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ESTIMATES FOR THE POISSON KERNELS ON HOMOGENEOUS MANIFOLDS OF NEGATIVE CURVATURE AND THE BOUNDARY HARNACK INEQUALITY IN THE NONCOERCIVE CASE

BY

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Abstract. Using a probabilistic technique we obtain upper and lower estimates for the Poisson kernels of the second order differential operators on a homogeneous manifold of negative curvature. Our results improve estimates obtained in the paper [5]. Moreover, for the noncoercive operator we proved the boundary Harnack inequality which turned out to be the same as in the coercive case.

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1. Introduction and main results. In this paper we consider a second order differential operator $\mathscr{L}$ on a connected, simply connected homogeneous manifold of negative curvature. Such a manifold is a solvable Lie group $S = NA$, being a semidirect product of a nilpotent Lie group N and an abelian group $A = \boldsymbol{R}^+$. Moreover, for an H belonging to the Lie algebra $\mathfrak{a}$ of A, the real parts of the eigenvalues of $\mathrm{Ad}_{\exp H}|_N$ are all greater than 0. Conversely, every such group equipped with a suitable left-invariant metric becomes a homogeneous Riemannian manifold with negative curvature [10].

On S we consider a second order left-invariant operator

$$\mathscr{L} = \sum_{j=0}^{m} Y_j^2 + Y.$$

We assume that $Y_0, Y_1, \ldots, Y_m$ generate the Lie algebra $\mathfrak{s}$ of S. Moreover, we can choose $Y_0, Y_1, \ldots, Y_m$ so that $Y_1(e), \ldots, Y_m(e)$ belong to the Lie algebra $\mathfrak{n}$ of N. Let π: $S \to A = S/N$ be the canonical homomorphism. Then the image of

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$\mathcal{L}$ under π is a second order left-invariant operator on $\boldsymbol{R}^+$,

$$(a\partial_a)^2 - \gamma a\partial_a,$$

where $\gamma \in \boldsymbol{R}$. The operator $\mathcal{L} = \mathcal{L}_\gamma$ is *noncoercive* (there is no $\varepsilon > 0$ such that $\mathcal{L} + \varepsilon I$ admits the Green function) if and only if $\gamma = 0$.

Finally, the operator we are interested in can be written in the form

$$\mathcal{L} = \mathcal{L}_\gamma = \sum_j \Phi_a(X_j)^2 + \Phi_a(X) + a^2\partial_a^2 + (1-\gamma)\,a\partial_a, \tag{1.1}$$

where $X, X_1, \ldots, X_m$ are left-invariant vector fields on N and vector fields $X_1, \ldots, X_m$ generate $\mathfrak{n}$,

$$\Phi_a = \mathrm{Ad}_{\exp(\log a)Y_0} = \exp\big((\log a)\,\mathrm{ad}_{Y_0}\big) = \exp\big((\log a)\,D\big),$$

and $D = \mathrm{ad}_{Y_0}$ is a derivation of the Lie algebra $\mathfrak{n}$ of the Lie group N such that the real parts d_j of the eigenvalues λ_j of D are positive. Multiplying $\mathcal{L}_\gamma$ by a constant we can make d_j arbitrarily large [5].

Our first result is about the Poisson kernels for the operators (1.1). Let μ_t^γ be the semigroup of measures generated by $\mathcal{L}_\gamma$. It is known (cf. [7]) that if $\gamma \geqslant 0$, then there exists a unique (up to a positive multiplicative constant) positive Radon measure ν_γ with a smooth density m_γ on N such that

$$\check{\mu}_t^\gamma * \nu_\gamma = \nu_\gamma, \qquad \gamma \geqslant 0.$$

ν_γ or its density m_γ is called the *Poisson kernel* for the operator $\mathcal{L}_\gamma$. For $\gamma > 0$ the measure ν_γ is bounded, while ν_0 is unbounded. These measures have been studied by many authors and in various contexts; see e.g. [5] and literature quoted therein. In particular in [5], Theorem 6.1, by using some probabilistic techniques, the following estimates of the Poisson kernels have been proved:

THEOREM 1.1. *Let m_γ be the Poisson kernel of $\mathcal{L}_\gamma$, $\gamma \geqslant 0$. Then there exists a constant C_γ such that, for all $x \in N$,*

$$C_\gamma^{-1}(|x|+1)^{-Q-\gamma} \leqslant m_\gamma(x) \leqslant C_\gamma(|x|+1)^{-Q-\gamma}, \tag{1.2}$$

where $|\cdot|$ denotes the "homogeneous norm" on N and Q denotes the "homogeneous dimension" of N (see Section 2 for precise definitions).

One of our main goals in this paper is to show that for small range of γ we can choose C_γ in (1.2) independent of γ. Namely, we are going to prove the following

THEOREM 1.2. *Let m_γ be the Poisson kernel of $\mathcal{L}_\gamma$. Then there exists a positive constant C such that, for every $x \in N$ and for every $\gamma \in [0, 1]$,*

$$C^{-1}(|x|+1)^{-Q-\gamma} \leqslant m_\gamma(x) \leqslant C(|x|+1)^{-Q-\gamma}.$$

It should be mentioned that the above uniform estimate has already found application in [12] for investigation into unbounded harmonic functions on

NA groups. Especially in the case $\gamma = 0$, as there have been no nonconstant bounded harmonic functions there.

Before stating our second result we need some definitions. Let $\tau(s_1, s_2)$ be the Riemannian distance between two points $s_1, s_2 \in NA$. For simplicity we very often write $\tau(s)$ instead of $\tau(e, s)$, where e denotes the identity element of NA.

Let $\Phi\colon [0, \infty) \to [c_0, \infty)$ with $\Phi(0) = c_0 \geqslant 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$ be a positive increasing function. A finite sequence of open sets $V_1 \supset \ldots \supset V_m$ with a sequence of points $s_1, \ldots, s_m$ such that $s_i \in \partial \bar{V}_i$, $i = 1, \ldots, m$, and with the properties that, for every $z \in \partial \bar{V}_{i+1}$,

$$\tau(z, \partial V_i) \geqslant \Phi\big(\tau(z, s_{i+1})\big) \tag{1.3}$$

and

$$c_0 \leqslant \tau(s_i, s_{i+1}) \leqslant c_0^{-1} \tag{1.4}$$

is called a *Φ-chain.*

As in [3], let $V_1 \supset V_2$ be two open sets and $B(p, r)$ be a Riemannian ball with center p and radius r contained in $V_1 \backslash \bar{V}_2$. The triple $\big(V_1, V_2, B(p, r)\big)$ is said to be a *(Φ, r)-triple* if for every $s_1 \in \partial V_1$ and every $s_2 \in \partial V_2$ there is a Φ-chain passing through s_1, s_2, p.

Now we are able to state some kind of the boundary Harnack inequality in the noncoercive case, i.e. for the operator $\mathscr{L}_0$ defined in (1.1).

THEOREM 1.3. *Given Φ and r there is a constant C such that for every (Φ, r)-triple $\big(V_1, V_2, B(p, r)\big)$ and any nonnegative $\mathscr{L}_0$-superharmonic functions f, g with the properties*:

(a) *f is $\mathscr{L}_0$-harmonic on $\bar{V}_2^c$ and is dominated by a potential there,*

(b) *g is $\mathscr{L}_0$-harmonic in $B(p, r)$,*

we have, for every $s \notin V_1$,

$$\frac{f(s)}{g(p)} \leqslant C \frac{g(s)}{g(p)}. \tag{1.5}$$

In the case $\gamma > 0$ there is a result analogous to Theorem 1.3 and has been proved by Damek in [3] by means of Ancona's theory [1]. Then Damek used it to give an estimate of the Green function for the operator $\mathscr{L}_\gamma$. The boundary Harnack inequality in [3] follows from Theorem 2.8 in [3]. Since our operator is noncoercive, we cannot apply Ancona's theory. The only thing we need in order to prove Theorem 1.3 is the analogue of Theorem 2.8 in [3]. Fortunately, it turns out that we are able to prove it (see Theorem 5.1 below) using an estimate of the Green function $\mathscr{G} = \mathscr{G}_0$ for $\mathscr{L}_0$ obtained by the author in [15]. To sum up, in [3] the boundary Harnack inequality was a tool to prove the estimate of the Green function; by contrast here, in the noncoercive case, we use the estimate of the Green function to get the boundary Harnack inequality.

The structure of the paper is as follows. In Section 2 we state precisely notation and all necessary definitions. Moreover, some theorems which will be used in the sequel are also cited.

In Section 3 we recall a definition of the Bessel process and prove some lemmas about its properties which generalize some standard ones. These lemmas are crucial in the proof of Theorem 1.2.

Finally, in Sections 4 and 5 we prove Theorem 1.2 and Theorem 1.3, respectively.

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2. Preliminaries. Let N be a connected and simply connected nilpotent Lie group. Let D be a derivation of the Lie algebra $\mathfrak{n}$ of N. For every $a \in \boldsymbol{R}^+$ we define an automorphism Φ_a of $\mathfrak{n}$ by the formula

$$\Phi_a = e^{(\log a)D}.$$

Writing $x = \exp X$ we put

$$\Phi_a(x) := \exp \Phi_a(X).$$

We assume that the real parts d_j of the eigenvalues λ_j of the matrix D are strictly greater than 0 and we define the number

$$Q = \sum_j \operatorname{Re} \lambda_j = \sum_j d_j, \tag{2.1}$$

which will be referred to as a *homogeneous dimension* of N. In this paper $D = \operatorname{ad}_{Y_0}$ (see Section 1).

We consider a group S which is a *semi-direct* product of N and the multiplicative group $A = \boldsymbol{R}^+ = \{\exp tY_0 \colon t \in \boldsymbol{R}\}$:

$$S = NA = \{xa\colon x \in N,\ a \in A\}$$

with multiplication given by the formula

$$(xa)(yb) = \big(x\Phi_a(y)\, ab\big).$$

In N we define a *homogeneous norm* $|\cdot|$ (cf. [6] and [5]) as follows. Let $(\cdot,\cdot)$ be a fixed inner product in $\mathfrak{n}$. We define a new inner product

$$\langle X, Y\rangle = \int_0^1 (\Phi_a(X), \Phi_a(Y))\frac{da}{a}$$

and the corresponding norm

$$\|X\| = \langle X, X\rangle^{1/2}.$$

We put

$$|X| = \left(\inf\{a > 0: \|\Phi_a(X)\| \geqslant 1\}\right)^{-1}.$$

One can easily show that for every $Y \neq 0$ there exists exactly one $a > 0$ such that $Y = \Phi_a(X)$ with $|X| = 1$. Then $|Y| = a$.

Finally, we define the homogeneous norm on N. For $x = \exp X$ we put

$$|x| = |X|.$$

Notice that if the action of $A = \boldsymbol{R}^+$ on N (given by Φ_a) is diagonal, the norm we have just defined is the usual homogeneous norm on N and the number Q in (2.1) is just the homogeneous dimension of N (cf. [8]).

Let $\mathscr{G}(xa, yb)$ be the *Green function* for $\mathscr{L}$. The function $\mathscr{G}$ is defined by two conditions:

(i) $\mathscr{L}\mathscr{G}(\cdot, yb) = -\delta_{yb}$ as distributions (functions are identified with distributions via the right Haar measure);

(ii) for every $yb \in S$, $\mathscr{G}(\cdot, yb)$ is a potential for $\mathscr{L}$.

Moreover, let $\mathscr{G}(xa) := \mathscr{G}(xa, e)$. Then

$$\mathscr{G}(xa, yb) = \mathscr{G}\left((yb)^{-1}xa, e\right) = \mathscr{G}\left((yb)^{-1}xa\right). \tag{2.2}$$

In the proof of Theorem 1.3 we will need the following estimate for the Riemannian distance τ, which is due to Guivarc'h [9].

LEMMA 2.1. *There is a positive constant C such that for every $x \in N$ and $a \in A$ we have*

$$\begin{aligned} C^{-1}\left(\ln(1+|x|)+|\ln a|\right) &\leqslant \tau(xa)+1 \\ &\leqslant C\left(\ln(1+|x|)+|\ln a|+1\right). \end{aligned} \tag{2.3}$$

For a multi-index $I = (i_1, \ldots, i_n)$, $i_j \in \boldsymbol{Z}^+$, and a basis $X_1, \ldots, X_n$ of the Lie algebra $\mathfrak{n}$ of N we write: $X^I = X_1^{i_1} \ldots X_n^{i_n}$ and $|I| = i_1 + \ldots + i_n$. For $k = 0, 1, \ldots, \infty$ we define

$$C^k = \{f\colon X^I f \in C(N) \text{ for } |I| < k+1\}$$

and

$$C^k_\infty = \{f \in C^k\colon \lim_{x \to \infty} X^I f(x) \text{ exists for } |I| < k+1\}.$$

For $k < \infty$ the space C^k_∞ is a Banach space with the norm

$$\|f\|_{C^k_\infty} = \sum_{|I| \leqslant k} \|X^I f\|_{C(N)}.$$

Let

$$L_{\sigma(t)} = \sigma(t)^{-2}\left(\sum \Phi_{\sigma(t)}(X_j)^2 + \Phi_{\sigma(t)}(X)\right). \tag{2.4}$$

For a continuous function σ: $[0, \infty) \to [0, \infty)$ let $\{U^\sigma(s, t)\colon 0 \leqslant s \leqslant t\}$ be the (unique) family of bounded operators on $C_\infty = C^0_\infty$ which satisfy

(i) $U^\sigma(s, s) = I$,

(ii) $U^\sigma(s, r)U^\sigma(r, t) = U^\sigma(s, t)$, $s < r < t$,

(iii) $\partial_s U^\sigma(s, t) f = -L_{\sigma(s)} U^\sigma(s, t) f$ for every $f \in C_\infty$,

(iv) $\partial_t U^\sigma(s, t) f = U^\sigma(s, t) L_{\sigma(t)} f$ for every $f \in C_\infty$,

(v) $U^\sigma(s, t)\colon C^2_\infty \to C^2_\infty$.

$U^\sigma(s, t)$ is a convolution operator, i.e. $U^\sigma(s, t) f = f * p^\sigma(t, s)$, where $p^\sigma(t, s)$ is a smooth density of the probability measure. By (ii) we have $p^\sigma(t, r) * p^\sigma(r, s) = p^\sigma(t, s)$ for $t > r > s$. Existence of the family $U^\sigma(s, t)$ follows from [14].

Later on we will see that the evolution generated by the operator (2.4) with σ being a trajectory of the Bessel process (see Section 4 for a definition) appears in the natural way in the explicit formula for the Green function and/or Poisson kernel of $\mathscr{L}$.

In [5], by using the Nash inequality, the following estimate of the evolution kernels $p^\sigma(t, 0)$ has been proved.

THEOREM 2.2. *For every compact set $K \subset N$ which does not contain the identity element e of N, there exist positive constants C, ξ, β_1, β_2 and $D \leqslant Q$ such that, for every $x \in K$ and for every $t > 0$,*

$$p^\sigma(t, 0) \leqslant C\Big(\int_0^t \sigma^{-2(1-Q/D)}(u)\, du\Big)^{-D/2} \exp\left(\frac{\xi}{A(0, t)}\right),$$

where $A(s, t) = \int_s^t \big(\sigma^{\beta_1}(u) + \sigma^{\beta_2}(u)\big)\, du$.

3. Bessel process. The main results of this section are Lemmas 3.6, 3.7 and the second part of Theorem 3.8. These results become very important in the proof of Theorem 1.2.

Let $b_\alpha(t)$ denote the Bessel process with a parameter $\alpha \geqslant 0$ (see [13]), i.e. a continuous Markov process with the state space $[0, +\infty)$ generated by $\partial^2_a + [(2\alpha+1)/a]\, \partial a$.

The transition function with respect to the measure $y^{2\alpha+1}\, dy$ is given (e.g. in [2]) by

$$(3.1)\qquad p_t(x, y) = \begin{cases} \dfrac{1}{2t} \exp\left(\dfrac{-x^2 - y^2}{4t}\right) I_\alpha\left(\dfrac{xy}{2t}\right) \dfrac{1}{(xy)^\alpha} & \text{for } x, y > 0, \\[2ex] \dfrac{1}{2^\alpha (2t)^{\alpha+1} \Gamma(\alpha+1)} \exp\left(\dfrac{-y^2}{4t}\right) & \text{for } x = 0, y > 0, \end{cases}$$

where

$$I_\alpha(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+\alpha}}{k!\,\Gamma(k+\alpha+1)}$$

is the Bessel function [11]. Therefore for $x \geqslant 0$ and a measurable set $B \subset (0, \infty)$:

$$P_x(b_\alpha(t) \in B) = \int_B p_t(x, y)\, y^{2\alpha+1}\, dy.$$

It turns out that the Bessel process has a property analogous with the Brownian Scaling Property. Namely, we have the following lemma in the formulation from [5] (cf. [13]):

LEMMA 3.1. *Let Ω be the space of trajectories of the Bessel process $b_\alpha(t)$. For $b_\alpha \in \Omega$ and $\lambda > 0$ define $\theta_\lambda(b_\alpha)(t) = \sqrt{\lambda}\, b_\alpha(t/\lambda)$. Assume that b_α starts from x, i.e. $b_\alpha(0) = x$. Then*:

(i) *for every $\lambda > 0$, $\tilde{b}(t) = \theta_\lambda(b_\alpha)(t)$ is the Bessel process (will a parameter α) starting from $\sqrt{\lambda} x$;*

(ii) *for every $\lambda > 0$, $x \geqslant 0$,*

$$E_x f \circ \theta_\lambda = E_{\sqrt{\lambda} x} f.$$

Now we are going to prove some lemmas concerning properties of the Bessel process. In fact, we want to show that in many cases of the very well-known estimates of some functionals of the Bessel process we can keep the constants which appear there independent of the parameter α of the Bessel process provided that α is between 0 and 1/2. If this happens, we say that estimates are *uniform.*

We are almost sure that most of those properties, even in such a general setting, should be known. However, since it is not easy to give references and our proofs depend heavily on them, we show the proofs or at least sketch them out for the reader's convenience.

In what follows we will very often need to use properties of the asymptotic behavior of the Bessel function $I_\alpha(x)$ for small and large values of x. They are described in the following lemma whose proof can be found e.g. in [11].

LEMMA 3.2. *Let I_α, $\alpha \geqslant 0$, be the Bessel function. Then*

$$I_\alpha(x) \asymp \frac{x^\alpha}{2^\alpha \Gamma(1+\alpha)}, \qquad x \to 0,$$

and

$$I_\alpha(x) \asymp \frac{\exp(x)}{(2\pi x)^{1/2}}, \qquad x \to \infty.$$

Using the formula (3.1) and the asymptotics given by Lemma 3.2 it is not difficult to obtain, after simple calculation, the next two lemmas. A little bit weaker results have been proved in [16]. In fact, a careful reading of the proofs in [16], together with some cosmetic changes there, gives desired estimates. Therefore we are going to omit the proofs.

LEMMA 3.3. *There exists a positive constant c such that, for every $0 \leqslant \alpha \leqslant 1/2$, every $\lambda > 0$, every $t > 0$ and every $x \geqslant 0$,*

$$P_x\big(b_\alpha(t) > x+\lambda\big) \leqslant c\exp(-\lambda^2/8t).$$

LEMMA 3.4. *There exists a positive constant c such that, for every $0 \leqslant \alpha \leqslant 1/2$, every $x > 0$, every $\lambda < x$ and every $t > 0$,*

$$P_x\big(b_\alpha(t) < x-\lambda\big) \leqslant c\exp(-\lambda^2/8t).$$

The next part of this section contains the uniform (with respect to the parameter $\alpha \in [0, 1/2]$) versions of the results from [16] and [5].

LEMMA 3.5. *There exist positive constants c_1, c_2 such that, for every $0 \leqslant \alpha \leqslant 1/2$ and for every $x \geqslant 0$, $r > 0$, $t > 0$,*

$$P_x\big(\max_{s\in[0,t]} b_\alpha(s) \leqslant r\big) \leqslant c_1 \exp\Big(-c_2 \frac{\sqrt{t}}{r}\Big).$$

Proof. Notice that it is enough to prove that there exists $\varepsilon > 0$ and $0 < \delta_0 < 1$ such that, for every $x \geqslant 0$ and for every $\alpha \in [0, 1/2]$,

$$P_x\big(\max_{s\in[0,t]} b_\alpha(s) \leqslant r\big) \leqslant \exp\Big(-\varepsilon \frac{\sqrt{t}}{r}\Big),$$

providing that $r/\sqrt{t} \leqslant \delta_0$.

At first we need to show that there exists $0 < \delta_0 < 1$ such that, for every $x \geqslant 0$, $0 \leqslant \alpha \leqslant 1/2$ and for every $\delta \leqslant \delta_0$,

$$P_x\big(b_\alpha(\delta) \leqslant \delta\big) \leqslant C\delta^{1+\alpha} \leqslant C\delta < 1,$$

where the constant C does not depend on x. In order to get this we use the asymptotic behavior of the Bessel function $I_\alpha(x)$ for small and large values of x (Lemma 3.2) and the formula for the transition function for the Bessel process (3.1).

Clearly,

$$P_x\big(\max_{s\in[0,\delta]} b_\alpha(s) \leqslant \delta\big) \leqslant C\delta^{1+\alpha} \leqslant C\delta.$$

Moreover, we will show that, for every $k \in N$,

$$P_x\big(\max_{s\in[0,k\delta]} b_\alpha(s) \leqslant \delta\big) \leqslant (C\delta^{1+\alpha})^k \leqslant C\delta^k.$$

Indeed, the case $k = 1$ is proved. Now, by induction, using the Markov property we get

$$P_x\big(\max_{s\in[0,k\delta]} b_\alpha(s) \leqslant \delta\big) = E_x 1_{\{b_\alpha:\max_{s\in[0,(k-1)\delta]} b_\alpha(s) \leqslant \delta\}} 1_{\{b_\alpha:\max_{s\in[(k-1)\delta,k\delta]} b_\alpha(s) \leqslant \delta\}}$$

$$= E_x E_{b_\alpha(\delta)} 1_{\{\sigma_\alpha:\max_{s\in[0,(k-1)\delta]} \sigma_\alpha(s) \leqslant \delta\}} 1_{\{b_\alpha:\max_{s\in[0,\delta]} b_\alpha(s) \leqslant \delta\}} \leqslant (C\delta^{1+\alpha})^k \leqslant (C\delta)^k.$$

Thus, if $k\delta = 1$, we get

$$P_x\big(\max_{s\in[0,1]} b_\alpha(s) \leqslant \delta\big) \leqslant (C\delta)^{1/\delta} = \exp\Big(-\varepsilon\frac{1}{\delta}\Big).$$

By the first part of Lemma 3.1 we get

$$P_x\big(\max_{s\in[0,t]} b_\alpha(s) \leqslant r\big) = P_{xt^{-1/2}}\big(\max_{s\in[0,1]} b_\alpha(s) \leqslant rt^{-1/2}\big) \leqslant \exp\Big(-\varepsilon\frac{\sqrt{t}}{r}\Big). \blacksquare$$

LEMMA 3.6. *There exist constants c_1, c_2 such that, for every $0 \leqslant \alpha \leqslant 1/2$, $R > 0$ and for every $t > 0$,*

$$P_R\big(\inf_{s\in[0,t]} b_\alpha(s) < R/2\big) \leqslant c_1 \exp(-c_2 R^2/t).$$

Proof. Clearly, we may assume that $R^2/t \geqslant 1$. Otherwise the inequality is trivial. Moreover, at first we prove this lemma for $R = 1$ and $t \leqslant 1$. For arbitrary R we use Lemma 3.1 which implies

$$P_R\big(\inf_{s\in[0,t]} b_\alpha(s) < R/2\big) = P_1\big(\inf_{s\in[0,t/R^2]} b_\alpha(s) < 1/2\big).$$

Let $\delta < 1/2$ and let A be such that $1/2 = A\sum_{n=1}^{\infty} 1/2^{n\delta}$. Notice that

$$\{b_\alpha\colon\ b_\alpha(0) = 1,\ \inf_{s\in[0,t]} b_\alpha(s) < 1/2\}$$
$$\subset \bigcup_{n=1}^{\infty}\bigcup_{k=0}^{2^n-1} \{b_\alpha\colon\ b_\alpha((k+1)t/2^n) < b_\alpha(kt/2^n) - A/2^{n\delta}\}.$$

Thus,

$$\begin{aligned}
P_1\big(\inf_{s\in[0,t]} b_\alpha(s) < 1/2\big) &\leqslant \sum_{n=1}^{\infty}\sum_{k=0}^{2^n-1} P_1\big(b_\alpha((k+1)t/2^n) < b_\alpha(kt/2^n) - A/2^{n\delta}\big)\\
&\leqslant \sum_{n=1}^{\infty}\sum_{k=0}^{2^n-1} E_1 E_{b_\alpha(kt/2^n)} 1_{\{\sigma_\alpha:\sigma_\alpha(t/2^n) < \sigma_\alpha(0) - A/2^{n\delta}\}}(\sigma_\alpha)\\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{2^n-1} \int_0^\infty \Phi(y)\, p_{kt/2^n}(1, y)\, y^{2\alpha+1}\, dy\\
&\leqslant \sum_{n=1}^{\infty}\sum_{k=0}^{2^n-1} \sup_{s\in(0,1]} \int_0^\infty \Phi(y)\, p_s(1, y) \max\{y, y^2\}\, dy,
\end{aligned}$$

where

$$\Phi(y) = E_y 1_{\{\sigma_\alpha:\sigma_\alpha(t/2^n) < y - A/2^{n\delta}\}} = P_y\big(\sigma_\alpha(t/2^n) < y - A/2^{n\delta}\big).$$

Using the estimate of $\Phi(y)$ and Lemma 3.4, we get

$$P_1\big(\inf_{s\in[0,t]} b_\alpha(s) < 1/2\big) \leqslant c_1 \exp(-c_2/t). \blacksquare$$

LEMMA 3.7. *There exist constants c_1, c_2 such that, for every $0 \leqslant \alpha \leqslant 1/2$, every $x \geqslant 0$, every $\lambda > 0$ and every $t > 0$,*

$$P_x\big(\sup_{s\in[0,t]} b_\alpha(s) > x+\lambda\big) \leqslant c_1 \exp(-c_2\lambda^2/t).$$

Proof. As before, it is enough to prove the lemma for $\lambda^2/t \geqslant 1$. Moreover, from the estimate for $x = 1$ by Lemma 3.1 we get this estimate for all $x > 1$. Therefore it is enough to consider $x \leqslant 1$. If so, let $\delta < 1/2$ and let A be such that $\lambda/2 = A\sum_{n=1}^{\infty} \lambda/2^{n\delta}$. Notice that

$$\{b_\alpha\colon\ b_\alpha(0) = x,\ \sup_{s\in[0,t]} b_\alpha(s) > \lambda\}$$
$$\subset \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{2^n-1} \{b_\alpha\colon\ b_\alpha\big((k+1)t/2^n\big) > b_\alpha(kt/2^n) + A\lambda/2^{n\delta}\}.$$

Thus,

$$\begin{aligned} P_x\big(\sup_{s\in[0,t]} b_\alpha(s) > x+\lambda\big) &\leqslant \sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} P_x\big(b_\alpha\big((k+1)t/2^n\big) > b_\alpha(kt/2^n) + A\lambda/2^{n\delta}\big) \\ &\leqslant \sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} E_x E_{b_\alpha(kt/2^n)} 1_{\{\sigma_\alpha:\sigma_\alpha(t/2^n) > \sigma_\alpha(0) + A\lambda/2^{n\delta}\}}(\sigma_\alpha) \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} \int_0^{\infty} \Psi(y)\, p_{kt/2^n}(x, y)\, y^{2\alpha+1}\, dy \\ &\leqslant \sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} \sup_{s\in(0,1],x\leqslant 1} \int_0^{\infty} \Psi(y)\, p_s(x, y) \max\{y, y^2\}\, dy, \end{aligned}$$

where

$$\Psi(y) = P_y\big(\sigma_\alpha(t/2^n) > y + A\lambda/2^{n\delta}\big).$$

Now we use the estimate of $\Psi(y)$, Lemma 3.3, and the assumption that $\lambda^2/t \geqslant 1$, and obtain

$$P_x\big(\sup_{s\in[0,t]} b_\alpha(s) > \lambda+x\big) \leqslant \sum_{n=1}^{\infty} 2^n \exp\big(-A\lambda^2\, 2^{n(1-2\delta)/8t}\big) \leqslant c_1 \exp(-A\lambda^2/16t). \ ■$$

THEOREM 3.8. *Let $\gamma > 0$. There exist positive constants β, C_1, C_2 such that, for every nonnegative a, t and λ and every $0 \leqslant \alpha \leqslant 1/2$,*

$$E_a \exp\big(-\lambda^{1+\gamma/2} \int_0^t b_\alpha^\gamma(s)\, ds\big) \leqslant C_1 \exp\big(-C_2(\lambda t)^\beta\big). \tag{3.2}$$

Moreover, let $D > 0$. Then, for every $a \geqslant 0$,

$$E_a\big(\int_0^t b_\alpha^\gamma(s)\, ds\big)^{-D} \leqslant Ct^{-D(1+\gamma/2)}, \tag{3.3}$$

where $C = C(D, \gamma, \beta)$.

Proof. At first, notice that it is enough to prove (3.2) for $\lambda = 1$. Indeed, assuming (3.2) for $\lambda = 1$, by Lemma 3.1 we can write

$$C_1 \exp\left(-C_2 (\lambda t)^{\beta}\right) \geqslant E_{\lambda^{1/2} a} \exp\left(-\int_0^{\lambda t} b_\alpha^\gamma(s)\, ds\right) = E_a \exp\left(-\int_0^{\lambda t} \lambda^{\gamma/2} b_\alpha^\gamma(s/\lambda)\, ds\right)$$

$$= E_a \exp\left(-\lambda^{1+\gamma/2} \int_0^t b_\alpha^\gamma(s)\, ds\right).$$

It is clear that it is enough to consider only $t \geqslant 1$.

Let $|\cdot|$ denote the Lebesgue measure on the real line. For a fixed but arbitrary $\alpha \in [0, 1/2]$ divide the set of all trajectories of the Bessel process b_α into three subsets

$$\Omega_1^\alpha = \{b_\alpha\colon \max_{s \in [0,t]} b_\alpha(s) \leqslant R\},$$

$$\Omega_2^\alpha = \{b_\alpha\colon \max_{s \in [0,t]} b_\alpha(s) > R,\ |\{s\colon b_\alpha(s) > R/2\}| \geqslant R^{-\xi/2}\},$$

$$\Omega_3^\alpha = \{b_\alpha\colon \max_{s \in [0,t]} b_s > R,\ |\{s\colon b_s > R/2\}| < R^{-\xi/2}\},$$

where $\xi > 0$ and $R = R(t) \geqslant 1$ will be chosen later.

First we consider the set Ω_1^α. By Lemma 3.5 we get

$$E_a \exp\left(-\int_0^t b_\alpha^\gamma(s)\, ds\right) 1_{\Omega_1^\alpha} \leqslant P_a(\Omega_1) \leqslant \exp\left(-\varepsilon t^{1/2}/R\right), \tag{3.4}$$

providing that $R/t^{1/2} \leqslant \delta_0$, where δ_0 is as in the proof of Lemma 3.5.

On the set Ω_2^α we have

$$E_a \exp\left(-\int_0^t b_\alpha^\gamma(s)\, ds\right) 1_{\Omega_2^\alpha} \leqslant E_a \exp\left(-R^{\gamma-\varepsilon/2}\right) = \exp\left(-R^{\gamma-\xi/2}\right). \tag{3.5}$$

At last, for Ω_3^α we have

$$E_a \exp\left(-\int_0^t b_\alpha^\gamma(s)\, ds\right) 1_{\Omega_3^\alpha} \leqslant P_a(\Omega_3^\alpha).$$

We define a stopping time $T = \inf\{s\colon b_\alpha(s) = R\}$. Then using the strong Markov property and the inequality $|\{s\colon b_\alpha(s) > R/2\}| < R^{-\xi/2}$ we have

$$P_a(\Omega_3^\alpha) \leqslant P_a\left(\inf_{s \in [T, T+R^{-\xi/2}]} b_\alpha(s) < R/2\right) = P_a E_{b_T} 1_{\{b_\alpha : \inf_{s \in [T, T+R^{-\xi/2}]} b_\alpha(s) < R/2\}}(b_\alpha)$$

$$\leqslant E_R 1_{\{b_\alpha : \inf_{s \in [0, R^{-\xi/2}]} b_\alpha(s) < R/2\}}(b_\alpha) = P_R\left(\inf_{s \in [0, R^{-\xi/2}]} b_\alpha(s) < R/2\right).$$

By Lemma 3.6 we get

$$P_R\left(\inf_{s \in [T, T+R^{-\xi/2}]} b_\alpha(s) < R/2\right) \leqslant c_1 \exp\left(-c_2 R^2/R^{-\xi/2}\right)$$

independent of $\alpha\in[0, 1/2]$ if $R^{-\xi/2}\leqslant 1$. Thus,

$$E_a\exp\Big(-\int_0^t b_\alpha^\gamma(s)\,ds\Big)1_{\Omega_3^\alpha}(b_\alpha)\leqslant C_1\exp\big(-C_2R^{2+\xi/2}\big). \tag{3.6}$$

Now, taking e.g. $\xi=\gamma$, and $R=t^{1/4}$, by (3.4)–(3.6) we get inequality (3.2).

Now we are going to prove inequality (3.3). We make use of (3.2).

Let ξ be such that $\lambda^{1+\gamma/2}=\xi^{1/D}$. Integrating the left-hand side of (3.2) with respect to ξ over $(0, +\infty)$ we get

$$\begin{aligned}\int_0^\infty E_a\exp\Big(-\xi^{1/D}\int_0^t b_\alpha^\gamma(s)\,ds\Big)d\xi &= E_a\int_0^\infty\exp\Big(-\xi^{1/D}\int_0^t b_\alpha^\gamma(s)\,ds\Big)d\xi\\ &= E_a\int_0^\infty\exp\Big(-\big[\xi\big(\int_0^t b_\alpha^\gamma(s)\,ds\big)^D\big]^{1/D}\Big)d\xi\\ &= E_a\Big(\int_0^t b_\alpha^\gamma(s)\,ds\Big)^{-D}\int_0^\infty\exp\big(-u^{1/D}\big)\,du\leqslant CE_a\Big(\int_0^t b_\alpha^\gamma(s)\,ds\Big)^{-D}.\end{aligned}\tag{3.7}$$

Similarly, integrating the right-hand side of (3.2) with respect to ξ over $(0, +\infty)$ and changing variables, $u=t\xi^{[D(1+\gamma/2)]^{-1}}$, we get

$$\begin{aligned}&C_1\int_0^\infty\exp\Big(-C_2\big(t\xi^{[D(1+\gamma/2)]^{-1}}\big)^\beta\Big)d\xi\\ &\quad=C_1D(1+\gamma/2)\,t^{-D(1+\gamma/2)}\int_0^\infty\exp\big(-C_2u^\beta\big)\,u^{D(1+\gamma/2)-1}\,du\leqslant Ct^{-D(1+\gamma/2)}.\end{aligned}\tag{3.8}$$

Now (3.2), (3.7) and (3.8) give us (3.3). ∎

4. Proof of Theorem 1.2. In the proof of Theorem 1.2 we make use of a probabilistic method introduced in [4] and then continued among others in [5], [6] and [15].

As in the previous papers [5] and [15], along with the operator $\mathscr{L}=\mathscr{L}_\gamma$, $\gamma\in\boldsymbol{R}$, defined in (1.1), we consider the corresponding operator on $N\times\boldsymbol{R}^+$:

$$L=L_\gamma=a^{-2}\mathscr{L}_\gamma=a^{-2}\sum_j\Phi_a(X_j)^2+a^{-2}\Phi_a(X)+\partial_a+\frac{1-\gamma}{a}\partial_a. \tag{4.1}$$

On $N\times\boldsymbol{R}^+$ we define *dilations*:

$$D_t(x, a)=\big(\Phi_t(x), ta\big),\qquad t>0.$$

It is easy to check that although the operator (4.1) is not left-invariant but is *homogeneous of degree* 2, i.e.

$$L_\gamma(f\circ D_t)=t^2L_\gamma f\circ D_t. \tag{4.2}$$

Let $\gamma \geqslant 0$. The Green function $G_{-\gamma}$ for $L_{-\gamma}$ is given by

$$G_{-\gamma}(x, a; y, b) = \int_0^\infty p_t(x, a; y, b)\,dt, \tag{4.3}$$

where $T_t f(x, a) = \int f(y, b) p_t(x, a; y, b)\,dy b^{1+\gamma} db$ is the heat semigroup on $L^2(N \times \boldsymbol{R}^+, dy b^{1+\gamma} db)$ with the infinitesimal generator $L_{-\gamma}$. In (4.3) we allow (y, b) to be $(e, 0)$ since

$$\lim_{(y,b)\to(e,0)} G_{-\gamma}(x, a; y, b) =: G_{-\gamma}(x, a; e, 0)$$

exists [5].

Notice that (4.2) implies that

$$G_{-\gamma}(x, a; y, b) = t^{-Q} G_{-\gamma}\big(D_{t^{-1}}(x, a); D_{t^{-1}}(y, b)\big). \tag{4.4}$$

It turns out (see [5]) that the Poisson kernel m_γ, $\gamma \geqslant 0$, can be expressed in terms of the Green function:

$$m_\gamma(x) = G_{-\gamma}(x^{-1}, 1; e, 0) = G^*_{-\gamma}(e, 0; x^{-1}, 1), \qquad \gamma \geqslant 0, \tag{4.5}$$

where $G^*_{-\gamma}$ is the Green function for the operator

$$L^*_{-\gamma} = a^{-2} \sum \Phi_a(X_j)^2 - a^{-2} \Phi_a(X) + \partial_a^2 + \frac{1+\gamma}{a} \partial_a$$

conjugate to $L_{-\gamma}$ with respect to the measure $dx\mu(da) = dx a^{1+\gamma} da$. Moreover, it has been proved in [5] that

$$G_{-\gamma}(x, a; e, 0) = \lim_{\eta\to 0} \int_0^\infty E_0\, p^\sigma(t, 0)(x)\, \mu([a-\eta, a+\eta])^{-1}\, 1_{[a-\eta, a+\eta]}\big(\sigma(t)\big)\,dt, \tag{4.6}$$

where the expectation is taken with respect to the distribution of the Bessel process $\sigma(t)$ with the parameter $\alpha = \gamma/2$ and $p^\sigma(t, s)$ are smooth probability densities of the evolution generated by the operator (2.4).

Sketch of the proof of Theorem 1.2. We are going to modify the proof of Theorem 1.1 from [5]. The idea in [5] of the proof of Theorem 1.1 was as follows.

Using the explicit formula (4.6) for $G_{-\gamma}(x, a; e, 0) = m_\gamma(x^{-1})$ we try to show that there exists a positive constant C which may depend on γ, so let us call it C_γ, such that

$$C_\gamma^{-1} \leqslant G_{-\gamma}(x, a; y, 0) \leqslant C_\gamma \tag{4.7}$$

for every (x, a) from the set $\{(x, a)\colon |x| + a = 1\}$. In order to do this we have used Theorem 2.2 and, by some standard techniques such as stopping times and properties of the Bessel process, we succeeded in proving (4.7). When (4.7) is established, the rest is just an application of dilations D_t and the homogeneity of the Green function (4.4).

All the above were done in [5]. A little bit earlier but only the upper bound in (1.2) has been proved in [16].

Now we are going to show how to prove Theorem 1.2. In fact, everything what is needed to do this has already been prepared in Section 2. All lemmas for the Bessel process are "uniform" provided that parameters α are in $[0, 1/2]$. Recall that "uniform" means that constants do not depend on α between 0 and 1/2. Therefore we can now rewrite the proof from [5] replacing appropriate probabilistic "nonuniform" lemmas and theorems by their "uniform" versions from Section 2 of this paper and get lower and upper bounds in (4.7) with C_γ's which do not depend on $0 \leqslant \gamma = 2\alpha \leqslant 1$. Now, as in [5], we use the homogeneity argument to complete the proof. (If we are interested only in the upper bound in Theorem 1.2 we can use the proof from [16], which seems to be a little bit longer but maybe more elementary.) ∎

5. Proof of Theorem 1.3. The main tool in the proof of Theorem 1.3 is the following theorem, being the analogue of Theorem 2.8 in [3], which gives us the behavior of the Green function $\mathscr{G} = \mathscr{G}_0$ for the operator $\mathscr{L}_0$ on the Φ-chain.

THEOREM 5.1. *Given a Φ-chain with points $s_1, \ldots, s_m$ there exists a positive constant C such that for every k $(1 < k < m)$*

$$C^{-1}\mathscr{G}(s_m, s_k)\mathscr{G}(s_k, s_1) \leqslant \mathscr{G}(s_m, s_1) \leqslant C\mathscr{G}(s_m, s_k)\mathscr{G}(s_k, s_1). \tag{5.1}$$

Proof. It has been proved in [15] that for every $0 < \delta < 1/2$ there exists a positive constant C such that, for every $xa \notin T_\delta := \{xa\colon\ 1-\delta < a < 1+\delta,\ |x| < \delta\}$,

$$C^{-1}w(xa) \leqslant \mathscr{G}(xa) \leqslant Cw(xa), \tag{5.2}$$

where

$$w(xa) = \begin{cases} 1 & \text{if } |x| \leqslant 1,\ a \leqslant 1 \quad \text{(region I)},\\ |x|^{-Q} & \text{if } |x| \geqslant 1,\ |x| \geqslant a \quad \text{(region II)},\\ a^{-Q} & \text{if } a \geqslant 1,\ a \geqslant |x| \quad \text{(region III)},\end{cases}$$

and $Q = \sum d_j = \sum \operatorname{Re}\lambda_j$.

By (2.2), inequality (5.1) is equivalent to

$$C^{-1}\mathscr{G}(s)\mathscr{G}(t) \leqslant \mathscr{G}(u) \leqslant C\mathscr{G}(s)\mathscr{G}(t), \tag{5.3}$$

where $s = s_k^{-1}s_m$, $t = s_1^{-1}s_k$, $u = s_1^{-1}s_m$.

Depending on the number of points in the regions I, II and III we have to consider 10 cases which we have indicated in the table below.

Case	1	2	3	4	5	6	7	8	9	10
Region I	3	2	2	1	1	1	0	0	0	0
Region II	0	1	0	2	0	1	3	0	2	1
Region III	0	0	1	0	2	1	0	3	1	2

In fact, there are more cases to consider because sometimes we have to take into account permutations of the set $\{s, t, u\}$. Since the proofs of some cases are very similar to each other, we are going to prove only some of them. Namely, these ones which are representative of all the methods sufficient to deal with the remaining cases.

Case 1. By (5.2),

$$C^{-3}\mathcal{G}(s)\mathcal{G}(t) \leqslant C^{-1} \leqslant \mathcal{G}(u) \leqslant C \leqslant C^3\mathcal{G}(s)\mathcal{G}(t).$$

Case 7. By (1.3), (1.4) and the property of the Riemannian distance, for every $1 \leqslant i \leqslant j \leqslant m$ we have

$$c_0^{-1} \leqslant \tau(s_j, \partial V_{j-1}) \leqslant \tau(s_j, s_i) \leqslant \sum_{k=i}^{j} \tau(s_{k+1}, s_k) \leqslant m\check{c}_0. \tag{5.4}$$

Now it follows from (2.3) that there exists a constant D such that, for $p = s, t, u$, $|\pi_N(p)| \leqslant D$, where π_N denotes the projection from S onto N. Therefore, using (5.2) we can write

$$\begin{aligned} C^{-3}D^{-Q}\mathcal{G}(s)\mathcal{G}(t) &\leqslant C^{-1}D^{-Q}|\pi_N(s)|^{-Q}|\pi_N(t)|^{-Q} \leqslant C^{-1}D^{-Q} \\ &\leqslant C^{-1}|\pi_N(u)|^{-Q} \leqslant \mathcal{G}(u), \\ \mathcal{G}(u) \leqslant C|\pi_N(u)|^{-Q} &\leqslant C = (C^3D^{2Q})C^{-2}D^{-2Q} \leqslant C^3D^{2Q}\mathcal{G}(s)\mathcal{G}(t). \end{aligned}$$

Case 8. Notice that (5.4) is still valid. Hence there exists a constant D such that, for $p = s, t, u$, $D^{-1} \leqslant \tau(p) \leqslant D$. Therefore there exists δ such that p does not belong to T_δ. Moreover, by (2.3), this implies that there exists a constant D such that $D^{-1} \leqslant \pi_A(p) \leqslant D$, where π_A denotes the projection of S onto A. In order to prove (5.3) we have to use (5.2) and proceed similarly to the previous case.

Case 6. Suppose that s, t and u are in the regions I, III and II, respectively. By (5.4) there is a constant C such that, for $p = s, t, u$, $D^{-1} \leqslant \tau(p) \leqslant D$. Therefore, there is a constant D such that $D^{-1} \leqslant \pi_A(p), |\pi_N(p)| \leqslant D$. Using (5.2) we can write

$$C^{-3}D^{-2Q}\mathcal{G}(s)\mathcal{G}(t) \leqslant C^{-1}D^{-2Q}\pi_A(t)^{-Q} \leqslant C^{-1}D^{-Q} \leqslant C^{-1}|\pi_N(u)|^{-Q} \leqslant \mathcal{G}(u),$$

$$\begin{aligned} \mathcal{G}(u) \leqslant C|\pi_N(u)|^{-Q} &\leqslant CD^{Q} = (C^3D^{2Q})C^{-1}(C^{-1}D^{-Q}) \\ &\leqslant C^3D^{2Q}\mathcal{G}(s)C^{-1}\pi_A(t)^{-Q} \leqslant C^3D^{2Q}\mathcal{G}(s)\mathcal{G}(t). \end{aligned}$$

Other locations of the points s, t, u are considered similarly.

To sum up, in order to prove inequality (5.3) in the remaining cases we proceed as in the above cases. The general strategy is as follows. We notice that inequality (5.4) is satisfied. This implies that $\pi_A(p)$ and $|\pi_N(p)|$ for $p = s, t, u$ are bounded from above and below. Then we use the estimate (5.2) for the Green function $\mathcal{G}$. ∎

Now we are able to prove Theorem 1.3.

Proof of Theorem 1.3. We have just to adopt the approach from [1] to our setting.

Let u be a potential which dominates f on $\bar{V}_2^c$. Thus there exists a positive measure μ on ∂V_2 such that $u(s) = \int_{\partial V_2} \mathscr{G}(s, t)\, d\mu(t)$ for every $s \in \bar{V}_2^c$. By Theorem 5.1 and the Riesz Representation Theorem, for every $s \in \partial V_1$ we obtain

$$f(s) \leqslant u(s) \leqslant C \int_{\partial V_2} \mathscr{G}(s, p)\mathscr{G}(p, t)\, d\mu(t) \tag{5.5}$$
$$\leqslant C\mathscr{G}(s, p)u(p) = C\mathscr{G}(s, p)f(p).$$

By assumptions on s and p and by (5.2), $\mathscr{G}(s, p)$ is bounded. Therefore (5.5) implies that

$$f(s) \leqslant Cf(p) \tag{5.6}$$

on ∂V_1, and so on V_1^c. On the other hand, the inequality

$$g(s) \geqslant Dg(p) \tag{5.7}$$

on V_1^c follows from the Harnack inequality. Combining (5.6) with (5.7) we get Theorem 1.3. ∎

REFERENCES

[1] A. Ancona, *Negatively curved manifolds, elliptic operators, and the Martin boundary*, Ann. of Math. 125 (1987), pp. 495–536.

[2] A. N. Borodin and P. Salminen, *Handbook of Brownian Motion — Facts and Formulae*, Birkhäuser Verlag, 1996.

[3] E. Damek, *Fundamental solutions of differential operators on homogeneous manifolds of negative curvature and related Riesz transforms*, Colloq. Math. 73 (2) (1997), pp. 229–249.

[4] E. Damek and A. Hulanicki, *Maximal functions related to subelliptic operators invariant under an action of nilpotent subgroup*, Studia Math. 103 (3) (1992), pp. 239–264.

[5] E. Damek, A. Hulanicki and R. Urban, *Martin boundary for homogeneous Riemannian manifolds of negative curvature at the bottom of the spectrum*, Rev. Mat. Iberoamericana (to appear).

[6] E. Damek, A. Hulanicki and J. Zienkiewicz, *Estimates for the Poisson kernels and their derivatives on rank one NA groups*, Studia Math. 126 (2) (1997), pp. 115–148.

[7] L. Élie, *Comportement asymptotique du noyau potentiel sur les groupes de lie*, Ann. Scient. École Norm. Sup. (4) (1982), pp. 257–364.

[8] G. B. Folland and E. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton Univ. Press, 1982.

[9] Y. Guivarc'h, *Sur la loi des grands nombres et la rayon spectral d'une marche aléatoire*, in: *Journées sur les marches aléatoires*, Vol. 74, Astérisque, 1980, pp. 47–98.

[10] E. Heintze, *On homogeneous manifolds of negative curvature*, Math. Ann. 211 (1974), pp. 23–34.

[11] N. N. Lebedev, *Special Functions and Their Applications*, Dover Publications, Inc., 1972.

[12] R. Penney and R. Urban, *Unbounded harmonic functions on homogeneous manifolds of negative curvature*, Colloq. Math. (to appear).

[13] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, 1991.

[14] H. Tanabe, *Equations of Evolutions*, Pitman, London 1979.

[15] R. Urban, *Noncoercive differential operators on homogeneous manifolds of negative curvature and their Green functions*, Colloq. Math. 88 (1) (2001), pp. 121–134.

[16] R. Urban, *Estimates for the Poisson kernels on NA groups. A probabilistic method*, PhD Thesis, Wrocław University, 1999.

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