PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 25, Fasc. 2 (2005), pp. 289–306

PERIODIC OBSERVATIONS OF HARMONIZABLE SYMMETRIC STABLE SEQUENCES

BY

LUTZ KLOTZ AND MANFRED RIEDEL (LEIPZIG)

Abstract. For harmonizable symmetric stable sequences we solve the following prediction problem: Assume that the values of the sequence are known at all odd integers. Compute the metric projection of an unknown value onto the space spanned by the known values as well as the corresponding approximation error. We study several questions related to this prediction problem such as regularity and singularity, Wold type decomposition, interrelations between the spaces spanned by the values at the even and odd integers, respectively.

2000 AMS Subject Classification: 60G25, 60G10, 42A10.

Key words and phrases: Harmonizable symmetric stable sequences, linear prediction, metric projection, Wold type decomposition.

1. INTRODUCTION

Let $\alpha \in (1, 2]$. A stochastic sequence $X := \{X_k: k \in Z\}$ of complex random variables is called a *symmetric α-stable* ($S\alpha S$) *sequence* if the elements of the linear span $l(X)$ of X_k, $k \in Z$, are $S\alpha S$ random variables.

In [4] and [1] concepts of a semi-inner product and a corresponding norm $\|\cdot\|_\alpha$ on $l(X)$ were introduced. Of course, random variables coinciding almost everywhere with respect to the underlying probability measure are identified. It turns out that convergence with respect to $\|\cdot\|_\alpha$ is equivalent to convergence in probability and that the closure of $l(X)$ with respect to $\|\cdot\|_\alpha$, which is denoted by $L(X)$ and is called the *time domain* of X, consists of jointly $S\alpha S$ random variables. One of the main goals of the study of $S\alpha S$ sequences or $S\alpha S$ processes is to extend properties of the well-known Gaussian processes, which correspond to the case $\alpha = 2$, to appropriate classes of $S\alpha S$ processes. We refer to [15] for a comprehensive treatise on stable processes or to the earlier survey paper [19] which is enough for our purposes.

Linear prediction theory is one of the most important achievements of the theory of stationary Gaussian sequences. In order to extend its results to stable processes with $\alpha \neq 2$, one needs a class of sequences which allows to apply

Fourier analysis. This leads to the notion of harmonizable $S\alpha S$ sequences. An $S\alpha S$ sequence X is called *harmonizable* if there exists an independently scattered $S\alpha S$ measure Z on the Borel σ-algebra of $[0, 2\pi) =: I$ such that $\mu(\cdot) := \|Z(\cdot)\|_\alpha^\alpha$ is finite and $X_k = \int_I e^{ikx} Z(dx)$, $k \in \boldsymbol{Z}$. The space $L^\alpha(\mu)$ of (equivalence classes of) measurable $\boldsymbol{C}$-valued functions α-integrable with respect to μ is called the *spectral domain* of X. According to Lemma 1.3 of [17] there exists an isometric isomorphism j between the time and the spectral domains of X such that $jX_k = e^{ik\cdot}$, $k \in \boldsymbol{Z}$. In case $\alpha = 2$, a harmonizable $S\alpha S$ sequence is stationary Gaussian and j is the well-known Kolmogorov's isomorphism.

The isometry j enables us to formulate linear prediction problems posed in terms of the rather abstract time domain of X as problems of trigonometric approximation in $L^\alpha(\mu)$. We emphasize that throughout the paper we shall work with spectral domains, i.e., we shall state and prove our results in the language of trigonometric approximation theory. So our assertions could be easily extended from $\alpha \in (1, 2]$ to $\alpha \in (1, \infty)$. However, since the case $\alpha \in (2, \infty)$ does not seem to be of statistical significance, we confine ourselves to $\alpha \in (1, 2]$.

Kolmogorov's extrapolation problem and the linear interpolation problem are the most extensively studied linear prediction problems. In Kolmogorov's problem it is assumed that the whole past of the sequence is known, i.e., X_k is known at any negative integer k. For harmonizable $S\alpha S$ sequences this problem was studied in [8], [5] and [3]. The linear interpolation problem deals with the case that all but one values of X can be observed. For $S\alpha S$ sequences it was investigated by Pourahmadi [13] and Weron [17]. In the latter paper, in the more general setting of processes on discrete groups the problem that an arbitrary finite number of values is unknown is discussed.

The present paper is devoted to another linear prediction problem which goes back to Yaglom [20]. He assumed that X_k is known for all odd (or, equivalently, for all even) integers k and computed the corresponding prediction error in the stationary Gaussian case ([20], Theorem 1′). Salehi ([14], Theorem 3.3) extended Yaglom's result to multivariate stationary Gaussian sequences and at the same time gave an explicit formula for the best linear approximation. One of the authors gave an extension of this result to Hilbert–Schmidt operator-valued sequences (see [10], Remark 4.11, and [11]). Section 5 of the present paper deals with the solution of Yaglom's prediction problem for harmonizable $S\alpha S$ sequences. Theorem 5.2 contains a description of the best linear prediction as well as a formula for the prediction error. However, we confess that all results just mentioned are somewhat unsatisfactory from point of view of applications since a series representation of predicted values by observed values, what is needed in practice, is unknown. Moreover, for stationary Gaussian sequences there exists a straightforward generalization to the case where X is observed at all integers of the lattice $n\boldsymbol{Z}$ for some integer $n > 1$. Such a generalization could not be given in case $\alpha < 2$.

Apart from a linear prediction problem itself there are related questions worth to investigate. For example, in Kolmogorov's extrapolation problem the concepts of regular and singular sequences play an important role. Section 3 of our paper deals with regularity and singularity with respect to the family $\mathscr{J}_{(n)}$ of translations of the set $n\boldsymbol{Z}$ for some integer $n > 1$. We characterize $\mathscr{J}_{(n)}$-regular and $\mathscr{J}_{(n)}$-singular harmonizable $S\alpha S$ sequences in terms of their spectral measures; see Theorems 3.2 and 3.4, respectively. In Section 4 these assertions are used to obtain a Wold type decomposition. If $\alpha = 2$, results of this kind are known even for multivariate sequences; see [11], Theorems 5.1 and 5.2.

Let $L_0(X)$ and $L_1(X)$ be the spaces spanned by X_k for even and odd integers k, respectively. The second part of Section 4 as well as Sections 6 and 7 are devoted to interrelations between $L_0(X)$ and $L_1(X)$. For example, we give conditions on μ guaranteeing that $L_0(X) \cap L_1(X)$ has finite dimension (Proposition 4.2 (ii)) or has finite codimension (Proposition 4.3), that the metric projection of $L_0(X)$ onto $L_1(X)$ has finite dimension (Proposition 6.2 (i)) or has finite codimensions with respect to $L_1(X)$ or $L(X)$ (Proposition 6.3), respectively.

In Section 7 we describe those sequences for which the gap between $L_0(X)$ and $L_1(X)$ is positive.

It would be hoped that these results tell us something about the degree of dependence between observations at even and odd integers. Unfortunately, it is likely that their rigorous probabilistic interpretation is difficult since by a theorem of Hardin, Jr. [6] the prediction in the sense of metric projection agrees with the prediction in the sense of conditional expectation only for α-sub-Gaussian processes.

In the concluding section we apply our results to sequences with rational spectral densities which are of particular practical interest. Finally, we mention that some of our assertions seem to be new even for stationary Gaussian sequences.

2. THE BASIC LEMMA

The main goal of this section is to prove Lemma 2.2 below which, despite its simplicity, is a cornerstone of our investigations.

Let $\alpha \in (1, 2]$. Let μ be a non-zero finite measure on the Borel σ-algebra $\mathfrak{B}$ of the interval $I := [0, 2\pi)$ and $L^\alpha(\mu)$ the corresponding Banach space of (equivalence classes of) measurable $\boldsymbol{C}$-valued functions α-integrable with respect to μ. The norm in $L^\alpha(\mu)$ is denoted by $\|\cdot\|_{\alpha,\mu}$. We write $\mathbf{1}_A$ for the indicator function of a set $A \subseteq I$. Let $\boldsymbol{N}$ and $\boldsymbol{N}_0$ denote the sets of positive and non-negative integers, respectively. Throughout Sections 2–4 of the present paper we assume that $n \in \boldsymbol{N}\backslash\{1\}$. We set $\boldsymbol{Z}_n := \{0, 1, \ldots, n-1\}$, $n \in \boldsymbol{N}\backslash\{1\}$, and consider it as an abelian group with addition modulo n. For $m \in \boldsymbol{Z}_n$ we set

$$\xi_{m,n} := \exp\left(\frac{2\pi i}{n} m\right).$$

Finally, for $n \in N\backslash\{1\}$ and $m \in \boldsymbol{Z}_n$ denote by $L^{\alpha}_{m,n}(\mu) =: L_{m,n}$ the subspace of $L^{\alpha}(\mu)$ spanned by functions $e^{i(nk+m)\cdot}$, $k \in \boldsymbol{Z}$.

Here and in the following the notion "subspace" means a closed linear subspace. Let us finally mention that the dependence on α and μ is frequently suppressed in the notation.

The following lemma is obvious.

LEMMA 2.1. *The operator of multiplication by the function $e^{i\cdot}$ maps $L^{\alpha}(\mu)$ onto $L^{\alpha}(\mu)$ and $L_{m,n}$ onto $L_{m+1,n}$, $m \in \boldsymbol{Z}_n$, isometrically.*

To formulate Lemma 2.2 we need some further definitions, notation and conventions. In what follows we shall consider I as an abelian group with addition modulo 2π. Let τ_n be the shift by $(2\pi)/n$, i.e., $\tau_n x := x + (2\pi)/n$, $x \in I$. Moreover, we set $\tau_n A := \{\tau_n x\colon x \in A\}$ for any subset A of I,

$$(\tau_n f)(x) := f(\tau_n^{-1} x) = f\left(x - \frac{2\pi}{n}\right), \qquad x \in I,$$

for any function f on I, and if $C \in \mathfrak{B}$ and ν is a measure on $\mathfrak{B} \cap C$, set

$$(\tau_n \nu)(B) := \nu(\tau_n^{-1} B), \qquad B \in \mathfrak{B} \cap \tau_n C.$$

For $m \in \boldsymbol{Z}_n$ denote by $I_{m,n}$ the interval $\big[(2\pi m)/n, (2\pi(m+1))/n\big)$ and by $\mu_{m,n}$ the restriction of μ to $\mathfrak{B} \cap I_{m,n}$. The measure

$$\tilde{\mu}_n := \sum_{m \in \boldsymbol{Z}_n} \tau_n^{-m} \mu_{m,n}$$

is a finite measure on $\mathfrak{B} \cap I_{0,n}$, and any $\tau_n^{-m} \mu_{m,n}$ is absolutely continuous to $\tilde{\mu}_n$. Denote the corresponding Radon–Nikodym derivative by $h_{m,n}$. Note that

$$\sum_{m \in \boldsymbol{Z}_n} h_{m,n}(x) = 1 \qquad \text{for } \tilde{\mu}_n\text{-a.a. } x \in I. \tag{2.1}$$

We draw the reader's attention to the fact that we shall not always indicate that certain relations are satisfied a.e. with respect to the underlying measures.

A function $\varphi \in L^{\alpha}(\tilde{\mu}_n)$ is extended to the whole interval I by setting it identically 0 outside of $I_{0,n}$.

LEMMA 2.2. *For $m \in \boldsymbol{Z}_n$ the map $V_{m,n}$ defined by*

$$V_{m,n}\varphi := \sum_{k \in \boldsymbol{Z}_n} \xi^k_{m,n} \tau^k_n \varphi, \qquad \varphi \in L^{\alpha}(\tilde{\mu}_n), \tag{2.2}$$

establishes an isometric isomorphism between $L^{\alpha}(\tilde{\mu}_n)$ and $L_{m,n}$. Moreover, if $f \in L^{\alpha}(\mu)$ is a linear combination of functions $e^{i(nk+m)\cdot}$, $k \in \boldsymbol{Z}$, and $\varphi = V^{-1}_{m,n} f$, then

$$f\,\mathbf{1}_{I_{0,n}} = \varphi \quad \tilde{\mu}_n\text{-a.e.} \tag{2.3}$$

Proof. Linearity of $V_{m,n}$ is clear. For $\varphi \in L^\alpha(\tilde{\mu}_n)$ we have

$$\|V_{m,n}\varphi\|_{\alpha,\mu}^\alpha = \int_I \Big|\sum_{k\in \mathbf{Z}_n} \xi_{m,n}^k \tau_n^k \varphi\Big|^\alpha d\mu = \sum_{k\in \mathbf{Z}_n} \int_{I_{k,n}} |\tau_n^k \varphi|^\alpha d\mu_{k,n}$$

$$= \sum_{k\in \mathbf{Z}_n} \int_{I_{0,n}} |\varphi|^\alpha d\tau_n^{-k} \mu_{k,n} = \int_{I_{0,n}} |\varphi|^\alpha d\tilde{\mu}_n,$$

which shows that $V_{m,n}$ is an isometry. Since

$$\tau_n^k e^{i(nl+m)\cdot} \mathbf{1}_{I_{0,n}} = \xi_{m,n}^{-k} e^{i(nl+m)\cdot} \mathbf{1}_{I_{k,n}},$$

we obtain

$$V_{m,n}(e^{i(nl+m)\cdot} \mathbf{1}_{I_{0,n}}) = e^{i(nl+m)\cdot} \in L_{m,n}, \qquad l \in \mathbf{Z}, \tag{2.4}$$

which yields

$$L_{m,n} \subseteq V_{m,n} L^\alpha(\tilde{\mu}_n) \tag{2.5}$$

as well as (2.3). Since the trigonometric polynomials are dense in $L^\alpha(\mu)$, it is not hard to see that for $m \in \mathbf{Z}_n$ the linear span of functions $e^{i(nl+m)\cdot}\mathbf{1}_{I_{0,n}}$, $l \in \mathbf{Z}$, is dense in $L^\alpha(\tilde{\mu}_n)$. Thus, by (2.4) and the continuity of $V_{m,n}$, we obtain $V_{m,n} L^\alpha(\tilde{\mu}_n) \subseteq L_{m,n}$, and taking into account (2.5) we conclude that $V_{m,n} L^\alpha(\tilde{\mu}_n) = L_{m,n}$. ∎

3. $\mathscr{I}_{(n)}$-REGULARITY AND $\mathscr{I}_{(n)}$-SINGULARITY

Let $\mathscr{I}_{(n)}$ be the family of translations of $n\mathbf{Z}$, i.e., $\mathscr{I}_{(n)} := \{n\mathbf{Z}+m\colon m \in \mathbf{Z}_n\}$. In this section we study $\mathscr{I}_{(n)}$-regularity and $\mathscr{I}_{(n)}$-singularity of a harmonizable $S\alpha S$ sequence or, equivalently, of its spectral measure μ.

DEFINITION 3.1. The measure μ is called *$\mathscr{I}_{(n)}$-regular* if $\bigcap_{m\in \mathbf{Z}_n} L_{m,n} = \{0\}$. It is called *$\mathscr{I}_{(n)}$-singular* if $L_{m,n} = L^\alpha(\mu)$ for some, and hence for any $m \in \mathbf{Z}_n$.

Formally, the definitions of $\mathscr{I}_{(n)}$-regularity and $\mathscr{I}_{(n)}$-singularity depend on α, but our results will show that these concepts depend only on μ and n. If $\alpha = 2$, characterizations of $\mathscr{I}_{(n)}$-regular and $\mathscr{I}_{(n)}$-singular measures can be found in Theorem 5.1 of [11]. These assertions can be extended to arbitrary $\alpha \in (1, 2]$. For convenience of the reader we give complete proofs.

To describe $\mathscr{I}_{(n)}$-regularity and $\mathscr{I}_{(n)}$-singularity in terms of μ we introduce the following conditions:

(R) For $\tilde{\mu}_n$-a.a. $x \in I_{0,n}$ there exists an $m \in \mathbf{Z}_n$ such that $0 < h_{m,n}(x) < 1$.

(S) For $\tilde{\mu}_n$-a.a. $x \in I_{0,n}$ there exists an $m \in \mathbf{Z}_n$ such that $h_{m,n}(x) = 1$.

Taking into account (2.1) we can state (R) in two equivalent ways:

(R′) For $\tilde{\mu}_n$-a.a. $x \in I_{0,n}$ there exist $k, m \in \mathbf{Z}_n$, $k \neq m$, such that

$$h_{k,n}(x)\, h_{m,n}(x) \neq 0.$$

(R″) For $\tilde{\mu}_n$-a.a. $x \in I_{0,n}$ and for any $m \in \mathbf{Z}_n$ one has $h_{m,n}(x) < 1$.

Similarly, condition (S) can be given in the following two equivalent forms:

(S′) For $\tilde{\mu}_n$-a.a. $x \in I_{0,n}$ there exists a unique $m \in \boldsymbol{Z}_n$ such that $h_{m,n}(x) = 1$.

(S″) The measures $\tau_n^{-m} \mu_{m,n}$, $m \in \boldsymbol{Z}_n$, are pairwise mutually singular.

THEOREM 3.2. *The measure* μ *is* $\mathscr{J}_{(n)}$*-regular if and only if condition* (R) *is satisfied.*

Proof. If (R″) is not satisfied, then there exist some $m \in \boldsymbol{Z}_n$ and $B \in \mathfrak{B} \cap I_{0,n}$ such that

$$\tilde{\mu}_n(B) > 0 \tag{3.1}$$

and

$$h_{m,n}(x) = 1 \quad \text{for } \tilde{\mu}_n\text{-a.a. } x \in B. \tag{3.2}$$

For $l \in \boldsymbol{Z}_n$ set $\varphi_l := \xi_{l,n}^{-1} \mathbf{1}_B$. By Lemma 2.2 and (3.1) it follows that

$$f_l := \sum_{k \in \boldsymbol{Z}_n} \xi_{l,n}^k \tau_n^k \varphi_l$$

belongs to $L_{l,n}$ and is not the zero function in $L^\alpha(\mu)$. From (3.2) and (2.1) we conclude that $h_{k,n} = 0$ on B for $k \in \boldsymbol{Z}_n$ and $k \neq m$. Thus, for all $l \in \boldsymbol{Z}_n$ the function f_l coincides with the non-zero element $\tau_n^m \mathbf{1}_B$ of $L^\alpha(\mu)$, which implies that μ is not $\mathscr{J}_{(n)}$-regular.

Now suppose that (R) is satisfied. We shall show that $f \in L_{0,n} \cap L_{1,n}$ implies that $f = 0$ in $L^\alpha(\mu)$, which, of course, yields the $\mathscr{J}_{(n)}$-regularity of μ. If $f \in L_{0,n} \cap L_{1,n}$, in view of Lemma 2.2 there exists $\varphi_j \in L^\alpha(\tilde{\mu}_n)$ such that

$$f = \sum_{k \in \boldsymbol{Z}_n} \xi_{j,n}^k \tau_n^k \varphi_j, \quad j \in \boldsymbol{Z}_2.$$

Hence we have

$$\tau_n^{-k}(f \mathbf{1}_{I_{k,n}}) = \xi_{j,n}^k \varphi_j \; \tau_n^{-k} \mu_k\text{-a.e.}, \quad k \in \boldsymbol{Z}_n. \tag{3.3}$$

For $m \in \boldsymbol{Z}_n$ set

$$B_m := \{x \in I_{0,n} \colon h_{m,n}(x) \neq 0\}. \tag{3.4}$$

Condition (R) implies that for $\tilde{\mu}_n$-a.a. $x \in B_m$ there exists some $l \in \boldsymbol{Z}_n \setminus \{m\}$ and $a \in \boldsymbol{R}$ such that

$$h_{m,n}(x) = a h_{l,n}(x) \tag{3.5}$$

and that

$$\tilde{\mu}_n(B_m) = \tilde{\mu}_n\Big(\bigcup_{l \in \boldsymbol{Z}_n \setminus \{m\}} B_m \cap B_l\Big). \tag{3.6}$$

From (3.3) we infer that

$$\xi_{j,n}^{-m} \tau_n^{-m}(f \mathbf{1}_{B_m}) = \xi_{j,n}^{-l}(f \mathbf{1}_{B_l}) \; \tilde{\mu}_n\text{-a.e.}$$

on $B_m \cap B_l$, $l \in \mathbf{Z}_n \setminus \{m\}$, $j \in \mathbf{Z}_2$. Hence, using (3.5) for $\tilde{\mu}_n$-a.a. $x \in B_m \cap B_l$, we get

$$\xi^m_{j,n} \big(\tau_n^{-m} (f\, \mathbf{1}_{B_m}) h_{m,n}\big)(x) - \xi^l_{j,k}\, a \big(\tau_n^{-l} (f\, \mathbf{1}_{B_l}) h_{l,n}\big)(x) = 0, \qquad j \in \mathbf{Z}_2,$$

which can be considered as a system of linear equations with invertible coefficient matrix

$$\begin{pmatrix} \xi^m_{0,n} & \xi^l_{0,n} \\ \xi^m_{1,n} & \xi^l_{1,n} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \xi^m_{1,n} & \xi^l_{1,n} \end{pmatrix}.$$

It follows that

$$\big(\tau_n^{-m} (f\, \mathbf{1}_{B_m}) h_{m,n}\big)(x) = 0 \qquad \text{for } \tilde{\mu}_n\text{-a.a. } x \in B_m \cap B_l.$$

Since $l \in \mathbf{Z}_n \setminus \{m\}$ was arbitrary, (3.6) yields $\tau_n^{-m} (f\, \mathbf{1}_{B_m}) h_{m,n} = 0$ $\tilde{\mu}_n$-a.e. on B_m, and since $m \in \mathbf{Z}_n$ was arbitrary, we get $f = 0$ in $L^\alpha(\mu)$. ■

Remark 3.3. The proof of Theorem 3.2 shows that μ is $\mathscr{J}_{(n)}$-regular if and only if the condition $L_{0,n} \cap L_{1,n} = \{0\}$ is satisfied.

THEOREM 3.4. *The measure μ is $\mathscr{J}_{(n)}$-singular if and only if condition* (S) *is satisfied.*

Proof. For $m \in \mathbf{Z}_n$, let B_m be defined by (3.4). Assume that $L_{0,n} = L^\alpha(\mu)$. Since $\tau_n^m \mathbf{1}_{B_m} \in L^\alpha(\mu)$, from Lemma 2.2 it follows that there exists some $\varphi_m \in L^\alpha(\tilde{\mu}_n)$ such that

$$\sum_{k \in \mathbf{Z}_n} \tau_n^k \varphi_m = \tau_n^m \mathbf{1}_{B_m}.$$

This means that $\varphi_m = 1$ $\tilde{\mu}_n$-a.e. on B_m and $\varphi_m = 0$ $\tilde{\mu}_n$-a.e. on B_l, $l \in \mathbf{Z}_n \setminus \{m\}$. Therefore, $\tilde{\mu}_n(B_m \cap B_l) = 0$, $l, m \in \mathbf{Z}_n$, $l \neq m$, which is equivalent to (S″).

Conversely, if (S) is satisfied, then for any $f \in L^\alpha(\mu)$, the function

$$\varphi := \sum_{k \in \mathbf{Z}_n} \tau_n^{-k} f\, \mathbf{1}_{I_{k,n}} \tag{3.7}$$

belongs to $L^\alpha(\tilde{\mu}_n)$. Since (3.7) implies $f = \sum_{k \in \mathbf{Z}_n} \tau_n^k \varphi$, an application of Lemma 2.2 gives $L_{0,n} = L^\alpha(\mu)$. ■

The space $L^2(\mu)$ is a Hilbert space. The following result contains necessary and sufficient conditions on the measure μ for the pairwise orthogonality of $L_{m,n}$, $m \in \mathbf{Z}_n$, in the case $\alpha = 2$.

THEOREM 3.5. *The subspaces $L_{m,n}$, $m \in \mathbf{Z}_n$, of $L^2(\mu)$ are pairwise orthogonal if and only if $h_{m,n} = n^{-1}$ $\tilde{\mu}_n$-a.e., i.e., if and only if μ is a periodic measure of period $2\pi/n$.*

Proof. Lemma 2.1 implies that the subspaces $L_{m,n}$, $m \in \mathbf{Z}_n$, are pairwise orthogonal if and only if $L_{0,n}$ is orthogonal to any $L_{l,n}$, $l \in \mathbf{Z}_n \setminus \{0\}$. From Lemma 2.2 we see that $L_{0,n}$ and $L_{l,n}$ are orthogonal if and only if for $\varphi, \psi \in L^2(\tilde{\mu}_n)$

we have

$$0=\int_I \sum_{k\in \mathbf{Z}_n} \tau_n^k \varphi\Big(\sum_{k\in \mathbf{Z}_n} \xi_{l,n}^k \tau_n^k \psi\Big)^* d\mu = \sum_{k\in \mathbf{Z}_n} \int_{I_{0,n}} \varphi\psi^* \xi_{l,n}^{-k}\, d\tau_n^{-k} \mu_{k,n}$$

$$= \int_{I_{0,n}} \varphi\psi^* \sum_{k\in \mathbf{Z}_n} \xi_{l,n}^{-k} h_{k,n}\, d\tilde{\mu}_n.$$

This in turn is equivalent to

$$\sum_{k\in \mathbf{Z}_n} \xi_{l,n}^{-k} h_{k,n}(x) = 0 \tag{3.8}$$

for $\tilde{\mu}_n$-a.a. $x\in I_{0,n}$, $l\in \mathbf{Z}_n\backslash\{0\}$. If $x\in I_{0,n}$, the equations (2.1) and (3.8) can be considered as a system of linear equations with unknown quantities $h_{k,n}(x)$. The identity $\sum_{k\in \mathbf{Z}_n} \xi_{l,n}^{-k} = 0$, $l\in \mathbf{Z}_n\backslash\{0\}$, implies that $h_{m,n}(x) = n^{-1}$, $m\in \mathbf{Z}_n$, is a solution. It is unique because the coefficient matrix of the system is invertible.

4. WOLD TYPE DECOMPOSITION AND DIMENSION RESULTS

The descriptions of $\mathscr{I}_{(n)}$-regular and $\mathscr{I}_{(n)}$-singular measures in the preceding section give rise to a Wold type decomposition of the spectral space $L^\alpha(\mu)$ of an $S\alpha S$ sequence into its $\mathscr{I}_{(n)}$-regular and $\mathscr{I}_{(n)}$-singular parts; see (4.1)–(4.3) and (4.5) below. We start with the following settings:

$$\tilde{R}_n := \{x\in I_{0,n}\colon\ h_{m,n}(x) < 1 \text{ for any } m\in \mathbf{Z}_n\},$$

$$\tilde{S}_n := \{x\in I_{0,n}\colon\ h_{m,n}(x) = 1 \text{ for some } m\in \mathbf{Z}_n\},$$

$$R_n := \bigcup_{k\in \mathbf{Z}_n} \tau_n^k \tilde{R}_n, \qquad S_n := \bigcup_{k\in \mathbf{Z}_n} \tau_n^k \tilde{S}_n.$$

Of course, the sets $\tilde{R}_n$ and $\tilde{S}_n$ are defined only within to $\tilde{\mu}_n$-equivalence. We shall not mention this fact in the following and hope this will not cause confusion.

Let ϱ_n and σ_n be restrictions of the measure μ to R_n and S_n, respectively, i.e., $\varrho_n(B) = \mu(B\cap R_n)$ and $\sigma_n(B) = \mu(B\cap S_n)$, $B\in \mathfrak{B}$. According to Theorems 3.2 and 3.4 the measures ϱ_n and σ_n are $\mathscr{I}_{(n)}$-regular and $\mathscr{I}_{(n)}$-singular, respectively. Moreover, ϱ_n and σ_n are mutually singular and $\mu = \varrho_n+\sigma_n$. In a canonical way $L^\alpha(\varrho_n)$ and $L^\alpha(\sigma_n)$ can be considered as subspaces of $L^\alpha(\mu)$, and $L^\alpha(\mu)$ becomes a direct sum of $L^\alpha(\varrho_n)$ and $L^\alpha(\sigma_n)$, i.e.,

$$L^\alpha(\mu) = L^\alpha(\varrho_n)\dot{+}L^\alpha(\sigma_n). \tag{4.1}$$

For $m\in \mathbf{Z}_n$, let $L_{m,n}^{(\varrho)}$ and $L_{m,n}^{(\sigma)}$ be the subspaces of $L^\alpha(\varrho_n)$ and $L^\alpha(\sigma_n)$, respectively, spanned by the functions $e^{i(nk+m)\cdot}$, $k\in \mathbf{Z}$. Note that $L_{m,n}^{(\varrho)}$ is spanned by functions $e^{i(nk+m)\cdot}\mathbf{1}_{R_n}$, and $L_{m,n}^{(\sigma)}$ by $e^{i(nk+m)\cdot}\mathbf{1}_{S_n}$, $k\in \mathbf{Z}$. Let us remark that

$$\bigcap_{k\in \mathbf{Z}_n} L_{k,n}^{(\varrho)} = \{0\} \tag{4.2}$$

and

$$L^{(\sigma)}_{m,n} = L^\alpha(\sigma_n), \qquad m \in \boldsymbol{Z}_n. \tag{4.3}$$

Let $f \in L^\alpha(\mu)$. Since $\int_I |f|^\alpha d\mu = \int_{R_n} |f|^\alpha d\mu + \int_{S_n} |f|^\alpha d\mu$, for $m \in \boldsymbol{Z}_n$ we get $\mathbf{1}_{R_n} L_{m,n} \subseteq L^{(\varrho)}_{m,n}$ and $\mathbf{1}_{S_n} L_{m,n} \subseteq L^{(\sigma)}_{m,n}$. Therefore, from the decomposition $f = f\,\mathbf{1}_{R_n} + f\,\mathbf{1}_{S_n}$ it follows that

$$L_{m,n} \subseteq L^{(\varrho)}_{m,n} \dot{+} L^{(\sigma)}_{m,n}, \qquad m \in \boldsymbol{Z}_n. \tag{4.4}$$

To see that in (4.4) one has equality, introduce measures $\tilde{\varrho}_n$ and $\tilde{\sigma}_n$, which are restrictions of $\tilde{\mu}_n$ to $\tilde{R}_n$ and $\tilde{S}_n$, respectively. If $f^{(\varrho)} \in L^{(\varrho)}_{m,n}$, according to Lemma 2.2 it can be written in the form

$$f^{(\varrho)} = \sum_{k \in \boldsymbol{Z}_n} \xi^k_{m,n} \tau^k_n \varphi^{(\varrho)}_m \qquad \text{for some } \varphi^{(\varrho)}_m \in L^\alpha(\tilde{\varrho}_n).$$

Defining the function $f \in L^{(\varrho)}_{m,n}$ by $f := \sum_{k \in \boldsymbol{Z}_n} \xi^k_{m,n} \tau^k_n \varphi^{(\varrho)}_m$, we have $f = f^{(\varrho)}$ on $\tilde{R}_n$ and $f = 0$ on $I \backslash \tilde{R}_n$. In other words, $L^{(\varrho)}_{m,n}$ can be considered as a subspace of $L_{m,n}$, $m \in \boldsymbol{Z}_n$. Similarly, $L^{(\sigma)}_{m,n} \subseteq L_{m,n}$. Combining these results with (4.4) we conclude that

$$L_{m,n} = L^{(\varrho)}_{m,n} \dot{+} L^{(\sigma)}_{m,n}, \qquad m \in \boldsymbol{Z}_n. \tag{4.5}$$

We shall show now that the relation

$$\bigcap_{m \in \boldsymbol{Z}_n} L_{m,n} = L^\alpha(\sigma_n) \tag{4.6}$$

is a consequence of the preceding constructions. To see that (4.6) is true note first that

$$\bigcap_{m \in \boldsymbol{Z}_n} \left(L^{(\varrho)}_{m,n} \dot{+} L^\alpha(\sigma_n)\right) = \bigcap_{m \in \boldsymbol{Z}_n} L^{(\varrho)}_{m,n} \dot{+} L^\alpha(\sigma_n),$$

since $L^\alpha(\sigma_n)$ and the algebraic sum of $L^{(\varrho)}_{m,n}$, $m \in \boldsymbol{Z}_n$, have intersection $\{0\}$. Now from (4.5), (4.3), and (4.2) we obtain (4.6).

For a subspace L of a Banach space X, denote by $\dim L$ its dimension and by $\operatorname{codim}(L \mid X)$ its codimension with respect to X. Since ϱ_n and σ_n are concentrated on disjoint sets, it is easy to derive from the Wold type decomposition of $L^\alpha(\mu)$ characterizations of those measures μ with finite dimensions or codimensions of $L_{m,n}$ or $\bigcap_{m \in \boldsymbol{Z}_n} L_{m,n}$, respectively. It is somewhat more convenient to state these results in terms of the support $\operatorname{supp} \tilde{\mu}_n$ of the measure $\tilde{\mu}_n$. Moreover, we use the following convention. We say that a certain subset $B \subseteq \operatorname{supp} \tilde{\mu}_n$ is a *finite set* or has *exactly d elements*, $d \in \boldsymbol{N}_0$, if B consists only of a finite number of jumps or has exactly d jumps of $\tilde{\mu}_n$, respectively, and does not have other mass points of $\tilde{\mu}_n$.

We start with a standard fact.

LEMMA 4.1. *Let* $d \in \boldsymbol{N}_0$. *Then* $\dim L^\alpha(\mu) = d$ *if and only if* $\operatorname{supp} \mu$ *consists of exactly* d *elements.*

By Lemma 2.1 it follows that $\dim L_{m,n}$ and $\operatorname{codim}(L_{m,n} \mid L^\alpha(\mu))$ do not depend on $m \in \boldsymbol{Z}_n$. Moreover, since $L_{m,n}$ is a subspace of $L^\alpha(\mu)$ and since the algebraic sum of $L_{m,n}$, $m \in \boldsymbol{Z}_n$, is dense in $L^\alpha(\mu)$, we obtain the inequalities

$$\dim L_{m,n} \leqslant \dim L^\alpha(\mu) \leqslant n \dim L_{m,n}. \tag{4.7}$$

In particular, $\dim L_{m,n} < \infty$ if and only if $\dim L^\alpha(\mu) < \infty$.

PROPOSITION 4.2. *Let $d \in \boldsymbol{N}_0$. Then*

(i) $\dim L_{m,n} = d$, $m \in \boldsymbol{Z}_n$, *if and only if* $\operatorname{supp} \tilde{\mu}_n$ *has exactly* d *elements.*

(ii) $\dim \bigcap_{m \in \boldsymbol{Z}_n} L_{m,n} = d$ *if and only if* $\tilde{S}_n$ *or, equivalently,* S_n *has exactly* d *elements.*

Proof. (i) is a consequence of Lemmas 2.2 and 4.1.

Using definitions of σ_n and $\tilde{\sigma}_n$ we see that $\dim L^\alpha(\sigma_n) = \dim L^\alpha(\tilde{\sigma}_n)$. Now from (4.6) and by virtue of Lemma 4.1 we get the assertion (ii). ∎

Proposition 4.2 (ii) is a refinement of Theorem 3.2. The next result contains a refinement of Theorem 3.4.

PROPOSITION 4.3. *The following assertions are equivalent:*

(i) $\operatorname{codim}(L_{m,n} \mid L^\alpha(\mu)) < \infty$, $m \in \boldsymbol{Z}_n$;

(ii) $\operatorname{codim}(\bigcap_{k \in \boldsymbol{Z}_n} L_{k,n} \mid L^\alpha(\mu)) < \infty$;

(iii) $\operatorname{codim}(\bigcap_{k \in \boldsymbol{Z}_n} L_{k,n} \mid L_{m,n}) < \infty$, $m \in \boldsymbol{Z}_n$;

(iv) R_n *or, equivalently,* $\tilde{R}_n$ *are finite sets.*

If for $d \in \boldsymbol{N}_0$ *the set* $\tilde{R}_n$ *has exactly* d *elements, then*

$$d \leqslant \operatorname{codim}(L_{m,n} \mid L^\alpha(\mu)) \leqslant (n-1)d, \quad m \in \boldsymbol{Z}_n, \tag{4.8}$$

$$2d \leqslant \operatorname{codim}\Big(\bigcap_{k \in \boldsymbol{Z}_n} L_{k,n} \mid L^\alpha(\mu)\Big) \leqslant nd, \tag{4.9}$$

$$\operatorname{codim}\Big(\bigcap_{k \in \boldsymbol{Z}_n} L_{k,n} \mid L_{m,n}\Big) = d, \quad m \in \boldsymbol{Z}_n. \tag{4.10}$$

Proof. Since all codimensions under consideration do not depend on the $\mathscr{I}_{(n)}$-singular part of μ, we can and shall assume that μ is $\mathscr{I}_{(n)}$-regular, i.e., $\tilde{S}_n$ is empty. Then (ii) and (iii), respectively, are equivalent to:

(ii′) $\dim L^\alpha(\mu) < \infty$;

(iii′) $\dim L_{m,n} < \infty$, $m \in \boldsymbol{Z}_n$.

(i) $\Rightarrow$ (ii′). By Remark 3.3, we have $L_{0,n} \cap L_{1,n} = \{0\}$, and hence

$$\dim L_{m,n} = \dim L_{1,n} \leqslant \operatorname{codim}(L_{0,n} \mid L^\alpha(\mu)) < \infty, \quad m \in \boldsymbol{Z}_n.$$

Now apply (4.7) to obtain (ii′).

(ii′) $\Rightarrow$ (iii′) is trivial.

(iii′) $\Rightarrow$ (iv) follows from Proposition 4.2 (i) and the fact that R_n is finite if and only if $\tilde{R}_n$ is finite.

(iv) $\Rightarrow$ (i) is an immediate consequence of Lemma 4.1.

Now we suppose that $\tilde{R}_n$ has exactly d elements. Then equality (4.10) is a consequence of Proposition 4.2 (i). From the definition of R_n it follows that R_n does not have less than $2d$ and more than nd elements. Applying Lemma 4.1 we get (4.9). Finally, it is easy to see that (4.8) follows from (4.9), (4.10) and the relation $\dim L^\alpha(\mu) = \dim L_{m,n} + \operatorname{codim}(L_{m,n} \mid L^\alpha(\mu))$. ∎

If the set $\tilde{R}_n$ is empty, $\operatorname{codim}(\bigcap_{k\in \mathbf{Z}_n} L_{k,n} \mid L^\alpha(\mu)) = 0$ by Theorem 3.2. Therefore, (4.9) leads to the following corollary.

COROLLARY 4.4. *The codimension* $\operatorname{codim}(\bigcap_{k\in \mathbf{Z}_n} L_{k,n} \mid L^\alpha(\mu))$ *cannot be equal to* 1.

5. METRIC PROJECTIONS ONTO L_1

To the end of the paper we are mainly concerned with the case $n = 2$. To simplify the notation we shall frequently suppress the dependence on $n = 2$ in the notation, i.e., we set $L_j := L_{j,2}$, $h_j := h_{j,2}$, $I_j := I_{j,2}$, $\mu_j := \mu_{j,2}$, $j \in \mathbf{Z}_2$, $\tau := \tau_2$, $\tilde{\mu} := \tilde{\mu}_2$, $\tilde{R} := \tilde{R}_2$, $\tilde{S} := \tilde{S}_2$, $R := R_2$, $S := S_2$, $\varrho := \varrho_2$, $\sigma := \sigma_2$.

Since $L^\alpha(\mu)$ is a strictly convex reflexive Banach space, for any subspace L of $L^\alpha(\mu)$ and any $f \in L^\alpha(\mu)$ there exists a unique $P_L f \in L$ such that

$$\min\{\|f-g\|_{\alpha,\mu} \colon g \in L\} = \|f - P_L f\|_{\alpha,\mu}$$

([16], Corollaries 2.4 and 3.3 of Chapter I). The operator P_L is called the *metric projection onto L* and the element $P_L f$ is called the *metric projection of f onto L*. Of course, if $\alpha = 2$, the operator P_L is the orthoprojector onto L. Recall, however, that for $\alpha \neq 2$ the operator P_L need not to be linear.

The present section deals with properties of the metric projection onto the space L_1. It will be denoted by P. We mention that because of Lemma 2.1 properties of P immediately lead to the corresponding properties of P_{L_0}. We wish to derive a formula for Pf, $f \in L^\alpha(\mu)$. Since $Pf \in L_1$, according to Lemma 2.2 there exists a unique $\psi \in L^\alpha(\tilde{\mu})$ such that $Pf = \psi - \tau\psi$ and it suffices to compute ψ. To do this we need the following lemma.

LEMMA 5.1. *Let* $a_0, a_1 \in \mathbf{R}$, $a_0 \geqslant 0$, $a_1 \geqslant 0$, $a_0 + a_1 > 0$, *and* $z_0, z_1 \in \mathbf{C}$. *The function*

$$u(z) := |z - z_0|^\alpha a_0 + |z + z_1|^\alpha a_1, \quad z \in \mathbf{C}, \tag{5.1}$$

attains its minimum at the point $(a_0^{1/(\alpha-1)} z_0 - a_1^{1/(\alpha-1)} z_1)(a_0^{1/(\alpha-1)} + a_1^{1/(\alpha-1)})^{-1}$.

Proof. Geometrical arguments show that the minimum of u is attained at a point of the form $z = tz_0 - (1-t)z_1$, $t \in [0, 1]$. Putting this expression into (5.1) and computing the minimum of the corresponding function of t, we get the assertion. ∎

THEOREM 5.2. *For any* $f \in L^\alpha(\mu)$, *the function* Pf *can be computed by* $Pf = \psi_f - \tau\psi_f$, *where*

$$\psi_f := \frac{f h_0^{1/(\alpha-1)} - (\tau^{-1} f) h_1^{1/(\alpha-1)}}{h_0^{1/(\alpha-1)} + h_1^{1/(\alpha-1)}} \quad \tilde{\mu}\text{-a.e.} \tag{5.2}$$

Moreover,

$$\|f-Pf\|^{\alpha}_{\alpha,\mu} = \int_{I_0} |f+\tau^{-1}f|^{\alpha} h_0 h_1 (h_0^{1/(\alpha-1)}+h_1^{1/(\alpha-1)})^{1-\alpha} d\tilde{\mu}. \tag{5.3}$$

Proof. By Lemma 2.2 for $g\in L_1$ there exists $\psi\in L^{\alpha}(\tilde{\mu})$ such that $g=\psi-\tau\psi$. It follows that

$$\begin{aligned}\|f-g\|^{\alpha}_{\alpha,\mu} &= \int_{I_0} |f-\psi|^{\alpha} d\mu_0 + \int_{I_1} |f+\tau\psi|^{\alpha} d\mu_1 \\ &= \int_{I_0} (|f-\psi|^{\alpha} h_0 + |\tau^{-1} f+\psi|^{\alpha} h_1)\, d\tilde{\mu}.\end{aligned}$$

To minimize the integral on the right-hand side it is enough to minimize the integrand for $\tilde{\mu}$-a.a. $x\in I_0$. From Lemma 5.1 we conclude that this minimum is attained at ψ_f given by (5.2). In order to prove that $Pf := \psi_f-\tau\psi_f$ belongs to L_1, according to Lemma 2.2 it suffices to show that $\psi_f\in L^{\alpha}(\tilde{\mu})$. Since $h_0+h_1=1$, we have $h_j^{\alpha/(\alpha-1)}\leqslant h_j$, $j\in \boldsymbol{Z}_2$, and there exists a positive constant c such that $h_0^{1/(\alpha-1)}+h_1^{1/(\alpha-1)}\geqslant c$. Therefore, we get

$$\begin{aligned}\int_{I_0} |\psi_f|^{\alpha} d\tilde{\mu} &\leqslant 2^{\alpha-1}\Big[\int_{I_0} |f|^{\alpha} h_0^{\alpha/(\alpha-1)} \big(h_0^{1/(\alpha-1)}+h_1^{1/(\alpha-1)}\big)^{-\alpha} d\tilde{\mu} \\ &\quad + \int_{I_0} |\tau^{-1} f|^{\alpha} h_1^{\alpha/(\alpha-1)} (h_0^{1/(\alpha-1)}+h_1^{1/(\alpha-1)})^{-\alpha} d\tilde{\mu}\Big] \\ &\leqslant 2^{\alpha-1} c^{-\alpha}\Big[\int_{I_0} |f|^{\alpha} h_0\, d\tilde{\mu} + \int_{I_0} |\tau^{-1} f|^{\alpha} h_1\, d\tilde{\mu}\Big] = 2^{\alpha-1} c^{-\alpha} \|f\|^{\alpha}_{\alpha,\mu} < \infty.\end{aligned}$$

Finally, the relation (5.3) follows by a straightforward computation. ■

If $\alpha=2$, the denominator on the right-hand side of (5.2) equals 1 and the formulas of Theorem 5.2 simplify themselves to known results (see [20], Theorem 1′; [14], Theorem 3.3; [10], Remark 4.11).

If $f\in L_0$, then we have $f=\varphi+\tau\varphi$ for a unique $\varphi\in L^{\alpha}(\tilde{\mu})$. Hence (5.2) and (5.3) take the forms

$$\psi_f = \frac{\varphi\,(h_0^{1/(\alpha-1)}-h_1^{1/(\alpha-1)})}{h_0^{1/(\alpha-1)}+h_1^{1/(\alpha-1)}} \tag{5.4}$$

and

$$\|f-Pf\|^{\alpha}_{\alpha,\mu} = 2^{\alpha}\int_{I_0} |\varphi|^{\alpha} h_0 h_1 (h_0^{1/(\alpha-1)}+h_1^{1/(\alpha-1)})^{1-\alpha} d\tilde{\mu}, \tag{5.5}$$

respectively.

We have another immediate consequence of Theorem 5.2.

COROLLARY 5.3. *The metric projection onto L_1 is a linear operator.*

It would be of some interest to determine the norm $\|P\|$ of P, but this problem seems to be difficult unless $\alpha=2$. If $\alpha\in(1, 2)$, all we can say is that from general Banach space geometry one obtains the estimate $1\leqslant\|P\|\leqslant 2^{2/\alpha-1}$

(see [12], Corollary 2.3). We mention that the results of [12] established for real Banach spaces remain also true for complex spaces.*

6. DIMENSION RESULTS FOR $\overline{PL_0}$

For a subset M of a Banach space, denote by $\overline{M}$ its closure. From Corollary 5.3 it follows that $\overline{PL_0}$ is a linear space. We have the inclusions

$$\{0\} = L_0 \cap L_1 \subseteq \overline{PL_0} \subseteq L_1 \subseteq L^\alpha(\mu).$$

It is of interest to describe those measures for which one or another space occurring in this chain of inclusions has a finite dimension or a finite codimension. We do not recall the results which can be obtained by choosing $n = 2$ in Propositions 4.2 and 4.3. We only mention that (4.9) leads to the following corollary.

COROLLARY 6.1. *The codimension* $\operatorname{codim}(L_0 \cap L_1 \mid L^\alpha(\mu))$ *is either infinite or equal to an even non-negative integer.*

Now we establish dimension results for $\overline{PL_0}$ using Theorem 5.2. Note that $\tilde{R}$ and $\tilde{S}$ determined in Section 4 can be defined in case $n = 2$ by

$$\tilde{R} := \{x \in \operatorname{supp} \tilde{\mu}\colon\ h_0(x) h_1(x) \neq 0\}, \qquad \tilde{S} := \{x \in \operatorname{supp} \tilde{\mu}\colon\ h_0(x) h_1(x) = 0\}.$$

Set $\tilde{R}_e := \{x \in \tilde{R}\colon\ h_0(x) = h_1(x)\}$.

PROPOSITION 6.2. *Let* $d \in N_0$. *Then*

(i) $\dim \overline{PL_0} = d$ *if and only if* $\operatorname{supp} \tilde{\mu} \backslash \tilde{R}_e$ *has exactly* d *elements.*

(ii) $\operatorname{codim}(L_0 \cap L_1 \mid \overline{PL_0}) = d$ *if and only if* $\tilde{R} \backslash \tilde{R}_e$ *has exactly* d *elements.*

Proof. (i) Since $\overline{PL_0}$ is a subspace of L_1, Lemma 2.2 shows that $\dim \overline{PL_0}$ is equal to dimension of the subspace $V_{1,2}^{-1} \overline{PL_0}$ of $L^\alpha(\tilde{\mu})$. Thus, the result immediately follows from (5.4) and Lemma 4.1.

(ii) From (5.2) we conclude that P is a direct sum of the corresponding metric projections in $L^\alpha(\varrho)$ and $L^\alpha(\sigma)$. Since for $\mathscr{I}_{(2)}$-singular measures the assertion is trivial, we can assume that μ is $\mathscr{I}_{(2)}$-regular, i.e., $\tilde{S}$ is empty. Then $\operatorname{codim}(L_0 \cap L_1 \mid \overline{PL_0}) = \dim \overline{PL_0}$ and an application of (i) completes the proof. ■

Note that by setting $d = 0$ in Proposition 6.2 (i) one obtains a result which can be considered as an extension of Theorem 3.5 to arbitrary $\alpha \in (1, 2]$ for the case $n = 2$.

PROPOSITION 6.3. (i) *Let* $d \in N_0$. *Then* $\operatorname{codim}(\overline{PL_0} \mid L_1) = d$ *if and only if* $\tilde{R}_e$ *consists of exactly* d *elements.*

* Helpful discussions with F. Mazzone and P. Wojtaszczyk are gratefully acknowledged.

(ii) $\operatorname{codim}(\overline{PL_0} \mid L^\alpha(\mu)) < \infty$ *if and only if* $\tilde{R}$ *or, equivalently,* R *is a finite set. If* $\tilde{R}_e$ *has* d_1 *elements and* $\tilde{R}\setminus\tilde{R}_e$ *has* d_2 *elements, then* $\operatorname{codim}(\overline{PL_0} \mid L^\alpha(\mu)) = 2d_1 + d_2$, $d_1, d_2 \in N_0$.

Proof. (i) As in the proof of Proposition 6.2 (i) we can and shall assume that μ is $\mathscr{J}_{(2)}$-regular. Since by Lemma 2.2

$$\operatorname{codim}(\overline{PL_0} \mid L_1) = \operatorname{codim}(V_{1,2}^{-1}\overline{PL_0} \mid L^\alpha(\tilde{\mu})),$$

from (5.4) we infer that $\operatorname{codim}(\overline{PL_0} \mid L_1)$ can be finite only if $\tilde{R}_e$ is finite. Moreover, if $\tilde{R}_e$ has exactly d elements, we obviously have $\operatorname{codim}(\overline{PL_0} \mid L_1) \geqslant d$. In order to complete the proof it is enough to show that any $\varphi \in L^\alpha(\tilde{\mu})$ such that $\varphi = 0$ on $\tilde{R}_e$ can be approximated by functions of $V_{1,2}^{-1}\overline{PL_0}$. Define $h := h_1/h_0$, which is possible because $h_0 \neq 0$ on $\tilde{R}$. For $k \in N$, set

$$B_k := \{x \in \tilde{R}\colon |1-h(x)| > 1/k \text{ and } |1+h(x)| < k\}.$$

Since $\dot{\bigcup}_{k\in N} B_k = \tilde{R}\setminus\tilde{R}_e$ and $\varphi = 0$ on $\tilde{R}_e$, for $\varepsilon > 0$ there exists $j \in N$ such that $\int_{\tilde{R}\setminus B_j} |\varphi|^\alpha d\tilde{\mu} < \varepsilon$. Define a function $\varphi_j \in L^\alpha(\mu)$ setting

$$\varphi_j = \begin{cases} \varphi(1+h^{1/(\alpha-1)})(1-h^{1/(\alpha-1)})^{-1} & \text{on } B_j, \\ 0 & \text{on } \tilde{R}\setminus B_j. \end{cases}$$

Then we have

$$\int_{\tilde{R}} |\varphi_j(h_0^{1/(\alpha-1)} - h_1^{1/(\alpha-1)})(h_0^{1/(\alpha-1)} + h_1^{1/(\alpha-1)})^{-1} - \varphi|^\alpha d\tilde{\mu} = \int_{\tilde{R}\setminus B_j} |\varphi|^\alpha d\tilde{\mu} < \varepsilon.$$

Now (5.4) shows that $V_{1,2}^{-1}P\varphi_j$ is the desired approximation.

(ii) Taking into account (i) and (4.8) we obtain all results from the equality

$$\operatorname{codim}(\overline{PL_0} \mid L^\alpha(\mu)) = \operatorname{codim}(\overline{PL_0} \mid L_1) + \operatorname{codim}(L_1 \mid L^\alpha(\mu)).$$

7. THE GAP BETWEEN L_0 AND L_1

Let M be a subset of a Banach space X, whose norm is denoted by $\|\cdot\|$. For $f \in X$ set $\operatorname{dist}(f, M) := \inf\{\|f-g\|\colon g \in M\}$. If L and N are subspaces of X such that L is not a subspace of N, the gap $\gamma(L, N)$ between L and N is defined by

$$\gamma(L, N) := \inf\left\{\frac{\operatorname{dist}(f, N)}{\operatorname{dist}(f, L\cap N)}\colon f \in L\setminus N\right\}. \tag{7.1}$$

We recall some facts on $\gamma(L, N)$ and refer to [9] for details and proofs. Clearly, $0 \leqslant \gamma(L, N) \leqslant 1$. According to Theorem 4.2 of Chapter IV in [9], $\gamma(L, N) > 0$ if and only if the algebraic sum of L and N is closed. Although in general $\gamma(L, N) \neq \gamma(N, L)$ (see [9]), the result just mentioned implies that $\gamma(L, N) > 0$ if and only if $\gamma(N, L) > 0$. To get an idea of what relation $\gamma(L, N) > 0$ means assume that X is a Hilbert space. If $L \cap N = \{0\}$, the angle $\vartheta \in [0, \pi/2]$

between L and N can be defined by

$$\cos\vartheta := \sup\left\{\frac{\|P_L f\|}{\|f\|}\colon f\in L,\ f\neq 0\right\}. \tag{7.2}$$

If $L\cap N\neq\{0\}$, let L' and N' be the orthogonal complements of $L\cap N$ in L and N, respectively. Since $\gamma(L, N)=\gamma(L', N')$ (see [9], p. 220), from (7.1) and (7.2) we easily obtain $\gamma(L, N)^2 = 1-\cos^2\vartheta'$, where ϑ' denotes the angle between L' and N'. Thus, $\gamma(L, N)>0$ if and only if the angle between L' and N' is positive.

Since the algebraic sum of L_0 and L_1 is dense in $L^\alpha(\mu)$, we obtain $L_0\subseteq L_1$ if and only if μ is $\mathscr{J}_{(2)}$-singular.

Now we wish to describe all non-$\mathscr{J}_{(2)}$-singular measures μ such that the inequality $\gamma(L_0, L_1)>0$ holds. From (7.1) and the definition of the metric projection we conclude

$$\gamma(L_0, L_1) = \inf\left\{\frac{\|f-Pf\|_{\alpha,\mu}}{\|f-P_{L_0\cap L_1}f\|_{\alpha,\mu}}\colon f\in L_0\backslash L_1\right\}. \tag{7.3}$$

Since $L_0\cap L_1 = L^\alpha(\sigma)$, it follows that

$$P_{L_0\cap L_1}f = f1_S,\qquad f\in L^\alpha(\mu). \tag{7.4}$$

By Lemma 2.2 a function $f\in L_0$ can be written as $f=\varphi+\tau\varphi$, $\varphi\in L^\alpha(\tilde{\mu})$. Hence (7.4) yields

$$\|f-P_{L_0\cap L_1}f\|^\alpha_{\alpha,\mu} = \int_{\tilde{R}}|\varphi|^\alpha\,d\tilde{\mu}.$$

Taking into account (7.3) and (5.5), we infer that $\gamma(L_0, L_1)>0$ if and only if

$$\inf\left\{\frac{2^\alpha\int_{\tilde{R}}|\varphi|^\alpha h_0 h_1\left(h_0^{1/(\alpha-1)}+h_1^{1/(\alpha-1)}\right)^{1-\alpha}d\tilde{\mu}}{\int_{\tilde{R}}|\varphi|^\alpha\,d\tilde{\mu}}\colon \varphi\in L^\alpha(\tilde{\mu})\right\}>0,$$

which in turn is equivalent to the existence of a positive constant c_1 such that

$$h_0 h_1\left(h_0^{1/(\alpha-1)}+h_1^{1/(\alpha-1)}\right)^{1-\alpha}\geqslant c_1\quad \tilde{\mu}\text{-a.e.} \tag{7.5}$$

on $\tilde{R}$. Setting $h := h_1/h_0$ on $\tilde{R}$ and using (2.1), we easily infer that (7.5) is satisfied if and only if there exists a positive constant c such that

$$c\leqslant h\leqslant c^{-1}\quad \tilde{\mu}\text{-a.e.} \tag{7.6}$$

or, equivalently, such that for any $B\in\mathfrak{B}\cap\tilde{R}$

$$c\mu_0(B)\leqslant(\tau^{-1}\mu_1)(B)\leqslant c^{-1}\mu_0(B). \tag{7.7}$$

Thus we have proved the following theorem.

THEOREM 7.1. *Assume that μ is not $\mathscr{J}_{(2)}$-singular. Then the gap between L_0 and L_1 is positive if and only if (7.7) is satisfied.*

THEOREM 7.2. *The relations* $L_0 \cap L_1 = \{0\}$ *and* $\gamma(L_0, L_1) > 0$ *hold if and only if* (7.7) *is true for any* $B \in \mathfrak{B} \cap \operatorname{supp} \tilde{\mu}$.

Proof. The assertion follows from Proposition 4.2 (ii) and Theorem 7.1.

8. RATIONAL SPECTRAL DENSITIES

A harmonizable $S\alpha S$ sequence as well as its spectral measure μ are called to have a *rational spectral density* if μ is of the form

$$d\mu = \left|\frac{p}{q}(e^{i\cdot})\right|^2 d\lambda, \tag{8.1}$$

where p and q are polynomials which can and will be assumed to have no common zeros, and λ denotes the normalized Lebesgue measure on I. Note that since μ is a finite measure, q does not have zeros on the unit circle $\boldsymbol{T}$. In the present section we specify some of the preceding results for sequences with rational spectral densities.

THEOREM 8.1. *If* μ *has a rational spectral density, then for* $n \in \boldsymbol{N}\backslash\{1\}$
(i) μ *is* $\mathscr{I}_{(n)}$*-regular*;
(ii) $\dim L_{m,n} = \infty, \ m \in \boldsymbol{Z}_n$;
(iii) $\operatorname{codim}(L_{m,n} \mid L^{\alpha}(\mu)) = \infty, \ m \in \boldsymbol{Z}_n$.

Proof. The assertions (i), (ii) and (iii) follow from Theorem 3.2, Proposition 4.2 (i) and Proposition 4.3, respectively.

THEOREM 8.2. *Let* μ *be of the form* (8.1). *Then* $\gamma(L_0, L_1) > 0$ *if and only if the set of zeros of* p *on* $\boldsymbol{T}$ *including multiplicities is symmetric with respect to the real axis.*

Proof. It is not hard to see that under the assumption (8.1) the function $h = h_1/h_0$ can be written as

$$h = \left|\frac{qp^-}{pq^-}(e^{i\cdot})\right|^2 \tag{8.2}$$

on I_0. Here for a function g on $\boldsymbol{C}$ the function g^- is defined by $g^-(z) := g(-z)$, $z \in \boldsymbol{C}$. Therefore, (7.6) is satisfied if and only if qp^- and pq^- have the same zeros on $\boldsymbol{T}$ including multiplicities. Since q does not have zeros on $\boldsymbol{T}$ and since z is a zero of p if and only if $-z$ is a zero of p^-, the assertion holds true. ■

To apply the results of Section 6 to sequences with rational spectral densities we need the following lemma.

LEMMA 8.3. *Let* r *and* s *be polynomials. If there exists a set* $A \subseteq I$ *such that* A *has an accumulation point and*

$$|r(e^{ix})| = |s(e^{ix})|, \qquad x \in A, \tag{8.3}$$

then

(i) $|r| = |s|$ *on* T;

(ii) *the zeros of* r *and* s *on* $C\backslash\{0\}$ *coincide including multiplicities.*

Proof. From (8.3) and the existence of an accumulation point of the set A we conclude that the trigonometric polynomials $|r(e^{i\cdot})|^2$ and $|s(e^{i\cdot})|^2$ are identical. This yields (i). Considering polynomials r and s as elements of a Hardy space, we see from (i) that their outer parts coincide. Since the outer part of a polynomial, whose zeros on $C\backslash\{0\}$ are denoted by z_j and w_l, where $|z_j| \geqslant 1$ and $|w_l| < 1$, respectively, is equal to $\prod_j (z-z_j) \prod_l (1-\overline{w}_l z)$, $z \in C$, the assertion (ii) follows. ∎

THEOREM 8.4. *If* μ *has a rational spectral density, then either* $PL_0 = \{0\}$ *or* $\overline{PL_0} = L_1$.

Proof. The relation (8.2) and Lemma 8.3 imply that either $h = 1_{I_0}$ or $h \neq 1_{I_0}$ $\tilde{\mu}$-a.e. According to Propositions 6.2 (i) and 6.3 (i) the first and the second cases are equivalent to $PL_0 = \{0\}$ and $\overline{PL_0} = L_1$, respectively. ∎

It is not difficult to describe the set of polynomials such that $h = 1_{I_0}$. In fact, from (8.2) and Lemma 8.3 we infer that in this case pq^- and qp^- have the same zeros on $C\backslash\{0\}$ including multiplicities. Since p and q do not have common zeros, the polynomials p and p^- have the same zeros on $C\backslash\{0\}$ including multiplicities, which implies that p is either an even polynomial or an odd one. Similarly, q is either even or odd. Since p and q can be odd polynomials only if $p(0) = q(0) = 0$, which is excluded, we get the following result.

PROPOSITION 8.5. *Suppose that* μ *has the form* (8.1). *Then* $PL_0 = \{0\}$ *if and only if either both polynomials* p *and* q *are even or one of them is even and the other is odd.*

Acknowledgment. We would like to thank Professor A. Weron for pointing out to us the papers [2], [7], and [18]. The paper [2] carefully examines the Wold decomposition. The papers [7] and [18] contain some disjointness results.

REFERENCES

[1] S. Cambanis, *Complex symmetric stable variables and processes*, in: *Contribution to Statistics*, P. K. Sen (Ed.), North-Holland, New York 1983, pp. 63–79.

[2] S. Cambanis, C. D. Hardin, and A. Weron, *Innovations and Wold decompositions of stable sequences*, Probab. Theory Related Fields 79 (1988), pp. 1–27.

[3] S. Cambanis and A. G. Miamee, *On prediction of harmonizable stable processes*, Sankhyā Ser. A 51 (1989), pp. 269–294.

[4] S. Cambanis and G. Miller, *Linear problems in pth order and stable processes*, SIAM J. Appl. Math. 41 (1981), pp. 43–69.

[5] S. Cambanis and A. R. Soltani, *Prediction of stable processes: Spectral and moving average representations*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 66 (1984), pp. 593–612.
[6] C. D. Hardin, Jr., *On the spectral representation of symmetric stable processes*, J. Multivariate Anal. 12 (1982), pp. 385–401.
[7] M. Hernández and C. Houdre, *Disjointness results for some classes of stable processes*, Studia Math. 105 (1993), pp. 235–252.
[8] Y. Hosoya, *Harmonizable stable processes*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 60 (1982), pp. 517–533.
[9] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin–Heidelberg–New York 1966.
[10] L. Klotz, *An interpolation problem for Hilbert–Schmidt operator-valued stationary processes*, Z. Anal. Anwendungen 20 (2001), pp. 525–535.
[11] L. Klotz, *Some remarks on an interpolation problem of A. M. Yaglom*, submitted to Teor. Veroyatnost. i Primenen. (2005).
[12] F. Mazzone, *On the norm of the metric projections*, J. Approx. Theory 97 (1999), pp. 214–219.
[13] M. Pourahmadi, *On minimality and interpolation of harmonizable stable processes*, SIAM J. Appl. Math. 44 (1984), pp. 1023–1030.
[14] H. Salehi, *On interpolation of q-variate stationary stochastic processes*, Pacific J. Math. 28 (1969), pp. 183–191.
[15] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes. Stochastic Models with Infinite Variance. Stochastic Modeling*, Chapman & Hall, New York 1994.
[16] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer, Berlin–Heidelberg–New York 1970.
[17] A. Weron, *Harmonizable stable processes on groups: Spectral, ergodic and interpolation properties*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 68 (1985), pp. 473–491.
[18] A. Weron, *A remark on disjointness results for stable processes*, Studia Math. 105 (1993), pp. 253–254.
[19] A. Weron, *Stable Processes and Measures: A Survey*, Lecture Notes in Math. 1080, Springer, Berlin–Heidelberg–New York 1984, pp. 306–364.
[20] A. M. Yaglom, *On problems about the linear interpolation of stationary random sequences and processes* (in Russian), Uspekhi Mat. Nauk 4 (1949), pp. 173–178.

Fakultät für Mathematik und Informatik
Universität Leipzig
04109 Leipzig, Germany
E-mail: riedel@rz.uni-leipzig.de

Received on 7.6.2005;
revised version on 20.10.2005