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LOCATIONS OF EXTREME VALUES OF THE EMPIRICAL PROCESS

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Abstract. Let α_n be a uniform empirical process and μ_n (respectively, ν_n) the unique location of its maximum (respectively, minimum). We establish a "liminf" iterated logarithm law for $|\mu_n - \nu_n|$.

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1. Introduction. Let $\{U_i\}_{i \geq 1}$ be a sequence of independent variables uniformly distributed on (0, 1). Consider the associated empirical process

$$\alpha_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\mathbb{1}_{U_i \leq t} - t), \quad 0 \leq t \leq 1. \tag{1.1}$$

For each $t \in (0, 1)$, we define

$$\mu_n = \inf\{0 \leq t \leq 1 \colon \alpha_n(t) = \sup_{0 \leq s \leq 1} \alpha_n(s)\}, \tag{1.2}$$

$$\nu_n = \inf\{0 \leq t \leq 1 \colon \alpha_n(t) = \inf_{0 \leq s \leq 1} \alpha_n(s)\}. \tag{1.3}$$

In words, μ_n and ν_n denote locations of the maximum and the minimum, respectively, of the empirical process over [0, 1]. We are interested in $|\mu_n - \nu_n|$, the time difference between the locations of the maximum and the minimum of $\alpha_n(t)$. It is easily seen that $\liminf_{n\to\infty} |\mu_n - \nu_n| = 0$ almost surely (a.s.). A natural question is to find the rate of growth $f(n)$ of the time difference, so that

$$\liminf_{n\to\infty} f(n)\,|\mu_n - \nu_n| = 1 \text{ a.s.}$$

Our result determines the exact rate of growth of the "liminf".

THEOREM 1. *Let μ_n and ν_n be defined as in* (1.2) *and* (1.3), *respectively. Then*

$$\liminf_{n\to\infty} \log_2^2(n)\,|\mu_n - \nu_n| = \pi^2 \quad a.s.$$

Let us say a few words about our method. Recall that a Kiefer process $\{K(t, n), 0 \leqslant t \leqslant 1, n \geqslant 0\}$ is a mean-zero Gaussian process with covariance

$$E\big(K(t, n)K(s, m)\big) = \big(\min(t, s) - ts\big)\min(n, m).$$

Our basic tool is the following strong approximation theorem due to Komlós, Major and Tusnády [3] (see also Csörgő and Révész [1], p. 141): after possible redefinitions of variables, there exists a coupling for $\alpha_n(t)$ and the Kiefer process $K(t, n)$, so that

$$\sup_{0 \leqslant t \leqslant 1} \left|\alpha_n(t) - \frac{1}{\sqrt{n}} K(t, n)\right| = O\left(\frac{\log^2 n}{\sqrt{n}}\right) \text{ a.s.} \tag{1.4}$$

Although (1.4) does not indicate how close the locations of the maxima of $\alpha_n(t)$ and $K(t, n)$ are to each other, our method, which is based on fine analysis of the sample paths of the Brownian bridge, reveals that accurate knowledge upon the location of the maximum (respectively, minimum) of the Kiefer process yields useful information upon μ_n (respectively, ν_n). This was also observed in Shi [4] in the study of the almost sure asymptotics of μ_n.

The lower bound in Theorem 1 is proved in Section 2 and the upper bound in Section 3.

Throughout the paper, $C > 1$ and $\tilde{C} > 1$ denote constants, $C_\varepsilon > 1$ and $\tilde{C}_\varepsilon > 1$ denote constants which only depend on ε. Their values may vary from one line to another (but not within the same line).

2. Proof of the lower bound in Theorem 1. The main ingredient in the proof of the lower bound in Theorem 1 is the following estimate:

Lemma 1. *Let $\{B(t), 0 \leqslant t \leqslant 1\}$ be a standard linear Brownian bridge, and define, for* $0 < u < 1$,

$$E \stackrel{\text{def}}{=} E(u) = \{\exists x \in [0, 1-u],\ \sup_{x \leqslant t \leqslant x+u} B(t) - \inf_{x \leqslant t \leqslant x+u} B(t) \geqslant B_R - u^2\},$$

where $B_R = \sup_{0 \leqslant t \leqslant 1} B(t) - \inf_{0 \leqslant t \leqslant 1} B(t)$. *Then, for any* $0 < \varepsilon < 1$, *we have*

$$P(E) \leqslant C_\varepsilon \exp\left(-\frac{(1-\varepsilon)\pi}{\sqrt{u}}\right).$$

Proof. We only need to consider small u. Recall that the Brownian bridge can be realized as $\{B(t) = W(t) - tW(1)\}_{0 \leqslant t \leqslant 1}$, where W is a standard Brownian motion. Moreover, $\{B(t)\}_{0 \leqslant t \leqslant 1}$ is independent of $W(1)$. Therefore

$$P(E) = P(E, |W(1)| < u)/P(|W(1)| < u). \tag{2.1}$$

In the event $\{|W(1)| < u\}$,

$$\sup_{x \leqslant t \leqslant x+u} B(t) - \inf_{x \leqslant t \leqslant x+u} B(t) \leqslant \sup_{x \leqslant t \leqslant x+u} W(t) - \inf_{x \leqslant t \leqslant x+u} W(t) + 2u^2,$$

and $B_R \leqslant W_R + u^2$, $B_R \geqslant W_R - u^2$, where $W_R = \sup_{0\leqslant t\leqslant 1} W(t) - \inf_{0\leqslant t\leqslant 1} W(t)$. Hence

$$\begin{aligned}(2.2)\quad & \boldsymbol{P}(E, |W(1)| < u) \\ & \leqslant \boldsymbol{P}\big(\exists x \in [0, 1-u],\ \sup_{x\leqslant t\leqslant x+u} W(t) - \inf_{x\leqslant t\leqslant x+u} W(t) \geqslant W_R - 3u^2\big) \\ & = \boldsymbol{P}\big(\sup_{0\leqslant s\leqslant 1-u}\ \sup_{0\leqslant t\leqslant u} |W(s+t) - W(s)| \geqslant W_R - 3u^2\big).\end{aligned}$$

Let us fix an integer r so that $u^{-1} < 2^r < 2u^{-1}$, and let, for any positive number s, $(s)_r = [2^r s]/2^r$. Then

$$|W(s+t) - W(s)| \leqslant \big|W\big((s+t)_r\big) - W\big((s)_r\big)\big| + \sum_{j=0}^{\infty} \big|W\big((s)_{r+j+1}\big) - W\big((s)_{r+j}\big)\big| + \sum_{j=0}^{\infty} \big|W\big((s+t)_{r+j+1}\big) - W\big((s+t)_{r+j}\big)\big|.$$

It follows from (2.2) that

$$\begin{aligned}& \boldsymbol{P}(E, |W(1)| < u) \\ & \leqslant \boldsymbol{P}\big(\sup_{0\leqslant s\leqslant 1-u}\ \sup_{0\leqslant t\leqslant u} \big|W\big((s+t)_r\big) - W\big((s)_r\big)\big| \geqslant (1-\varepsilon)(W_R - 3u^2)\big) \\ & \quad + \boldsymbol{P}\big(2 \sup_{0\leqslant s\leqslant 1-u}\ \sup_{0\leqslant t\leqslant u} \sum_{j=0}^{\infty} \big|W\big((s+t)_{r+j+1}\big) - W\big((s+t)_{r+j}\big)\big| \geqslant \varepsilon (W_R - 3u^2)\big).\end{aligned}$$

Let us define a constant γ so that $0 < \gamma < (\pi/8)^{1/2}$. Hence, if $W_R > \gamma u^{1/4}$, we have $\varepsilon(W_R - 3u^2) \geqslant \varepsilon W_R/2$, so that

$$\begin{aligned}(2.3)\quad & \boldsymbol{P}(E, |W(1)| < u) \\ & \leqslant \boldsymbol{P}\big(\sup_{0\leqslant s\leqslant 1-u}\ \sup_{0\leqslant t\leqslant u} \big|W\big((s+t)_r\big) - W\big((s)_r\big)\big| \geqslant (1-\varepsilon)(W_R - 3u^2)\big) \\ & \quad + \boldsymbol{P}\big(\sup_{0\leqslant s\leqslant 1-u}\ \sup_{0\leqslant t\leqslant u} \sum_{j=0}^{\infty} \big|W\big((s+t)_{r+j+1}\big) - W\big((s+t)_{r+j}\big)\big| \geqslant \varepsilon W_R/4,\ W_R > \gamma u^{1/4}\big) \\ & \quad + \boldsymbol{P}(W_R < \gamma u^{1/4}) \\ & \stackrel{\text{def}}{=} \Delta_1 + \Delta_2 + \Delta_3.\end{aligned}$$

We first estimate Δ_1. Let us define $t_i = i/2^r$ for any integer i, so that $0 \leqslant i \leqslant 2^r - 2$. Since $2^{-r} \leqslant u \leqslant 2^{1-r}$, from the Markov property it follows that

$$\begin{aligned}(2.4)\quad \Delta_1 & = \boldsymbol{P}\Big(\bigcup_{i=0}^{2^r-2} \bigcup_{j=0}^{1} |W(t_{i+j+1}) - W(t_i)| \geqslant (1-\varepsilon)(W_R - 3u^2)\Big) \\ & \leqslant \sum_{i=0}^{2^r-2} \boldsymbol{P}\big(X_1 < (1-\varepsilon)^{-1} X_2 + 3u^2,\ X_3 < (1-\varepsilon)^{-1} X_2 + 3u^2\big) \stackrel{\text{def}}{=} \sum_{i=0}^{2^r-2} P_i,\end{aligned}$$

where

$$X_1 = \sup_{0 \leqslant t \leqslant t_i} |W(t)|, \quad X_2 = \sup_{t_i \leqslant t \leqslant t_i + u} (W(t) - W(t_i)) - \inf_{t_i \leqslant t \leqslant t_i + u} (W(t) - W(t_i)),$$

$$X_3 = \sup_{t_i + u \leqslant t \leqslant 1} |W(1) - W(t)|.$$

Recall that (see Shorack and Wellner [5], p. 34)

$$\text{(2.5)} \quad P\Big(\sup_{0 \leqslant t \leqslant 1} |W(t)| < x\Big) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \exp\Big(-\frac{(2k-1)^2 \pi^2}{8x^2}\Big), \quad x > 0.$$

Consequently, for any $x > 0$,

$$P(W_R < x) \leqslant P\Big(\sup_{0 \leqslant t \leqslant 1} |W(t)| < x\Big) \leqslant \frac{4}{\pi} \exp\Big(-\frac{\pi^2}{8x^2}\Big).$$

Hence, by putting

$$\alpha_i = \alpha_i(u) = \frac{1}{t_i} + \frac{1}{1 - t_i - u},$$

and conditioning upon X_2 combined with the independence between X_1, X_2 and X_3, it follows that

$$P_i \leqslant \frac{16}{\pi^2} E\Big(\exp\Big(-\frac{\pi^2 \alpha_i (1-\varepsilon)^2}{8(X_2 + 3(1-\varepsilon) u^2)^2}\Big)\Big)$$

$$\leqslant \frac{16}{\pi^2} E\Big(\exp\Big(-\frac{\pi^2 (1-\varepsilon)^2 \alpha_i}{8(1+\varepsilon)^2 X_2}\Big)\Big) + \frac{16}{\pi^2} P\Big(X_2 < \frac{3u^2}{(1-\varepsilon)\varepsilon}\Big)$$

$$= \frac{16}{\pi^2} \exp\Big(-\frac{\pi(1-\varepsilon)\sqrt{\alpha_i}}{2(1+\varepsilon)\sqrt{u}}\Big) + \frac{16}{\pi^2} P\Big(X_2 < \frac{3u^2}{(1-\varepsilon)\varepsilon}\Big).$$

Another application of (2.5) combined with the fact that $\alpha_i > 4$ (for any i) yields that

$$P_i \leqslant \frac{16}{\pi^2} \exp\Big(-\frac{\pi(1-\varepsilon)}{2(1+\varepsilon)\sqrt{u}}\Big) + \frac{64}{\pi^3} \exp\Big(-\frac{\pi^2 \varepsilon^2}{72u^3}\Big).$$

Putting this into (2.4) we obtain

$$\text{(2.6)} \qquad \Delta_1 \leqslant C \exp\Big(-\frac{\pi(1 - 3\varepsilon/2)}{\sqrt{u}}\Big).$$

To estimate Δ_2 note that $\sup_{0 < t \leqslant u} |(s+t)_{r+j+1} - (s+t)_{r+j}| \leqslant 2^{-(r+j+1)}$ and that $W((s+t)_{r+j+1}) - W((s+t)_{r+j})$ is a Gaussian variable with variance

$(s+t)_{r+j+1}-(s+t)_{r+j}$. For any random $x_j > 1$, we have

$$(2.7)\quad P\Big(\sup_{0\leqslant s\leqslant 1-u}\ \sup_{0\leqslant t\leqslant u}\big|W\big((s+t)_{r+j+1}\big)-W\big((s+t)_{r+j}\big)\big| \geqslant \frac{x_j}{\sqrt{2^{r+j+1}}}\Big)$$

$$\leqslant \sum_{i=0}^{2^{r+j+1}-1} P\Big(|W(t'_{i+1})-W(t'_i)| \geqslant \frac{x_j}{\sqrt{2^{r+j+1}}}\Big)$$

$$\leqslant 2^{r+j+1}\max_i P\Big(|W(t'_{i+1})-W(t'_i)| \geqslant \frac{x_j}{\sqrt{2^{r+j+1}}}\Big),$$

where $t'_i = i/2^{r+j+1}$. Let us define

$$c_1 = \sum_{j=0}^{\infty}\sqrt{\frac{j}{2^{j+1}}},\qquad c_2 = \sum_{j=0}^{\infty} 2^{-(j+1)/2},\qquad \beta_1 = (\gamma/8c_1)^2,\qquad \beta_2 = (8c_2)^{-2}.$$

We choose now

$$x_j = \varepsilon\Big(\beta_1 j + \frac{\beta_2}{u} W_R^2\Big)^{1/2}.$$

Let us define $Y_1 = \sup_{0\leqslant t\leqslant t'_i}|W(t)|$ and $Y_2 = \sup_{t'_{i+1}\leqslant t\leqslant 1}|W(t)-W(t'_{i+1})|$. One can mention that W, Y_1 and Y_2 are clearly independent. Furthermore, using the inequality $W_R \geqslant \max(Y_1, Y_2)$ and the Markov property we obtain

$$P\Big(|W(t'_{i+1})-W(t'_i)| \geqslant \frac{x_j}{\sqrt{2^{r+j+1}}}\Big)$$

$$\leqslant P\Big(|W(t'_{i+1})-W(t'_i)| \geqslant \frac{\varepsilon}{\sqrt{2^{r+j+1}}}\Big(\beta_1 j + \frac{\beta_2}{u}\max{}^2(Y_1, Y_2)\Big)^{1/2}\Big)$$

$$\leqslant P\Big(|W(1)| \geqslant \varepsilon\Big(\beta_1 j + \frac{\beta_2}{8u}(Y_1^2+Y_2^2)\Big)^{1/2}\Big).$$

Therefore,

$$(2.8)\quad P\Big(\sup_{0\leqslant s\leqslant 1-u}\ \sup_{0\leqslant t\leqslant u}\sum_{j=0}^{\infty}\big|W\big((s+t)_{r+j+1}\big)-W\big((s+t)_{r+j}\big)\big| \geqslant \sum_{j=0}^{\infty}\frac{x_j}{\sqrt{2^{r+j+1}}},\ W_R > \gamma u^{1/4}\Big)$$

$$\leqslant 2^{r+1}\sum_{j=0}^{\infty} 2^j P\Big(|W(1)| \geqslant \varepsilon\Big(\beta_1 j + \frac{\beta_2}{8u}(Y_1^2+Y_2^2)\Big)^{1/2}\Big) \overset{\text{def}}{=} 2^{r+1}\sum_{j=0}^{\infty} 2^j P'_j.$$

Furthermore, it is easily seen that

$$\sum_{j=0}^{\infty} x_j\frac{1}{\sqrt{2^{r+j+1}}} \leqslant \varepsilon\Big(\sqrt{\beta_1}\sum_{j=0}^{\infty}\sqrt{\frac{j}{2^{r+j+1}}} + \sqrt{\frac{\beta_2}{u}}\,W_R\sum_{j=0}^{\infty}\frac{1}{\sqrt{2^{r+j+1}}}\Big).$$

Since $u^{-1} \leqslant 2^r$, we have

$$\sum_{j=0}^{\infty} x_j \frac{1}{\sqrt{2^{r+j+1}}} \leqslant \varepsilon(c_1\sqrt{\beta_1 u}+c_2\sqrt{\beta_2}\, W_R).$$

In the event $\{W_R > \gamma u^{1/4}\}$, $c_1\sqrt{\beta_1 u} \leqslant W_R/8$. Hence

$$\sum_{j=0}^{\infty} x_j \frac{1}{\sqrt{2^{r+j+1}}} \leqslant \frac{\varepsilon}{4} W_R. \tag{2.9}$$

Let us focus now on the estimate of P'_i. By conditioning upon (Y_1, Y_2) and using Mill's ratio (see, for example, Shorack and Wellner [5], p. 850) combined with the independence between Y_1 and Y_2, we obtain the existence of an absolute constant C so that

$$P'_j \leqslant C\exp\left(-\frac{\varepsilon\beta_1^2 j}{2}\right) E\left(\exp\left(-\frac{\varepsilon\beta_2^2}{8u} Y_1^2\right)\right) E\left(\exp\left(-\frac{\varepsilon\beta_2^2}{8u} Y_2^2\right)\right).$$

Expectations in the previous equation can be easily bounded by using the following statement for any positive number λ:

$$E\left(\exp\left(-\lambda\left(\sup_{0\leqslant t\leqslant 1} |W(t)|\right)^2\right)\right) \leqslant \tilde{C}\int_1^2 \exp(-\lambda x^2)\,dx \leqslant \tilde{C}\exp(-\lambda).$$

Combining this with the scaling property we get

$$P'_j \leqslant C\exp\left(-\frac{\varepsilon\beta_1^2 j}{2}\right)\exp\left(-\frac{\varepsilon\beta_2^2(t'_i+1-t'_{i+1})}{8u}\right).$$

Putting this into (2.7) and using the fact that $t'_{i+1}-t'_i \leqslant 1/2$, we obtain

$$P\left(\sup_{0\leqslant s\leqslant 1-u}\ \sup_{0\leqslant t\leqslant u} |W((s+t)_{r+j+1})-W((s+t)_{r+j})| \geqslant x_j\frac{1}{\sqrt{2^{r+j+1}}},\ W_R > \gamma u^{1/4}\right) \leqslant 2^{r+1}C\exp\left(-\frac{\varepsilon\beta_2^2}{16u}\right)2^j\exp\left(-\frac{\varepsilon\beta_1^2 j}{2}\right).$$

Hence, taking the sum we have

$$P\left(\sup_{0\leqslant s\leqslant 1-u}\ \sup_{0\leqslant t\leqslant u} \sum_{j=0}^{\infty} |W((s+t)_{r+j+1})-W((s+t)_{r+j})| \geqslant \sum_{j=0}^{\infty} x_j\frac{1}{\sqrt{2^{r+j+1}}},\ W_R > \gamma u^{1/4}\right) \tag{2.10}$$
$$\leqslant 2^{r+1}C\exp\left(-\frac{\varepsilon\beta_2^2}{16u}\right)\sum_{j=0}^{\infty} 2^j\exp\left(-\frac{\varepsilon\beta_1^2 j}{2}\right) = A\frac{\tilde{C}_\varepsilon}{u}\exp\left(-\frac{\varepsilon\beta_2^2}{16u}\right),$$

where $A = 2C\sum_{j=0}^{\infty} 2^j\exp(-\varepsilon\beta_1^2 j/2)$.

Combining (2.3), (2.9) and (2.10) we obtain

$$\Delta_2 \leqslant A\frac{\tilde{C}_\varepsilon}{u}\exp\left(-\frac{\varepsilon\beta_2^2}{16u}\right) \leqslant \frac{C_\varepsilon}{u}\exp\left(-\frac{\varepsilon\beta_2^2}{16u}\right). \tag{2.11}$$

As far as Δ_3 is concerned, an application of (2.5) combined with the definition of γ yields

$$\Delta_3 \leqslant \frac{4}{\pi}\exp\left(-\frac{\pi^2}{8\gamma^2\sqrt{u}}\right) \leqslant \frac{4}{\pi}\exp\left(-\frac{\pi}{\sqrt{u}}\right). \tag{2.12}$$

Going back to (2.2) and using (2.3), (2.6), (2.11) and (2.12) we get

$$\begin{aligned} P(E, |W(1)| < u) &\leqslant C\exp\left(-\frac{\pi(1-3\varepsilon/2)}{\sqrt{u}}\right) + \frac{C_\varepsilon}{u}\exp\left(-\frac{\varepsilon\beta_2^2}{16u}\right) + \frac{4}{\pi}\exp\left(-\frac{\pi}{\sqrt{u}}\right) \\ &\leqslant \tilde{C}_\varepsilon\exp\left(-\frac{\pi(1-3\varepsilon/2)}{\sqrt{u}}\right). \end{aligned}$$

Going back to (2.1) and applying the inequality $P(|W(1)| < u) \geqslant u\exp(-u^2/2)/2$, we have

$$P(E) \leqslant \frac{2\tilde{C}_\varepsilon}{u}\exp\left(\frac{u^2}{2}\right)\exp\left(-\frac{\pi(1-3\varepsilon/2)}{\sqrt{u}}\right) \leqslant C_\varepsilon\exp\left(-\frac{\pi(1-2\varepsilon)}{\sqrt{u}}\right).$$

Replacing ε by $\varepsilon/2$ and taking large values for C and C_ε we complete the proof of Lemma 1. ∎

Proof of the lower bound in Theorem 1. Fix a small number $\varepsilon_1 \in (0, 1)$. Then one can easily find a positive number $\varepsilon > 0$ so that

$$\theta_1 \overset{\text{def}}{=} (1-\varepsilon_1)/(1+\varepsilon)(1-5\varepsilon_1)^{1/2} > 1.$$

Let furthermore define, for $k \geqslant 1$ and $m \geqslant 1$,

$$n_k = [\exp(k^{1-\varepsilon_1})]+1, \qquad u(m) = \frac{(1-5\varepsilon_1)\pi^2}{\log_2^2(m)},$$

and δ so that

$$\delta > 12\sqrt{\frac{n_{k+1}-n_k}{n_k}\log k} + 4\frac{\log n}{n^{1/4}}.$$

Let

$$\begin{aligned} E_k = \{\exists x \in [0, 1-u(n_k)], \quad &\sup_{x\leqslant t\leqslant x+u(n_k)} K(t, n_k) - \inf_{x\leqslant t\leqslant x+u(n_k)} K(t, n_k) \\ &\geqslant \sup_{0\leqslant t\leqslant 1} K(t, n_k) - \inf_{0\leqslant t\leqslant 1} K(t, n_k) - \delta\sqrt{n_k}\}. \end{aligned}$$

By means of Lemma 1 (noticing that $\delta \leqslant u^2(n_k)$) we have

$$P(E_k) = P\Big(\exists x \in [0, 1-u(n_k)], \sup_{x \leqslant t \leqslant x+u(n_k)} B(t) - \inf_{x \leqslant t \leqslant x+u(n_k)} B(t) \geqslant B_R - \delta\Big)$$
$$\leqslant C \exp\left(-\frac{\pi(1-\varepsilon)}{\sqrt{u(n_k)}}\right) \leqslant Ck^{-\theta_1},$$

which is summable for k. Then, according to the Borel–Cantelli lemma, almost surely, for k sufficiently large, we have, for any $x \in [0, 1-u(n_k)]$,

$$\sup_{x \leqslant t \leqslant x+u(n_k)} K(t, n_k) - \inf_{x \leqslant t \leqslant x+u(n_k)} K(t, n_k) \leqslant \sup_{0 \leqslant t \leqslant 1} K(t, n_k) - \inf_{0 \leqslant t \leqslant 1} K(t, n_k) - \delta\sqrt{n_k}.$$

At this step of the proof, we need to show that oscillations of the Kiefer process between n_k and n_{k+1} are relatively small. Let $\{W(t, y), 0 \leqslant t \leqslant 1, y \geqslant 0\}$ be a two-parameter Brownian sheet. Then applying Corollary 1.12.4 of Csörgő and Révész [1] to $T = n_{k+1}$ and $a_T = (1-\varepsilon_1) T (\log T)^{-\varepsilon_1/(1-\varepsilon_1)}$ we obtain (noticing that $n_{k+1} - n_k \sim (1-\varepsilon_1) n_{k+1} (\log n_{k+1})^{-\varepsilon_1/(1-\varepsilon_1)}$ as $k \to \infty$)

$$\limsup_{k \to \infty} \big(2(n_{k+1}-n_k)\log k\big)^{-1/2} \max_{n_k \leqslant n \leqslant n_{k+1}} \sup_{0 \leqslant t \leqslant 1} |W(t, n) - W(t, n_k)| \leqslant 1 \text{ a.s.}$$

In particular,

$$\limsup_{k \to \infty} \big(2(n_{k+1}-n_k)\log k\big)^{-1/2} \max_{n_k \leqslant n \leqslant n_{k+1}} |W(1, n) - W(1, n_k)| \leqslant 1 \text{ a.s.}$$

Since the Kiefer process $K(t, n)$ can be realized as $K(t, n) = W(t, n) - tW(1, n)$, combining this with the previous estimates we obtain

$$\limsup_{k \to \infty} \big((n_{k+1}-n_k)\log k\big)^{-1/2} \max_{n_k \leqslant n \leqslant n_{k+1}} \sup_{0 \leqslant t \leqslant 1} |K(t, n) - K(t, n_k)| \leqslant \sqrt{8} \text{ a.s.}$$

Let $n_k \leqslant n \leqslant n_{k+1}$. Then we have, for any $x \in [0, 1-u(n)]$,

$$\sup_{x \leqslant t \leqslant x+u(n)} K(t, n) - \inf_{x \leqslant t \leqslant x+u(n)} K(t, n)$$
$$\leqslant \sup_{x \leqslant t \leqslant x+u(n_k)} K(t, n) - \inf_{x \leqslant t \leqslant x+u(n_k)} K(t, n)$$
$$\leqslant \sup_{x \leqslant t \leqslant x+u(n_k)} K(t, n_k) - \inf_{x \leqslant t \leqslant x+u(n_k)} K(t, n_k) + 2 \sup_{0 \leqslant t \leqslant 1} |K(t, n) - K(t, n_k)|$$
$$\leqslant \sup_{0 \leqslant t \leqslant 1} K(t, n_k) - \inf_{0 \leqslant t \leqslant 1} K(t, n_k) - \delta\sqrt{n_k} + 6\sqrt{(n_{k+1}-n_k)\log k}.$$

Consequently, for all $x \in [0, 1-u(n)]$, we have

$$\sup_{x \leqslant t \leqslant x+u(n)} \alpha_n(t) - \inf_{x \leqslant t \leqslant x+u(n)} \alpha_n(t) - \big(\sup_{0 \leqslant t \leqslant 1} \alpha_n(t) - \inf_{0 \leqslant t \leqslant 1} \alpha_n(t)\big)$$
$$\leqslant n^{-1/2} \Big(\sup_{x \leqslant t \leqslant x+u(n)} K(t, n) - \inf_{x \leqslant t \leqslant x+u(n)} K(t, n)$$

$$-\big(\sup_{0\leqslant t\leqslant 1} K(t,n)-\inf_{0\leqslant t\leqslant 1} K(t,n)\big)\big)+4n^{-1/4}\log n$$

$$\leqslant n_k^{-1/2}\big(-\delta\sqrt{n_k}+12\sqrt{(n_{k+1}-n_k)\log k}\big)+4n_k^{-1/4}\log n_k<0.$$

This yields that $|\mu_n-\nu_n|\geqslant u(n)$. Hence

$$\liminf_{n\to\infty}\log_2^2(n)\,|\mu_n-\nu_n|\geqslant(1-5\varepsilon_1)\,\pi^2.$$

Letting ε tend to 0, and then taking ε_1 close to 0, we complete the proof of the lower bound of Theorem 1. ∎

3. Proof of the upper bound in Theorem 1. The proof of the upper bound is based on the following lemma:

LEMMA 2. *Let $\{B(t),\,0\leqslant t\leqslant 1\}$ be a standard linear Brownian bridge, and define, for $0<u<1$,*

$$F\overset{\text{def}}{=}F(u)=\{\max_{1/2\leqslant t\leqslant 1/2+u} B(t)>\max_{t\notin[1/2,1/2+u]} B(t)+u,$$
$$\min_{1/2\leqslant t\leqslant 1/2+u} B(t)<\min_{t\notin[1/2,1/2+u]} B(t)-u\}.$$

Then, for any $0<\varepsilon<1$, we have

$$P(F)\geqslant C_\varepsilon\exp\left(-\frac{(1+\varepsilon)\pi}{\sqrt{u}}\right).$$

Proof. Let $\{W(t),\ t\geqslant 0\}$ be a Wiener process. Then recalling that $\{B(t)=W(t)-tW(1)\}_{0\leqslant t\leqslant 1}$ we have

$$\max_{1/2\leqslant t\leqslant 1/2+u} B_t\geqslant\max_{1/2\leqslant t\leqslant 1/2+u} W_t-(1/2+u)\,u,$$

$$\max_{t\notin[1/2,1/2+u]} B_t\leqslant\max_{t\notin[1/2,1/2+u]} W_t+u,$$

$$\min_{1/2\leqslant t\leqslant 1/2+u} B_t\leqslant\min_{1/2\leqslant t\leqslant 1/2+u} W_t+(1/2+u)\,u,$$

$$\min_{t\notin[1/2,1/2+u]} B_t\geqslant\min_{t\notin[1/2,1/2+u]} W_t-u.$$

Hence

$$\text{(3.1)}\qquad P\big(F,\,|W(1)|<u\big)\geqslant P\big(\max_{1/2\leqslant t\leqslant 1/2+u} W_t>\max_{t\notin[1/2,1/2+u]} W_t+4u,$$
$$\min_{1/2\leqslant t\leqslant 1/2+u} W_t<\min_{t\notin[1/2,1/2+u]} W_t-4u,\ |W(1)|<u\big).$$

Let $0 < \varepsilon < 1$, $0 < u < u_0$ and $A = A(u) = \pi\sqrt{u}/4$, where $u_0 = u_0(\varepsilon) > 0$, is so small that

$$\varepsilon A > 4u, \tag{3.2}$$

$$2\exp\left(-\frac{2\pi^2}{c^2}\right) \leqslant \frac{\varepsilon u}{4A}\exp\left(-\frac{\pi^2}{2c^2}\right). \tag{3.3}$$

We also define the following measurable events:

$$E_1 = \{-(1+\varepsilon)A < \inf_{0\leqslant t\leqslant 1/2} W(t),\ \sup_{0\leqslant t\leqslant 1/2} W(t) < (1+\varepsilon)A,$$
$$W(1/2)\in[A, (1+\varepsilon)A]\},$$
$$E_2 = \{W(1/2+\varepsilon u)\in[(1+2\varepsilon)A, (1+3\varepsilon)A]\},$$
$$E_3 = \{W(1/2+(1-\varepsilon)u)\in[-(1+3\varepsilon)A, -(1+2\varepsilon)A]\},$$
$$E_4 = \{W(1/2+u)\in[-(1+\varepsilon)A, -A]\},$$
$$E_5 = \{-(1+2\varepsilon)A < \inf_{1/2+u\leqslant t\leqslant 1} W(t),\ \sup_{1/2+u\leqslant t\leqslant 1} W(t) < (1+2\varepsilon)A, |W(1)| < u\}.$$

In view of (3.1) and (3.2), we have

$$P(F, |W(1)| < u) \geqslant P(\bigcap_{i=1}^{5} E_i).$$

Using the Markov property, it is easily seen that

$$P(\bigcap_{i=1}^{5} E_i) = E[E(\mathbb{1}_{\{\bigcap_{i=1}^{5} E_i\}} | \mathscr{F}_{1/2+u})]$$
$$= E[\mathbb{1}_{\{\bigcap_{i=1}^{4} E_i\}} E(\mathbb{1}_{E_5} | \mathscr{F}_{1/2+u})] \geqslant \inf_{x\in[-(1+\varepsilon)A, -A]} P(\tilde{E}_5) P(\bigcap_{i=1}^{4} E_i),$$

where

$$\tilde{E}_5 = \tilde{E}_5(x) = \left\{\frac{-(1+2\varepsilon)A - x}{\sqrt{1/2-u}} < \inf_{0\leqslant t\leqslant 1} W(t),\ \sup_{0\leqslant t\leqslant 1} W(t) < \frac{(1+2\varepsilon)A - x}{\sqrt{1/2-u}},\right.$$
$$\left. W(1)\in\left[\frac{-u-x}{\sqrt{1/2-u}}, \frac{u-x}{\sqrt{1/2-u}}\right]\right\}.$$

Iterating this procedure we obtain

$$P(\bigcap_{i=1}^{5} E_i) \geqslant \inf_{x\in[-(1+\varepsilon)A, -A]} P(\tilde{E}_5) \inf_{x\in[-(1+3\varepsilon)A, -(1+2\varepsilon)A]} P(\tilde{E}_4)$$
$$\times \inf_{x\in[(1+2\varepsilon)A, (1+3\varepsilon)A]} P(\tilde{E}_3) \inf_{x\in[A, (1+\varepsilon)A]} P(\tilde{E}_2) P(E_1),$$

where

$$\tilde{E}_4 = \tilde{E}_4(x) = \left\{W(1) \in \left[\frac{-(1+\varepsilon)A-x}{\sqrt{\varepsilon u}}, \frac{-A-x}{\sqrt{\varepsilon u}}\right]\right\},$$

$$\tilde{E}_3 = \tilde{E}_3(x) = \left\{W(1) \in \left[\frac{-(1+3\varepsilon)A-x}{\sqrt{(1-2\varepsilon)u}}, \frac{-(1+2\varepsilon)A-x}{\sqrt{(1-2\varepsilon)u}}\right]\right\},$$

$$\tilde{E}_2 = \tilde{E}_2(x) = \left\{W(1) \in \left[\frac{(1+2\varepsilon)A-x}{\sqrt{\varepsilon u}}, \frac{(1+3\varepsilon)A-x}{\sqrt{\varepsilon u}}\right]\right\}.$$

Let us begin by estimating $P(\tilde{E}_5)$. For notational convenience we define

$$a = \frac{(1+2\varepsilon)A-x}{\sqrt{1/2-u}}, \tag{3.4}$$

$$b = \frac{-(1+2\varepsilon)A-x}{\sqrt{1/2-u}}, \tag{3.5}$$

$$c = a-b = \frac{2(1+2\varepsilon)A}{\sqrt{1/2-u}}. \tag{3.6}$$

Then, recalling the joint density of the infimum, supremum and the terminal value of Brownian motion (see Itô and McKean [2], p. 31): for all $a > 0$, $b < 0$, $y \in [b, a]$, $c \stackrel{\text{def}}{=} a-b$,

$$\begin{aligned} \frac{1}{dy} P_0\big(b < \inf_{0 \leqslant t \leqslant 1} W(t), \sup_{0 \leqslant t \leqslant 1} W(t) < a, W(1) \in dy\big) \\ = \frac{2}{c} \sum_{k=1}^{\infty} \exp\left(-\frac{k^2\pi^2}{2c^2}\right) \sin\left(\frac{k\pi a}{c}\right) \sin\left(\frac{k\pi(a-y)}{c}\right). \end{aligned} \tag{3.7}$$

Accordingly, we have

$$\begin{aligned} P(\tilde{E}_5) &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \exp\left(-\frac{(2k-1)^2\pi^2}{2c^2}\right) \sin\left(\frac{(2k-1)\pi a}{c}\right) \sin\left(\frac{(2k-1)\pi u}{2(1+2\varepsilon)A}\right) \\ &= \frac{4}{\pi} \exp\left(-\frac{\pi^2}{2c^2}\right) \sin\left(\frac{\pi a}{c}\right) \sin\left(\frac{\pi u}{2(1+2\varepsilon)A}\right) + R, \end{aligned}$$

where

$$R \stackrel{\text{def}}{=} \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{2k-1} \exp\left(-\frac{(2k-1)^2\pi^2}{2c^2}\right) \sin\left(\frac{(2k-1)\pi a}{c}\right) \sin\left(\frac{(2k-1)\pi u}{2(1+2\varepsilon)A}\right).$$

Recalling a and c defined in (3.4) and (3.6), respectively, and given the fact that, for all $x \in [0, \pi/4]$, $\sin(x) \geqslant x/\sqrt{2}$, $a/c \leqslant 1-\varepsilon/2$ and $u/A < \varepsilon$, we have

$$\sin\left(\frac{\pi a}{c}\right) \geqslant \sin\left(\frac{\pi\varepsilon}{2}\right) \geqslant \frac{\pi\varepsilon}{2\sqrt{2}} \quad \text{and} \quad \sin\left(\frac{\pi u}{2(1+2\varepsilon)A}\right) \geqslant \frac{u}{3A}.$$

On the other hand, by (3.3),

$$R \leqslant \sum_{k=2}^{\infty} \exp\left(-\frac{2k\pi^2}{2c^2}\right) \leqslant \frac{\exp(-2\pi^2/c^2)}{1-\exp(-\pi^2/c^2)} \leqslant 2\exp\left(-\frac{2\pi^2}{c^2}\right).$$

Thus

$$P(\tilde{E}_5) \geqslant \frac{\varepsilon u}{3A}\exp\left(-\frac{\pi^2}{2c^2}\right) - \frac{\varepsilon u}{4A}\exp\left(-\frac{\pi^2}{2c^2}\right).$$

As a consequence,

$$\inf_{x\in[-(1+\varepsilon)A,\,-A]} P(\tilde{E}_5) \geqslant \frac{\varepsilon u}{12A}\exp\left(-\frac{\pi^2}{16A^2}\right). \tag{3.8}$$

Similarly, by means of (3.7) and using the scaling property, we have

$$P(E_1) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1}\exp\left(-\frac{(2k-1)^2\pi^2}{16(1+\varepsilon)^2A^2}\right)\left(1-\cos\left(\frac{(2k-1)\pi\varepsilon}{2(1+\varepsilon)}\right)\right) \tag{3.9}$$

$$\geqslant \frac{2}{\pi}\exp\left(-\frac{\pi^2}{16(1+\varepsilon)^2A^2}\right)\left(1-\cos\left(\frac{\pi\varepsilon}{2(1+\varepsilon)}\right)\right) - \frac{4}{\pi}\sum_{k=2}^{\infty}\exp\left(-\frac{2k\pi^2}{16(1+\varepsilon)^2A^2}\right)$$

$$\geqslant C_\varepsilon \exp\left(-\frac{\pi^2}{16A^2}\right).$$

It is easy to estimate $P(\tilde{E}_i)$ for $2 \leqslant i \leqslant 4$. Indeed, we have, for any $x \in [-(1+3\varepsilon)A,\ -(1+2\varepsilon)A]$,

$$P(\tilde{E}_4) = \frac{1}{\sqrt{2\pi}}\int_{(-(1+\varepsilon)A-x)/\sqrt{\varepsilon u}}^{(-A-x)/\sqrt{\varepsilon u}} \exp(-v^2/2)\,dv \geqslant \frac{\varepsilon A}{\sqrt{2\pi}\sqrt{\varepsilon u}}\exp\left(-\frac{((1+\varepsilon)A+x)^2}{2\varepsilon u}\right).$$

Hence, it follows that

$$\inf_{x\in[-(1+3\varepsilon)A,\,-(1+2\varepsilon)A]} P(\tilde{E}_4) \geqslant \frac{\sqrt{\varepsilon}A}{\sqrt{2\pi}\sqrt{u}}\exp\left(-\frac{\varepsilon A^2}{2u}\right). \tag{3.10}$$

A similar argument yields

$$\inf_{x\in[(1+2\varepsilon)A,(1+3\varepsilon)A]} P(\tilde{E}_3) \geqslant \frac{\varepsilon A}{\sqrt{2\pi}\sqrt{(1-2\varepsilon)u}}\exp\left(-\frac{2(1+3\varepsilon)^2A^2}{u}\right), \tag{3.11}$$

$$\inf_{x\in[A,(1+\varepsilon)A]} P(\tilde{E}_2) \geqslant \frac{\sqrt{\varepsilon}A}{\sqrt{2\pi}\sqrt{u}}\exp\left(-\frac{9\varepsilon A^2}{2u}\right). \tag{3.12}$$

Combining (3.8)–(3.12) we obtain

$$P\left(\bigcap_{i=1}^{5} E_i\right) \geqslant C_\varepsilon \frac{A}{\sqrt{u}}\exp\left(-\frac{2(1+3\varepsilon)^2A^2}{u} - \frac{\pi^2}{8A^2}\right).$$

Since $A = \pi\sqrt{u}/4$, the expression on the right-hand side is

$$\pi C_\varepsilon \exp\left(-\frac{(1+4\varepsilon)\pi}{\sqrt{u}}\right).$$

Replacing ε by $\varepsilon/4$ readily completes the proof of Lemma 2. ∎

Proof of the upper bound in Theorem 1. Fix a small number $\varepsilon \in (0, 1)$. Then one can easily find a positive number $\varepsilon_1 > 0$ so that

$$\theta_2 \stackrel{\text{def}}{=} (1+\varepsilon)(1+\varepsilon_1)/(1+5\varepsilon_1)^{1/2} < 1.$$

Let us furthermore define, for $k \geqslant 1$ and $m \geqslant 1$,

$$n_k = [\exp(k^{1+\varepsilon_1})]+1, \qquad u(m) = \frac{(1+5\varepsilon_1)\pi^2}{\log_2^2(m)}.$$

Let

$$F_k = \Big\{ \sup_{1/2 \leqslant t \leqslant 1/2+u(n_k)} K_k(t) > \sup_{t \notin [1/2, 1/2+u(n_k)]} K_k(t) + u(n_k)\sqrt{n_k - n_{k-1}},$$

$$\inf_{1/2 \leqslant t \leqslant 1/2+u(n_k)} K_k(t) < \inf_{t \notin [1/2, 1/2+u(n_k)]} K_k(t) - u(n_k)\sqrt{n_k - n_{k-1}} \Big\},$$

where $K_k(t) = K(t, n_k) - K(t, n_{k-1})$. For each $n \geqslant 1$, $t \to (n_k - n_{k-1})^{-1/2} K_k(t)$ is a Brownian bridge. It follows from Lemma 2 that

$$P(F_k) \geqslant C_\varepsilon \exp\left(-\frac{(1+\varepsilon)\pi}{\sqrt{u(n_k)}}\right) \geqslant C_\varepsilon k^{-\theta_2},$$

which is the general term of a divergent series. Since the F_k's are independent, by the Borel–Cantelli lemma, almost surely there exist infinitely many k's so that

$$\sup_{1/2 \leqslant t \leqslant 1/2+u(n_k)} K_k(t) > \sup_{t \notin [1/2, 1/2+u(n_k)]} K_k(t) + u(n_k)\sqrt{n_k - n_{k-1}}, \tag{3.13}$$

$$\inf_{1/2 \leqslant t \leqslant 1/2+u(n_k)} K_k(t) < \inf_{t \notin [1/2, 1/2+u(n_k)]} K_k(t) - u(n_k)\sqrt{n_k - n_{k-1}}. \tag{3.14}$$

Furthermore, applying Corollary 1.15.1 of Csörgő and Révész [1] to $y = n_{k-1}$ yields

$$\limsup_{k\to\infty} \left(n_{k-1}\log_2(n_{k-1})\right)^{-1/2} \sup_{0 \leqslant t \leqslant 1} |K(t, n_{k-1})| = 1/\sqrt{2} \text{ a.s.}$$

Combining this with (3.13) and (3.14) we obtain

$$\sup_{1/2 \leqslant t \leqslant 1/2+u(n_k)} K(t, n_k) > \sup_{t \notin [1/2, 1/2+u(n_k)]} K(t, n_k)$$

$$-2\sqrt{n_{k-1}\log_2(n_{k-1})} + u(n_k)\sqrt{n_k - n_{k-1}},$$

$$\inf_{1/2\leqslant t\leqslant 1/2+u(n_k)} K(t, n_k) < \inf_{t\notin[1/2,1/2+u(n_k)]} K(t, n_k) + 2\sqrt{n_{k-1}\log_2(n_{k-1})} - u(n_k)\sqrt{n_k - n_{k-1}}.$$

Consequently,

$$\sup_{1/2\leqslant t\leqslant 1/2+u(n_k)} \alpha_{n_k}(t) - \inf_{1/2\leqslant t\leqslant 1/2+u(n)} \alpha_{n_k}(t) - \Big(\sup_{t\notin[1/2,1/2+u(n)]} \alpha_{n_k}(t) - \inf_{t\notin[1/2,1/2+u(n)]} \alpha_{n_k}(t)\Big)$$

$$\geqslant n_k^{-1/2}\Big(\sup_{1/2\leqslant t\leqslant 1/2+u(n_k)} K(t, n_k) - \inf_{1/2\leqslant t\leqslant 1/2+u(n_k)} K(t, n_k) - \Big(\sup_{t\notin[1/2,1/2+u(n_k)]} K(t, n_k) - \inf_{t\notin[1/2,1/2+u(n_k)]} K(t, n_k)\Big)\Big) - 4n_k^{-1/4}\log n_k$$

$$\geqslant 2n_k^{-1/2}\big(u(n_k)\sqrt{n_k - n_{k-1}} - 2\sqrt{n_{k-1}\log k}\big) - 4n_k^{-1/4}\log n_k > 0.$$

This yields that $|\mu_n - \nu_n| \leqslant u(n)$. Hence

$$\liminf_{n\to\infty} \log_2^2(n)\,|\mu_n - \nu_n| \leqslant (1+5\varepsilon_1)\pi^2.$$

Letting ε tend to 0 and then letting ε_1 also tend to 0 we complete the proof of the upper bound of Theorem 1.

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