PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 23, Fasc. 1 (2003), pp. 1–5

LINEARLY ADDITIVE RANDOM FIELDS WITH INDEPENDENT INCREMENTS ON TIME-LIKE CURVES

BY

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Abstract. Let V be a convex cone in $\boldsymbol{R}^n$. A curve $L = \{l(t);\ t \in \boldsymbol{R}_+\} \subset \boldsymbol{R}^n$ is called a *time-like curve* if $\{l(s);\ s \geqslant t\} \subset l(t)+V$ holds for any t. A random field $\{X(t);\ t \in \boldsymbol{R}^n\}$ whose restriction $X|_L(t) = X(l(t))$ on time-like curve L becomes an additive process is considered and it is characterized as a set-indexed random field on the dual cone V^*.

0. SOME FACTS FROM PROJECTIVE GEOMETRY

0.1. Local coordinate. Let $\boldsymbol{P}^n = (\boldsymbol{R}^{n+1}\backslash\{0\})/(\boldsymbol{R}\backslash\{0\})$ be the n-dimensional real projective space and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n, x_0)$ be its homogeneous coordinate. That is, two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are *identified* in $\boldsymbol{P}^n$ if there exists a non-zero real number c such that $\boldsymbol{x} = c \cdot \boldsymbol{y}$.

The subset $H_\infty = \{(x_1, x_2, \ldots, x_n, 0)\}$ is called the (*hyper-*)*plane at infinity*. We can introduce a local coordinate in the set $\boldsymbol{P}^n\backslash H_\infty$ as $\pi(\boldsymbol{x}) = (x_1/x_0, x_2/x_0, \ldots, x_n/x_0)$, and identify $\boldsymbol{P}^n\backslash H_\infty$ with $\boldsymbol{R}^n$. In this report, we will use these local coordinates and use $\boldsymbol{x}$ instead of $\pi(\boldsymbol{x})$ to simplify the notation.

0.2. Hyperplanes. To any vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ (or $(a_1, a_2, \ldots, a_n, 1)$ in homogeneous coordinates) there corresponds a hyperplane

$$H_{\boldsymbol{a}} = \{\boldsymbol{x} \in \boldsymbol{P}^n;\ \boldsymbol{a} \cdot \boldsymbol{x} = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = -1\}$$

(in homogeneous coordinates the condition inside the brackets takes the form $\boldsymbol{a} \cdot \boldsymbol{x} = a_1 x_1 + \ldots + a_n x_n + a_0 x_0 = 0$). Let us introduce a map * from $\boldsymbol{P}^n$ onto the set of all hyperplanes as $\boldsymbol{a}^* = H_{\boldsymbol{a}}$. Let us use the same notation to denote the inverse map of *, that is, $\boldsymbol{a}^{**} = (H_{\boldsymbol{a}})^* = \boldsymbol{a}$ (the map * is called the *duality map* between points and hyperplanes in $\boldsymbol{P}^n$). Note that $\boldsymbol{a}^*$ is the hyperplane which is perpendicular to the vector $\boldsymbol{a}$ and the distance from the origin $\boldsymbol{O}$ is $-1/|\boldsymbol{a}|$ (i.e., located on the opposite side of the origin).

0.3. Some properties. Let us fix a point $\boldsymbol{a}$. Take any point $\boldsymbol{y} \in \boldsymbol{a}^*$ and consider the following set of hyperplanes:

$$\{\boldsymbol{y}^*;\ \boldsymbol{y} \in \boldsymbol{a}^*\} = \{\{\boldsymbol{z};\ \boldsymbol{z} \cdot \boldsymbol{y} = -1\};\ \boldsymbol{y} \cdot \boldsymbol{a} = -1\}.$$

It follows that $\bigcup_{y \in a^*} y^* = \{a\}$. This means that the dual hyperplane a^* of a point a is identified with the set of all hyperplanes $H_y = y^*$ which contain the point a.

For any point a, define the set $S(a) = \{x;\ a \cdot x \leqslant -1\}$ which can mean the set of all hyperplanes crossing the line segment $\overline{Oa}$ (see [8]).

1. RANDOM MEASURES AND LINEARLY ADDITIVE RANDOM FIELDS

1.1. Random measure. Let $(E, \mathscr{B}, \mu)$ be a σ-finite measure space. Fix an infinitely divisible law and write its characteristic function as $e^{-c\varphi(z)}$.

DEFINITION 1.1. A random field $\mathscr{Y} = \{Y(B);\ B \in \mathscr{B},\ \mu(B) < \infty\}$ is called a *random measure with controlled measure* $(E, \mathscr{B}, \mu)$ if the following conditions are satisfied:

1. $E[e^{izY(B)}] = e^{-\mu(B)\varphi(z)}$;
2. for any $A, B \in \mathscr{B}$ such that $A \cap B = \emptyset$, $Y(A)$ and $Y(B)$ are independent and $Y(A \cup B) = Y(A) + Y(B)$ holds a.s.;
3. for any disjoint family $A_n \in \mathscr{B}$, $n = 1, 2, \ldots$,

$$Y\Big(\bigcup_n A_n\Big) = \sum_n Y(A_n) \text{ a.s.}$$

1.2. Linearly additive random fields.

DEFINITION 1.2 (linearly additive random fields). An R^n-parameter random field $\{X(t);\ t \in R^n\}$ is called *linearly additive* if it is additive on any line. That is, the process $Z(s) \equiv X(sv + v_0)$ has independent increments on any line $\{sv + v_0\}$.

Mori [2] obtained the following theorem:

THEOREM 1.1 (Mori [2]). *Let $\{X(t)\}$ be an R^n-parameter linearly additive random field. Then there exists uniquely a measure μ on the set of all hyperplanes in R^n and the field has the representation*

$$X(t) = Y(S_t),$$

where the set S_t is defined in Section 0.3, and $\{Y(B);\ B$ a measurable set in $R^n\}$ is the random measure which corresponds to the infinite divisible law of $\{X(t)\}$ controlled by (R^n, μ).

Let us call this measure the *Chentsov–Mori measure* of $\{X\}$.

2. MULTIPARAMETER ADDITIVE RANDOM FIELDS

2.1. R^n_+-parameter case.

DEFINITION 2.1 (Rocha-Arteaga and Sato [3]). An R^n-parameter random field $\{X(t);\ t \in R^n\}$ is called a *multiparameter additive random field* if the following conditions hold:

1. For any points $s_1 \preceq s_2 \preceq \ldots \preceq s_m$, the differences $X(s_n) - X(s_{n-1})$, $n = 2, 3, \ldots, m$, make an independent system, where $u \preceq t$ means that $u_k \leqslant t_k$ hold for any k-th coordinate.

2. If $s_1 \preceq s_2$, $s_3 \preceq s_4$ and $s_2 - s_1 = s_4 - s_3$, then $X(s_2) - X(s_1)$ and $X(s_4) - X(s_3)$ are subject to the same law (i.e. $\{X\}$ is invariant under the parallel transforms).

3. $X(0) = 0$ a.s.

4. $X(s)$ is right-continuous and has left limits with respect to the order $\preceq$.

THEOREM 2.1 (Takenaka [11]). *Let $\{X(t);\ t \in \boldsymbol{R}^n\}$ be a linearly additive multiparameter random field. Then there exists uniquely a measure μ which concentrates on $(\boldsymbol{R}_-)^n$, the field $\{X\}$ has the following Chentsov type representation:*

$$X(t) = Y(S(t)),$$

where $Y = \{Y\}$ is the random measure controlled by the measure $\mu(x)$ $(= ((dr)/r^{n+1})\, dv(\mathrm{u}),\ x = r \cdot \mathrm{u},\ \mathrm{u} \in S^{n-1})$, *which is invariant under the dual actions of the parallel transforms.*

2.2. The case of convex cone parameter.

2.2.1. *Convex cone.*

DEFINITION 2.2. A closed subset $V \subset \boldsymbol{R}^n$ is called a *convex cone* if the following conditions are satisfied:

1. There exists v_0 such that $v_0 \cdot v \geqslant 0$ for all $v \in V$.

2. V is a convex set, that is for any $v_1, v_2 \in V$ and for all c, $0 \leqslant c \leqslant 1$, it follows that $cv_1 + (1-c)v_2 \in V$.

3. For any $v \in V$ and for all $c \geqslant 0$, $cv \in V$.

DEFINITION 2.3. Let V be a convex cone. The *dual cone* V^* is defined as:

$$V^* = \{u \in \boldsymbol{R}^n;\ u \cdot v \leqslant 0,\ \forall v \in V\}.$$

Note that the dual cone V^* is also a convex cone, and we have $(V^*)^* = V$.

2.2.2. EXAMPLES.

- Let us set $V = \boldsymbol{R}^n_+$. Then $V^* = \boldsymbol{R}^n_-$.
- Let $V_f = \{(x_1, \ldots, x_n);\ \sqrt{|x_2^2 + \ldots + x_n^2|} \leqslant x_1\}$ be the future cone. Then the dual cone $V_f^* = \{\sqrt{x_2^2 + \ldots + x_n^2} \leqslant -x_1\}$ is the past cone in physics where we take the speed of light equal to 1.

2.3. V-parameter additive random fields. Let $V \subset \boldsymbol{R}^n_+$ be a convex cone. A curve $L = \{l(t);\ t \in \boldsymbol{R}_+\} \subset V$ is called a *time-like curve* if $\{l(s);\ s \geqslant t\} \subset l(t) + V$ holds for any $t \in \boldsymbol{R}_+$.

DEFINITION 2.4. A random field $\{X(t);\ t \in V\}$ is called an *additive random field with respect to V* if the parameter restriction $\{X_L(t) = X(l(t));\ t \in \boldsymbol{R}_+\}$ becomes an additive process for any time-like curve L.

Note that $\boldsymbol{R}^n_+$ is a convex cone. Thus the multiparameter additive field considered in Section 2.1 is an example of an additive field with respect to the cone $\boldsymbol{R}^n_+$.

THEOREM 2.2 (Takenaka [12]). *Let $\{X(\boldsymbol{t});\ \boldsymbol{t}\in \boldsymbol{R}^n\}$ be a linearly additive random field and V be a convex cone. If $\{X(\boldsymbol{t});\ \boldsymbol{t}\in V\}$ is an additive random field with respect to V, then there exists uniquely a measure μ which concentrates on the dual cone V^* and the process $\{X\}$ has the following Chentsov type representation:*

$$X(\boldsymbol{t}) = Y(S(\boldsymbol{t})),$$

where $\{Y(\cdot)\}$ is the random measure controlled by μ.

2.4. Proof. Let us take $v_1, v_2, v_3, \ldots \in V$. Consider the differences

$$\begin{aligned} X(v_1) &= Y(S(v_1)),\\ X(v_2+v_1)-X(v_1) &= Y(S(v_2+v_1))-Y(S(v_1)),\\ X(v_3+v_2+v_1)-X(v_2+v_1) &= Y(S(v_3+v_2+v_1))-Y(S(v_2+v_1)),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

If the corresponding set $S(\cdot)\cap V^*$ with respect to the increasing sequence of points $v_1, v_1+v_2, v_1+v_2+v_3, \ldots$ is an increasing sequence of subsets, the above differences are of the form

$$\begin{aligned} X(v_1) &= Y(S(v_1)),\\ X(v_2+v_1)-X(v_1) &= Y(S(v_2+v_1)\backslash S(v_1)),\\ X(v_3+v_2+v_1)-X(v_2+v_1) &= Y(S(v_3+v_2+v_1)\backslash S(v_2+v_1)),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and are independent of each other.

Let us show the above. To make it simple consider the following two sets: $S(v_1) = \{v_1\cdot x \leqslant -1\}$ and $S(v_1+v_2) = \{(v_1+v_2)\cdot x \leqslant -1\}$. We are now concerned with the intersection of these two sets. Let us consider the intersection of the boundary of two sets:

$$B = \{x;\ v_1\cdot x = -1,\ (v_1+v_2)\cdot x = -1\}.$$

It follows that for any $z\in B$, $z\cdot v_2 = 0$. Recall the definition: $V^* = \{u;\ u\cdot v \leqslant 0,\ \forall v\in V\}$; the equation $z\cdot v_2 = 0$ means that the set B is located outside of the dual cone V^*. The distances of two boundary hyperplanes from the origin are $1/\|v_1\|$, $1/\|v_1+v_2\|$. Consequently, we have $\|v_1\| < \|v_1+v_2\|$. These facts prove that the following relation of the sets holds:

$$\big(S(v_1+v_2)\cap V^*\big)\supset \big(S(v_1)\cap V^*\big).$$

Since the support of the Chentsov–Mori measure related to this field X is V^*, $X(v_1+v_2)-X(v_1)$ is independent of $X(v_1)$.

In the same manner we can prove the main result.

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Received on 7.6.2002