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A GENERALIZED LAW OF THE ITERATED LOGARITHM FOR THE LARGEST OBSERVATION OF A TRIANGULAR ARRAY

BY

ANDRÉ ADLER (CHICAGO, ILLINOIS)

Abstract. Consider independent and identically distributed random variables $\{X, X_{kj}, 1 \leqslant j \leqslant k, k \geqslant 1\}$ from a particular distribution with $EX = \infty$. We show that there exists an unusual generalized Law of the Iterated Logarithm involving $\max_{1 \leqslant j \leqslant k} X_{kj}$.

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This paper explores the asymptotic behavior of weighted partial sums of random variables. These random variables are the largest observations from each row of a triangular array. The techniques used in proving our theorems are similar to those found in [1] and [5] in the sense that we first obtain a weak law to conclude that the lower limit is almost surely bounded above by $1/(\alpha+2)$. As for obtaining equality the proofs differ in the sense that we actually exhibit a random variable that achieves this bound. Furthermore, it should be pointed out that our random variables, $\{X_k, k \geqslant 1\}$, are not identically distributed.

Let $\{X, X_{kj}, 1 \leqslant j \leqslant k, k \geqslant 1\}$ be independent and identically distributed random variables with common density $f(x) = x^{-2} I(x \geqslant 1)$. Set $X_k = \max_{1 \leqslant j \leqslant k} X_{kj}$. Note that since $EX_{kj} = \infty$, it follows that $EX_k = \infty$ for all $k \geqslant 1$. As for notation we set $a_n = n^{\alpha}$ and $b_n = n^{\alpha+2} \lg n$, where $\lg x = \max\{1, \log x\}$. To expedite matters we also set $c_n = b_n/a_n = n^2 \lg n$. We use the constant C to denote a generic bound that is not necessarily the same in each appearance.

THEOREM 1.

$$\frac{\sum_{k=1}^{n} a_k X_k}{b_n} \xrightarrow{P} \frac{1}{\alpha+2} \quad \textit{for all } \alpha > -2.$$

Proof. We will use the Degenerate Convergence Theorem, which can be found on p. 338 of [3]. For all $1 \leqslant k \leqslant n$,

$$\begin{aligned} P\{X_k > \varepsilon b_n/a_k\} & \\ &= 1 - P\{X_k \leqslant \varepsilon b_n/a_k\} = 1 - [P\{X \leqslant \varepsilon b_n/a_k\}]^k = 1 - [F(\varepsilon b_n/a_k)]^k \\ &= 1 - \left[1 - \frac{a_k}{\varepsilon b_n}\right]^k = 1 - \sum_{j=0}^{k} \binom{k}{j} \left(\frac{-a_k}{\varepsilon b_n}\right)^j \\ &= \sum_{j=1}^{k} \binom{k}{j} (-1)^{j+1} \left(\frac{a_k}{\varepsilon b_n}\right)^j \leqslant \sum_{j=1}^{k} k^j \left(\frac{a_k}{\varepsilon b_n}\right)^j. \end{aligned}$$

So for $\alpha \geqslant -1$ and all $\varepsilon > 0$

$$\begin{aligned} \sum_{k=1}^{n} P\{X_k > \varepsilon b_n/a_k\} & \\ &\leqslant \sum_{k=1}^{n} \sum_{j=1}^{k} k^j \left(\frac{a_k}{\varepsilon b_n}\right)^j \leqslant \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{k^j k^{\alpha j}}{(\varepsilon n^{\alpha+2} \lg n)^j} = \sum_{j=1}^{n} \left(\frac{1}{\varepsilon n^{\alpha+2} \lg n}\right)^j \sum_{k=1}^{n} k^{(\alpha+1)j} \\ &\leqslant C \sum_{j=1}^{n} \left(\frac{1}{\varepsilon n^{\alpha+2} \lg n}\right)^j n^{(\alpha+1)j+1} \leqslant Cn \sum_{j=1}^{\infty} \left(\frac{1}{\varepsilon n \lg n}\right)^j = \frac{Cn}{\varepsilon n \lg n - 1} \to 0. \end{aligned}$$

When $-2 < \alpha < -1$ we need to partition j into three cases. Let

$$A_j = \{j\colon j < -1/(\alpha+1)\}, \quad B_j = \{j\colon j = -1/(\alpha+1)\}$$

and

$$C_j = \{j\colon j > -1/(\alpha+1)\}.$$

Then, as above,

$$\begin{aligned} \sum_{k=1}^{n} P\{X_k > \varepsilon b_n/a_k\} & \\ &\leqslant \sum_{j=1}^{n} \left(\frac{1}{\varepsilon n^{\alpha+2} \lg n}\right)^j \sum_{k=1}^{n} k^{(\alpha+1)j} = \sum_{A_j} \left(\frac{1}{\varepsilon n^{\alpha+2} \lg n}\right)^j \sum_{k=1}^{n} k^{(\alpha+1)j} \\ &\quad + \sum_{B_j} \left(\frac{1}{\varepsilon n^{\alpha+2} \lg n}\right)^j \sum_{k=1}^{n} k^{(\alpha+1)j} + \sum_{C_j} \left(\frac{1}{\varepsilon n^{\alpha+2} \lg n}\right)^j \sum_{k=1}^{n} k^{(\alpha+1)j}. \end{aligned}$$

The first series goes to zero since

$$\sum_{A_j} \left(\frac{1}{\varepsilon n^{\alpha+2} \lg n}\right)^j \sum_{k=1}^{n} k^{(\alpha+1)j} \leqslant Cn \sum_{A_j} \left(\frac{1}{n \lg n}\right)^j$$

and there are only a finite number of terms in A_j. In the event of $B_j \neq \emptyset$, in which case the second term would be zero, this series consists of one term, which is bounded above by

$$C(n \lg n)^{(\alpha+2)/(\alpha+1)}$$

which goes to zero since the exponent is negative. Finally, the last series

$$\sum_{C_j}\left(\frac{1}{\varepsilon n^{\alpha+2}\lg n}\right)^j \sum_{k=1}^{n} k^{(\alpha+1)j}$$

$$\leqslant C\sum_{C_j}\left(\frac{1}{n^{\alpha+2}\lg n}\right)^j \leqslant C\sum_{j=1}^{\infty}\left(\frac{1}{n^{\alpha+2}\lg n}\right)^j = \frac{C}{n^{\alpha+2}\lg n-1}\to 0.$$

Next, we need to show that

$$\sum_{k=1}^{n} \operatorname{Var}\left(\frac{a_k X_k}{b_n} I(X_k < b_n/a_k)\right) \to 0.$$

This sequence is bounded above by

$$b_n^{-2}\sum_{k=1}^{n} a_k^2 EX_k^2 I(X_k < b_n/a_k)$$

$$= b_n^{-2}\sum_{k=1}^{n} a_k^2 \int_1^{b_n/a_k} k\left(1-\frac{1}{x}\right)^{k-1} dx \leqslant b_n^{-2}\sum_{k=1}^{n} a_k^2 k \int_1^{b_n/a_k} dx$$

$$\leqslant b_n^{-2}\sum_{k=1}^{n} a_k^2 k\left(\frac{b_n}{a_k}\right) = b_n^{-1}\sum_{k=1}^{n} a_k k = \frac{\sum_{k=1}^{n} k^{\alpha+1}}{n^{\alpha+2}\lg n} \leqslant \frac{C}{\lg n}\to 0.$$

Lastly, we need to see where our sequence is going:

$$\sum_{k=1}^{n} E\left(\frac{a_k X_k}{b_n} I(X_k < b_n/a_k)\right)$$

$$= b_n^{-1}\sum_{k=1}^{n} a_k k \int_1^{b_n/a_k}\left(1-\frac{1}{x}\right)^{k-1}\frac{dx}{x}$$

$$= b_n^{-1}\sum_{k=1}^{n} a_k k \int_1^{b_n/a_k}\left[\frac{1}{x}+\sum_{j=1}^{k-1}\binom{k-1}{j}(-1)^j x^{-j-1}\right]dx$$

$$= b_n^{-1}\sum_{k=1}^{n} a_k k\left[\lg b_n - \lg a_k + \sum_{j=1}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{j} - \sum_{j=1}^{k-1}\binom{k-1}{j}\frac{(-1)^j (b_n/a_k)^{-j}}{j}\right]$$

$$= b_n^{-1}\sum_{k=1}^{n} a_k k\lg b_n - b_n^{-1}\sum_{k=1}^{n} a_k k\lg a_k + b_n^{-1}\sum_{k=1}^{n} a_k k\sum_{j=1}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{j}$$

$$-b_n^{-1}\sum_{k=1}^{n} a_k k\sum_{j=1}^{k-1}\binom{k-1}{j}\frac{(-1)^j (b_n/a_k)^{-j}}{j}.$$

The first sequence

$$\text{(1)}\qquad \frac{\lg b_n}{b_n}\sum_{k=1}^{n} a_k k \sim \frac{(\alpha+2)\sum_{k=1}^{n} k^{\alpha+1}}{n^{\alpha+2}} \to 1.$$

The second sequence

$$(2)\qquad \frac{1}{b_n}\sum_{k=1}^{n} a_k k \lg a_k = \frac{\alpha\sum_{k=1}^{n} k^{\alpha+1}\lg k}{n^{\alpha+2}\lg n} \to \frac{\alpha}{\alpha+2}.$$

Using equation 0.155, #4 from [4], p. 4, we have

$$\sum_{j=1}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{j} \sim -\lg k.$$

Hence our third sequence

$$(3)\qquad b_n^{-1}\sum_{k=1}^{n} a_k k \sum_{j=1}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{j} \sim \frac{-\sum_{k=1}^{n} k^{\alpha+1}\lg k}{n^{\alpha+2}\lg n} \to \frac{-1}{\alpha+2}.$$

Next we will show that the last sequence converges to zero. In doing so, we again need to observe two different cases. If $\alpha \geqslant -1$, then

$$\left|b_n^{-1}\sum_{k=1}^{n} a_k k \sum_{j=1}^{k-1}\binom{k-1}{j}\frac{(-1)^j (b_n/a_k)^{-j}}{j}\right|$$

$$\leqslant b_n^{-1}\sum_{k=1}^{n} a_k k \sum_{j=1}^{k-1}\binom{k-1}{j}(b_n/a_k)^{-j} \leqslant \frac{1}{n^{\alpha+2}\lg n}\sum_{k=1}^{n} k^{\alpha+1}\sum_{j=1}^{k} k^j\left(\frac{k^\alpha}{n^{\alpha+2}\lg n}\right)^j$$

$$= \frac{1}{n^{\alpha+2}\lg n}\sum_{j=1}^{n}\sum_{k=j}^{n}\frac{k^{\alpha+1+j+\alpha j}}{(n^{\alpha+2}\lg n)^j} \leqslant \frac{1}{n^{\alpha+2}\lg n}\sum_{j=1}^{n}\frac{1}{(n^{\alpha+2}\lg n)^j}\sum_{k=1}^{n} k^{\alpha+1+j+\alpha j}$$

$$\leqslant \frac{C}{n^{\alpha+2}\lg n}\sum_{j=1}^{n}\frac{n^{\alpha+2+j+\alpha j}}{(n^{\alpha+2}\lg n)^j} = \frac{C}{n^{\alpha+2}\lg n}\sum_{j=1}^{n}\frac{n^{\alpha+2-j}}{(\lg n)^j} = \frac{C}{\lg n}\sum_{j=1}^{n}\left(\frac{1}{n\lg n}\right)^j$$

$$\leqslant \frac{C}{\lg n}\left(\frac{n}{n\lg n}\right) = \frac{C}{(\lg n)^2} \to 0.$$

When $-2 < \alpha < -1$ we need to partition j into three cases. Let

$$A_j = \{j:\ (\alpha+1)(j+1) > -1\},\qquad B_j = \{j:\ (\alpha+1)(j+1) = -1\}$$

and

$$C_j = \{j:\ (\alpha+1)(j+1) < -1\}.$$

Then, as in the last calculation,

$$\left|b_n^{-1}\sum_{k=1}^{n} a_k k \sum_{j=1}^{k-1}\binom{k-1}{j}\frac{(-1)^j (b_n/a_k)^{-j}}{j}\right|$$

$$\leqslant \frac{1}{n^{\alpha+2}\lg n}\sum_{j=1}^{n}\left(\frac{1}{n^{\alpha+2}\lg n}\right)^j\sum_{k=1}^{n} k^{(\alpha+1)(j+1)}$$

$$= \frac{1}{n^{\alpha+2}\lg n}\sum_{A_j}\left(\frac{1}{n^{\alpha+2}\lg n}\right)^j\sum_{k=1}^{n} k^{(\alpha+1)(j+1)}$$

$$+\frac{1}{n^{\alpha+2}\lg n}\sum_{B_j}\left(\frac{1}{n^{\alpha+2}\lg n}\right)^j\sum_{k=1}^n k^{(\alpha+1)(j+1)}$$
$$+\frac{1}{n^{\alpha+2}\lg n}\sum_{C_j}\left(\frac{1}{n^{\alpha+2}\lg n}\right)^j\sum_{k=1}^n k^{(\alpha+1)(j+1)}.$$

The first sequence

$$\frac{1}{n^{\alpha+2}\lg n}\sum_{A_j}\left(\frac{1}{n^{\alpha+2}\lg n}\right)^j\sum_{k=1}^n k^{(\alpha+1)(j+1)}\leqslant\frac{C}{n^{\alpha+2}\lg n}\sum_{A_j}\left(\frac{1}{n^{\alpha+2}\lg n}\right)^j n^{(\alpha+1)(j+1)+1}$$
$$=\frac{C}{n^{\alpha+2}\lg n}\sum_{A_j}\frac{n^{\alpha+2-j}}{(\lg n)^j}=\frac{C}{\lg n}\sum_{A_j}\left(\frac{1}{n\lg n}\right)^j<\frac{C}{(\lg n)^2}\to 0.$$

The second sequence, which consists of at most one term,

$$\frac{1}{n^{\alpha+2}\lg n}\sum_{B_j}\left(\frac{1}{n^{\alpha+2}\lg n}\right)^j\sum_{k=1}^n k^{(\alpha+1)(j+1)}\leqslant\frac{C}{n^{\alpha+2}\lg n}\sum_{B_j}\left(\frac{1}{n^{\alpha+2}\lg n}\right)^j\lg n$$
$$=\frac{C}{n^{\alpha+2}}(n^{\alpha+2}\lg n)^{(\alpha+2)/(\alpha+1)}\to 0$$

since $-2<\alpha<-1$. As for the final sequence,

$$\frac{1}{n^{\alpha+2}\lg n}\sum_{C_j}\left(\frac{1}{n^{\alpha+2}\lg n}\right)^j\sum_{k=1}^n k^{(\alpha+1)(j+1)}\leqslant\frac{C}{n^{\alpha+2}\lg n}\sum_{C_j}\left(\frac{1}{n^{\alpha+2}\lg n}\right)^j$$
$$\leqslant\frac{C}{n^{\alpha+2}\lg n}\to 0.$$

Combining (1), (2) and (3) we have

$$\sum_{k=1}^n E\left(\frac{a_k X_k}{b_n}I(X_k<b_n/a_k)\right)\to 1-\frac{\alpha}{\alpha+2}-\frac{1}{\alpha+2}=\frac{1}{\alpha+2},$$

completing the proof. ■

CLAIM. *For all* $M>0$

$$1-\left[1-\frac{1}{Mn^2\lg n}\right]^n\sim\frac{1}{Mn\lg n}.$$

Proof. From the Binomial Theorem we have

$$1-\left[1-\frac{1}{Mn^2\lg n}\right]^n=1-\sum_{j=0}^n\binom{n}{j}\left(\frac{-1}{Mn^2\lg n}\right)^j$$
$$=\frac{1}{Mn\lg n}+\sum_{j=2}^n\binom{n}{j}\left(\frac{-1}{Mn^2\lg n}\right)^j.$$

Thus, we need to show that

$$Mn\lg n \sum_{j=2}^{n} \binom{n}{j} \left(\frac{-1}{Mn^2 \lg n}\right)^j \to 0,$$

$$\left| Mn\lg n \sum_{j=2}^{n} \binom{n}{j} \left(\frac{-1}{Mn^2 \lg n}\right)^j \right| \leqslant Mn\lg n \sum_{j=2}^{n} \binom{n}{j} \left(\frac{1}{Mn^2 \lg n}\right)^j$$

$$< Mn\lg n \sum_{j=2}^{n} n^j \left(\frac{1}{Mn^2 \lg n}\right)^j = Mn\lg n \sum_{j=2}^{n} \left(\frac{1}{Mn \lg n}\right)^j$$

$$< Mn\lg n \left[\frac{n}{(Mn\lg n)^2}\right] = \frac{1}{M \lg n} \to 0.$$

THEOREM 2.

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n} a_k X_k}{b_n} = \frac{1}{\alpha+2} \textit{ almost surely for all } \alpha > -2$$

and

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} a_k X_k}{b_n} = \infty \textit{ almost surely for all } \alpha > -2.$$

Proof. Using our Claim, for all $M > 0$ we have

$$\sum_{n=1}^{\infty} P\{X_n > Mc_n\}$$

$$= \sum_{n=1}^{\infty} [1 - P\{X_n < Mc_n\}] = \sum_{n=1}^{\infty} [1 - (P\{X < Mc_n\})^n] = \sum_{n=1}^{\infty} [1 - (F(Mc_n))^n]$$

$$= \sum_{n=1}^{\infty} \left[1 - \left(1 - \frac{1}{Mc_n}\right)^n\right] = \sum_{n=1}^{\infty} \left[1 - \left(1 - \frac{1}{Mn^2 \lg n}\right)^n\right] \geqslant \sum_{n=1}^{\infty} \frac{C}{n \lg n} = \infty.$$

This implies that

$$\limsup_{n\to\infty} \frac{a_n X_n}{b_n} = \infty \text{ almost surely,}$$

whence

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} a_k X_k}{b_n} = \infty \text{ almost surely.}$$

In view of Theorem 1 we need only show that

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n} a_k X_k}{b_n} \geqslant \frac{1}{\alpha+2} \text{ almost surely for all } \alpha > -2.$$

To this end we need to find a new truncation to our random variables. Note that

$$b_n^{-1}\sum_{k=1}^{n} a_k X_k \geqslant b_n^{-1}\sum_{k=1}^{n} a_k X_k I(1\leqslant X_k\leqslant k^2)$$

$$= b_n^{-1}\sum_{k=1}^{n} a_k [X_k I(1\leqslant X_k\leqslant k^2)-EX_k I(1\leqslant X_k\leqslant k^2)]$$

$$+b_n^{-1}\sum_{k=1}^{n} a_k EX_k I(1\leqslant X_k\leqslant k^2).$$

The first term vanishes almost surely by the usual Khintchine–Kolmogorov Convergence Theorem (see [3]) and Kronecker's lemma since

$$\sum_{n=1}^{\infty} c_n^{-2} EX_n^2 I(1\leqslant X_n\leqslant n^2) = \sum_{n=1}^{\infty}\left(\frac{1}{n^4(\lg n)^2}\right)\int_1^{n^2} n\left(1-\frac{1}{x}\right)^{n-1} dx$$

$$\leqslant \sum_{n=1}^{\infty}\left(\frac{1}{n^4(\lg n)^2}\right)\cdot n^3 = \sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2} < \infty.$$

As for the second term,

$$b_n^{-1}\sum_{k=1}^{n} a_k EX_k I(1\leqslant X_k\leqslant k^2) = b_n^{-1}\sum_{k=1}^{n} a_k\int_1^{k^2} k\left(1-\frac{1}{x}\right)^{k-1}\frac{dx}{x}$$

$$=\frac{1}{n^{\alpha+2}\lg n}\sum_{k=1}^{n} k^{\alpha+1}\int_1^{k^2}\left[\frac{1}{x}+\sum_{j=1}^{k-1}\binom{k-1}{j}(-1)^j x^{-j-1}\right]dx$$

$$=\frac{1}{n^{\alpha+2}\lg n}\sum_{k=1}^{n} 2k^{\alpha+1}\lg k+\frac{1}{n^{\alpha+2}\lg n}\sum_{k=1}^{n} k^{\alpha+1}\sum_{j=1}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{j}$$

$$-\frac{1}{n^{\alpha+2}\lg n}\sum_{k=1}^{n} k^{\alpha+1}\sum_{j=1}^{k-1}\binom{k-1}{j}\frac{(-1)^j k^{-2j}}{j}.$$

The first sequence

$$\frac{2}{n^{\alpha+2}\lg n}\sum_{k=1}^{n} k^{\alpha+1}\lg k\to\frac{2}{\alpha+2}. \tag{4}$$

The second sequence

$$\frac{1}{n^{\alpha+2}\lg n}\sum_{k=1}^{n} k^{\alpha+1}\sum_{j=1}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{j}\sim\frac{-1}{n^{\alpha+2}\lg n}\sum_{k=1}^{n} k^{\alpha+1}\lg k\to\frac{-1}{\alpha+2}. \tag{5}$$

The last sequence approaches zero since

$$\left|\frac{1}{n^{\alpha+2}\lg n}\sum_{k=1}^{n} k^{\alpha+1}\sum_{j=1}^{k-1}\binom{k-1}{j}\frac{(-1)^j k^{-2j}}{j}\right|$$

$$\leqslant \frac{1}{n^{\alpha+2}\lg n}\sum_{k=1}^{n} k^{\alpha+1}\sum_{j=1}^{k} k^j k^{-2j} = \frac{1}{n^{\alpha+2}\lg n}\sum_{k=1}^{n} k^{\alpha+1}\sum_{j=1}^{k} k^{-j}$$

$$< \frac{1}{n^{\alpha+2}\lg n}\sum_{k=1}^{n} k^{\alpha+1}\sum_{j=1}^{k} k^{-1}$$

$$= \frac{1}{n^{\alpha+2}\lg n}\sum_{k=1}^{n} k^{\alpha+1} < \frac{1}{n^{\alpha+2}\lg n}\cdot(Cn^{\alpha+2}) = \frac{C}{\lg n}\to 0.$$

Combining (4) and (5) we have

$$\liminf_{n\to\infty}\frac{\sum_{k=1}^{n} a_k X_k}{b_n} \geqslant \liminf_{n\to\infty}\frac{\sum_{k=1}^{n} a_k X_k(1\leqslant X_k\leqslant k^2)}{b_n} = \frac{2}{\alpha+2}-\frac{1}{\alpha+2} = \frac{1}{\alpha+2},$$

which completes the proof. ∎

In the case of $\alpha = -2$, a Strong Law of Large Numbers does exist. Naturally, the norming sequence differs from our sequence $b_n = n^{\alpha+2}\lg n$. This result can be found in [2]. If $\alpha < -2$, then our partial sum $\sum_{k=1}^{n} a_k X_k$ converges. So if we divide it by any sequence approaching infinity, the limit will be zero, which is quite uninteresting.

THEOREM 3. *The partial sum $\sum_{k=1}^{n} a_k X_k$ converges for all $\alpha < -2$.*

Proof. Here we partition our sum in a fourth and final way:

$$\sum_{n=1}^{\infty} a_n X_n = \sum_{n=1}^{\infty} a_n X_n I(1\leqslant X_n\leqslant n^2(\lg n)^2) + \sum_{n=1}^{\infty} a_n X_n I(X_n > n^2(\lg n)^2).$$

Observe that

$$P\{X_n > n^2(\lg n)^2\} = 1 - P\{X_n < n^2(\lg n)^2\}$$

$$= 1-[P\{X < n^2(\lg n)^2\}]^n = 1-[F(n^2(\lg n)^2)]^n = 1-\left[1-\frac{1}{n^2(\lg n)^2}\right]^n$$

$$= 1-\sum_{j=0}^{n}\binom{n}{j}(-1)^j\left(\frac{1}{n^2(\lg n)^2}\right)^j = \frac{1}{n(\lg n)^2}-\sum_{j=2}^{n}\binom{n}{j}(-1)^j\left(\frac{1}{n^2(\lg n)^2}\right)^j.$$

Thus

$$P\{X_n > n^2(\lg n)^2\} \sim \frac{1}{n(\lg n)^2}$$

since

$$\left|n(\lg n)^2\sum_{j=2}^{n}\binom{n}{j}(-1)^j\left(\frac{1}{n^2(\lg n)^2}\right)^j\right| \leqslant n(\lg n)^2\sum_{j=2}^{n} n^j\left(\frac{1}{n^2(\lg n)^2}\right)^j$$

$$= n(\lg n)^2\sum_{j=2}^{n}\left(\frac{1}{n(\lg n)^2}\right)^j \leqslant (n(\lg n)^2)\cdot\left(\frac{n}{(n(\lg n)^2)^2}\right) = \frac{1}{(\lg n)^2}\to 0.$$

So by the Borel–Cantelli lemma the second series is finite almost surely. As for the first series,

$$E\sum_{n=1}^{\infty} a_n X_n I\left(1 \leqslant X_n \leqslant n^2 (\lg n)^2\right) = \sum_{n=1}^{\infty} a_n \int_1^{n^2(\lg n)^2} n\left(1-\frac{1}{x}\right)^{n-1}\frac{dx}{x}$$

$$\leqslant \sum_{n=1}^{\infty} n^{\alpha+1} \int_1^{n^2(\lg n)^2} \frac{dx}{x} \leqslant C \sum_{n=1}^{\infty} n^{\alpha+1} \lg n < \infty$$

since $\alpha < -2$. Hence our series is convergent almost surely. ∎

A couple of comments about the underlying distribution used in this paper should be mentioned. We used $f(x) = x^{-2} I(x \geqslant 1)$, but it should be possible to work with any distribution in which $P\{X > x\} \sim L(x)/x$ for all slowly varying functions $L(x)$. However, each case must be treated separately due to the intricate calculations that must be performed, as shown in this paper. Also, on a much simpler note, it does not matter where our distribution starts. What always matters is the tail behavior. For example, if we let $Y_{nk} = X_{nk} + c$ for some constant c, then $Y_n = X_n + c$, and since $\sum_{k=1}^{n} a_k = o(b_n)$, our conclusions also hold for the partial sum $\sum_{k=1}^{n} a_k Y_k$.

REFERENCES

[1] A. Adler, *Generalized one-sided laws of the iterated logarithm for random variables barely with or without finite mean*, J. Theoret. Probab. 3 (1990), pp. 587–598.

[2] A. Adler, *An unusual strong law of large numbers for the largest observation in a triangular array*, Bull. Inst. Math. Acad. Sinica 29 (2001), to appear.

[3] Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, 2nd edition, Springer, New York 1988.

[4] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 4th edition, Academic Press, New York 1980.

[5] M. Klass and H. Teicher, *Iterated logarithm laws for asymmetric random variables barely with or without finite mean*, Ann. Probab. 5 (1977), pp. 861–874.

Department of Mathematics
Illinois Institute of Technology
Chicago, Illinois, 60616, U.S.A.
E-mail: adler@iit.edu

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