6.1. *If $f(t) \in C^1([0, \infty))$ is increasing, concave and $f(0) = 0$, then for each $u > 0$*

$$P\left(\sup_{t\geqslant 0}\left(B(f(t))-t\right) > u\right) \geqslant \exp\left(-M_f(u)\right). \tag{6.2}$$

Proof. Let $g(t)$ be the inverse function to $f(t)$. Note that $g(t)$ is increasing and convex.

The proof is based on the observation that $k_x(t) = g'(x)t+g(x)-g'(x)x$ is the tangent function to $g(t)$ at point $x\,(x > 0)$. Using a geometric approach we have

$$\begin{aligned} P\left(\sup_{t\geqslant 0}\left(B(f(t))-t\right) > u\right) &= P\left(\sup_{t\geqslant 0}\left(B(t)-g(t)\right) > u\right) \\ &\geqslant P\left(\sup_{t\geqslant 0}\left(B(t)-k_x(t)\right) > u\right) &&(6.3)\\ &= P\left(\sup_{t\geqslant 0}\left(B(t)-g'(x)t-g(x)+g'(x)x\right) > u\right) \\ &= P\left(\sup_{t\geqslant 0}\left(B(t)-g'(x)t\right) > g(x)-g'(x)x+u\right) \\ &= \exp\left(-2g'(x)\left(g(x)-g'(x)x+u\right)\right) \\ &\geqslant \exp\left(-M_f(u)\right) &&(6.4) \end{aligned}$$

for all $u > 0$. Inequality (6.3) is a consequence of concavity of $g(t)$. Inequality (6.4) follows from the fact that the function $-2g'(x)\left(g(x)-g'(x)x+u\right)$ takes its maximum at x_u such that $g(x_u)-g'(x_u)x_u+u = 0$. Moreover, since $f(t)$ is the inverse function to $g(x)$, we obtain $-2g'(x_u)\left(g(x_u)-g'(x_u)x_u+u\right) = -M_f(u)$. ∎

Remark 6.1. The supremum of the Brownian motion with nonlinear drift $B(t)-t^{1/(2H)}$ for $H \in (0, 1)$ was studied by Dębicki et al. [5]. Other papers with nonlinear drift are Ferebee [9], [10] and Jennen [12].

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