PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 30, Fasc. 1 (2010), pp. 73–85

LIMIT THEOREMS FOR PRODUCTS OF SUMS OF INDEPENDENT RANDOM VARIABLES*

BY

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Abstract. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with finite second moments and $\{N_n,\ n \geqslant 1\}$ be a sequence of positive integer-valued random variables. Write $S_n = \sum_{k=1}^{n}(X_k - EX_k)$, $n \geqslant 1$, and let N be a standard normal random variable. In the paper the convergences

$$\Big(\prod_{k=1}^{n}(S_k/a_k+1)\Big)^{\gamma_n} \xrightarrow{\mathcal{D}} e^N \quad \text{and} \quad \Big(\prod_{k=1}^{N_n}(S_k/a_k+1)\Big)^{\gamma_n} \xrightarrow{\mathcal{D}} e^N$$

are considered for some sequences $\{a_n\}$ and $\{\gamma_n\}$ of positive integer numbers such that $S_n + a_n \geqslant 0$ a.e. The case when γ_n are random variables is also considered. The main results generalize the main theorems presented by Pang et al. [3].

2000 AMS Mathematics Subject Classification: Primary: 60F05; Secondary: 60G50.

Key words and phrases: Lognormal distribution; randomly indexed product of sums; central limit theorem; self-normalized.

1. INTRODUCTION

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent identically distributed random variables with mean μ and variance σ. In this paper we are interested in limit theorems for products $\prod_{j=1}^{n} S_j$. This study was begun by Arnold and Villaseñor in [1]. They obtained, for the sequence $\{X_n, n \geqslant 1\}$ of i.i.d. exponentially distributed random variables with $\mu = 1$, the following convergence:

$$\Big(\prod_{j=1}^{n}(S_j/j)\Big)^{1/\sqrt{n}} \xrightarrow{\mathcal{D}} e^{\sqrt{2}N}.$$

* Research supported by the Deutsche Forschungsgemeinschaft through the German-Polish project 436 POL 125/0-1 and by TODEQ MTKD-CT-2005-030042.

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This result was generalized by Rempała and Wesołowski. In the paper [7] they omit the assumptions that X_n are exponentially distributed and obtain the convergence

$$\left(\frac{\prod_{j=1}^{n} S_j}{n!\mu^n}\right)^{1/\gamma\sqrt{n}} \xrightarrow{\mathcal{D}} e^{\sqrt{2}N}.$$

On the other hand, this result was generalized by Qi [6] and Lu and Qi [2] to the case of a stable limit law. Furthermore, in [3] the following result was obtained:

THEOREM 1.1. *Assume that the positive random variable X has mean μ $(\mu > 0)$ and is in the domain of attraction of the normal law and let t_n be a positive integer-valued random variable. In addition, assume that there is a positive constant sequence $\{b_n\}$ tending to infinity as $n \to \infty$ such that $t_n/b_n \xrightarrow{\mathcal{P}} \nu$, where ν is a positive random variable and independent of $\{X_i, i \geqslant 1\}$. Then*

$$\left(\frac{\prod_{k=1}^{t_n} S_k}{t_n!\mu^{t_n}}\right)^{\mu/V_{t_n}} \xrightarrow{\mathcal{D}} e^{\sqrt{2}N},$$

where $V_n^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2, \overline{X} = n^{-1}\sum_{i=1}^{n} X_i, n \geqslant 1.$

It is worthwhile to remark that all the above-mentioned results deal with the case of independent identically distributed random variables. If we omit the assumption that $\{X_n, n \geqslant 1\}$ are identically distributed, the computations become very complicated and so far there are no results for products of sums of random variables, which are only independent. In this paper we fill this gap. Furthermore, we consider random normalization similar to that in Theorem 1.1 (selfnormalization) and also randomly indexed products.

2. MAIN RESULTS

Let $\{X_n,\ n \geqslant 1\}$ be a sequence of independent random variables such that $\sigma_n^2 = \mathrm{Var}(X_n) < \infty$, $V_n^2 = \sum_{k=1}^{n} X_k^2$, $S_n = \sum_{k=1}^{n} X_k$, $s_n^2 = \sum_{k=1}^{n} \sigma_k^2 = EV_n^2$. For simplicity of the notation we assume that the sequence $\{X_n,\ n \geqslant 1\}$ is centered, i.e. $EX_n = 0,\ n \geqslant 1$ (except for Corollaries 2.1 and 2.3). In the whole paper $\{a_n,\ n \geqslant 1\}$ denotes a sequence of positive reals divergent to infinity (we put $a_0 = 0$).

THEOREM 2.1. *Let $X_n \geqslant a_{n-1} - a_n$ a.s., $n \geqslant 1$, and*

$$\sum_{n=1}^{\infty} E|X_n|^p/a_n^p < \infty \tag{2.1}$$

for some p, $0 < p \leqslant 2$, and let $\{\gamma_n,\ n \geqslant 1\}$ be a sequence of positive real numbers convergent to zero such that

$$\gamma_n \sum_{k=1}^{n} (s_k^2/a_k^2) \to 0 \quad \textit{as } n \to \infty, \tag{2.2}$$

$$\gamma_n^2 \sum_{k=1}^{n} (A_k^n)^2 \sigma_k^2 \to 1 \quad \textit{as } n \to \infty, \tag{2.3}$$

and for all $\varepsilon > 0$

$$\gamma_n^2 \sum_{k=1}^{n} (A_k^n)^2 E\big[X_k^2 I[\gamma_n A_k^n |X_k| > \varepsilon]\big] \to 0 \quad \textit{as } n \to \infty, \tag{2.4}$$

where $A_i^j = \sum_{k=i}^{j} (1/a_k), 1 \leqslant i \leqslant j$. *Then*

$$\Big(\prod_{k=1}^{n} (S_k/a_k + 1)\Big)^{\gamma_n} \xrightarrow{\mathcal{D}} e^N \quad \textit{as } n \to \infty. \tag{2.5}$$

Furthermore, if the sequence of random variables $\{\lambda_n, n \geqslant 1\}$ *satisfies*

$$\lambda_n^2 \sum_{k=1}^{n} (A_k^n)^2 X_k^2 \xrightarrow{\mathcal{P}} 1 \quad \textit{as } n \to \infty,$$

then

$$\Big(\prod_{k=1}^{n} (S_k/a_k + 1)\Big)^{\lambda_n} \xrightarrow{\mathcal{D}} e^N \quad \textit{as } n \to \infty. \tag{2.6}$$

COROLLARY 2.1. *Let* $\{X_n,\ n \geqslant 1\}$ *be a sequence of independent, positive random variables such that* $EX_n = \mu_n$, *and*

$$\sum_{n=1}^{\infty} (\sigma_n^2/L_n^2) < \infty,$$

$$\gamma_n \sum_{k=1}^{n} \sigma_k^2 \sum_{j=k}^{n} (1/L_j^2) \to 0 \quad \textit{as } n \to \infty,$$

$$\gamma_n^2 \sum_{k=1}^{n} \Big(\sum_{j=k}^{n} (1/L_j)\Big)^2 E\Big[(X_k - \mu_k)^2 I\big[\gamma_n\Big(\sum_{j=k}^{n} (1/L_j)\Big)|X_k - \mu_k| > \varepsilon\big]\Big] \to 0 \quad \textit{as } n \to \infty,$$

where

$$L_n = \sum_{k=1}^{n} \mu_k, \quad \gamma_n = \Big(\sum_{k=1}^{n} \sigma_k^2 \Big(\sum_{j=k}^{n} (1/L_j)\Big)^2\Big)^{-1/2}.$$

Then

$$\Big(\prod_{k=1}^{n} (S_k/L_k)\Big)^{\gamma_n} \xrightarrow{\mathcal{D}} e^N \quad \textit{as } n \to \infty.$$

COROLLARY 2.2. *Let* $\{X_n,\ n \geqslant 1\}$ *be a sequence of independent random variables such that* $E|X_n|^{2+\delta} < \infty$ *for some* $\delta > 0,\ n \geqslant 1$. *Condition* (2.4) *may be then replaced by*

$$\gamma_n^{2+\delta} \sum_{k=1}^{n} (A_k^n)^{2+\delta} E|X_k|^{2+\delta} \to 0 \quad \textit{as } n \to \infty.$$

COROLLARY 2.3. *Let* $\{X_n,\ n \geqslant 1\}$ *be a sequence of independent identically distributed random variables with* $EX_1 = \mu$. *Then Theorem* 2.1 *holds for the sequence* $\{X_n - \mu,\ n \geqslant 1\}$ *with*

$$\gamma_n = \frac{\mu}{\sigma\sqrt{2n}},$$

where $\sigma^2 = \sigma_1^2$, $a_k = \mu k$.

Thus our Theorem 2.1 generalizes Theorem 1 given in [7].

Let us consider $\gamma_n = s_n^{-1}$, $n \geqslant 1$, as the normalizing sequence. We will assume that $s_n \to \infty$ as $n \to \infty$. We get

THEOREM 2.2. *Let* $X_n \geqslant a_{n-1} - a_n$ *a.s. and assume that* (2.1) *holds with some* p, $0 < p \leqslant 2$. *If*

$$\frac{1}{s_n} \sum_{k=1}^{n} (s_k^2/a_k^2) \to 0 \quad \textit{as } n \to \infty \tag{2.7}$$

and

$$\frac{1}{s_n^2} \sum_{k=1}^{n} (A_k^n - 1)^2 \sigma_k^2 \to 0 \quad \textit{as } n \to \infty, \tag{2.8}$$

and, for every $\varepsilon > 0$ (*the Lindeberg condition*):

$$\frac{1}{s_n^2} \sum_{k=1}^{n} EX_k^2 I[|X_k| > \varepsilon s_n] \to 0 \quad \textit{as } n \to \infty. \tag{2.9}$$

Then

$$\Big(\prod_{k=1}^{n} (S_k/a_k + 1)\Big)^{1/s_n} \xrightarrow{\mathcal{D}} e^N \quad \textit{as } n \to \infty. \tag{2.10}$$

Furthermore, if

$$\sum_{n=1}^{\infty} (\sigma_n^2/s_n^2) < \infty,$$

then

$$\Big(\prod_{k=1}^{n} (S_k/a_k + 1)\Big)^{1/V_n} \xrightarrow{\mathcal{D}} e^N \quad \textit{as } n \to \infty. \tag{2.11}$$

Now, let $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued random variables and let $\{k_n, n \geqslant 1\}$ be a sequence of positive integers divergent to infinity. We assume that

$$N_n \xrightarrow{\mathcal{P}} \infty \quad \text{as } n \to \infty$$

and that $s_n \to \infty$ as $n \to \infty$. We put $M(n) = k_n \vee N_n$, $m(n) = k_n \wedge N_n$. Under the notation of Theorem 2.1 we will consider the following conditions:

$$\text{(A)} \qquad \frac{\gamma_{N_n}}{\gamma_{k_n}} \xrightarrow{\mathcal{P}} 1 \quad \text{as } n \to \infty,$$

$$\text{(B)} \qquad \gamma_{k_n} s_{m(n)} A_{m(n)+1}^{M(n)} \xrightarrow{\mathcal{P}} 0 \quad \text{as } n \to \infty,$$

$$\text{(C)} \qquad \gamma_{k_n}^2 \sum_{i=m(n)+1}^{M(n)} (A_i^{M(n)})^2 \sigma_i^2 \xrightarrow{\mathcal{P}} 0 \quad \text{as } n \to \infty,$$

$$\text{(D)} \qquad \gamma_{k_n} s_{k_n} A_{m(n)+1}^{M(n)} \xrightarrow{\mathcal{P}} 0 \quad \text{as } n \to \infty,$$

$$\text{(E)} \qquad P[|s_{N_n}^2 - s_{k_n}^2| > c s_{m(n)}^2] \to 0 \quad \text{as } n \to \infty \text{ for some } 0 < c < 1.$$

LEMMA 2.1. *Let the assumptions of Theorem* 2.1 *be satisfied and consider the following cases*:

(i) (A) *jointly with* (D),

(ii) (C) *jointly with* (D),

(iii) (B) *jointly with* (E).

Then (i) *and* (ii) *are equivalent and* (iii) *implies* (i) *and* (ii).

The stronger result that (iii) is equivalent to (i) and (ii) cannot be proved. Indeed, if we take $a_k = k(k+1)$, $\sigma_k^2 = k$, $k_n = n$, and

$$P[N_n = 2n] = P[N_n = n/2] = 1/2,$$

then conditions (A)–(D) are true, but condition (E) fails.

THEOREM 2.3. *Let the assumptions of Theorem* 2.1 *hold. Assume that*

$$\gamma_{k_n} \sum_{i=m(n)+1}^{M(n)} |X_i| A_{m(n)+1}^{M(n)} \xrightarrow{\mathcal{L}_1} 0 \quad \textit{as } n \to \infty. \tag{2.12}$$

Then under the conditions (A) *and* (D) *the following convergences hold*:

$$\Big(\prod_{k=1}^{N_n} (S_k/a_k + 1)\Big)^{\gamma_{k_n}} \xrightarrow{\mathcal{D}} e^N \quad \textit{as } n \to \infty \tag{2.13}$$

and

$$\Big(\prod_{k=1}^{N_n} (S_k/a_k + 1)\Big)^{\gamma_{N_n}} \xrightarrow{\mathcal{D}} e^N \quad \textit{as } n \to \infty. \tag{2.14}$$

3. PROOFS

LEMMA 3.1. *Let* $\{X_n, n \geqslant 1\}, \{Y_n, n \geqslant 1\}$ *and* $\{Z_n, n \geqslant 1\}$ *be sequences of random variables such that*

$$X_n \xrightarrow{\mathcal{D}} F(\cdot), \quad Y_n \xrightarrow{\mathcal{P}} 0, \quad Z_n \xrightarrow{\mathcal{P}} 1 \quad \text{as } n \to \infty$$

with continuous distribution function $F(\cdot)$. *Then*

$$X_n Z_n + Y_n \xrightarrow{\mathcal{D}} F(\cdot) \quad \text{as } n \to \infty.$$

Proof of Lemma 3.1 is elementary and will be omitted.

Proof of Theorem 2.1. Let us put

$$C_n = S_n/a_n + 1, \quad n \geqslant 1.$$

Then from (2.1) and Theorem 6.6 in [4] we have

$$C_n - 1 = S_n/a_n \xrightarrow{\text{a.s.}} 0 \quad \text{as } n \to \infty.$$

Thus

$$\forall_{\delta>0} \exists_R \forall_{r>R}\, P[\sup_{k \geqslant r} |C_k - 1| > \delta] < \delta,$$

and hence there exist two sequences of real numbers, $\delta_m \downarrow 0$ ($\delta_1 = \frac{1}{2}$) and $R_m \uparrow \infty$, such that

$$P[\sup_{k \geqslant R_m} |C_k - 1| > \delta_m] < \delta_m.$$

Proceeding as in the proof of Theorem 1 in [7] we see that it is enough to show that:

$$\gamma_n \sum_{k=1}^{R_m} \big(\log(C_k) - C_k + 1\big) \xrightarrow{\mathcal{P}} 0 \quad \text{as } n \to \infty, \tag{3.1}$$

$$4\gamma_n \sum_{k=1}^{n} (C_k - 1)^2 \xrightarrow{\mathcal{P}} 0 \quad \text{as } n \to \infty, \tag{3.2}$$

$$\gamma_n \sum_{k=1}^{n} (C_k - 1) \xrightarrow{\mathcal{D}} N \quad \text{as } n \to \infty, \tag{3.3}$$

but the first follows from the fact that $\gamma_n \to 0$ as $n \to \infty$, and the second from Markov's inequality and (2.2):

$$P\big[4\gamma_n \sum_{k=1}^{n} (C_k - 1)^2 > \varepsilon\big] \leqslant \frac{4\gamma_n}{\varepsilon} \sum_{k=1}^{n} E(C_k - 1)^2 = \frac{4\gamma_n}{\varepsilon} \sum_{k=1}^{n} (s_k^2/a_k^2) \to 0.$$

Now we prove (3.3). We have

$$\gamma_n \sum_{k=1}^{n} (S_k/a_k) = \gamma_n \sum_{k=1}^{n} \sum_{i=1}^{k} (X_i/a_k) = \gamma_n \sum_{i=1}^{n} A_i^n X_i.$$

Let us define

$$Y_{i,n} = A_i^n X_i, \quad 1 \leqslant i \leqslant n.$$

Thus we have

$$Z_n = \gamma_n \sum_{k=1}^{n} (C_k - 1) = \gamma_n \sum_{k=1}^{n} Y_{k,n}.$$

Let us observe that $EY_{k,n} = 0$ and $\mathrm{Var}(Y_{k,n}) = (A_k^n)^2 \mathrm{Var}(X_k) = (A_k^n)^2 \sigma_k^2$ imply by (2.3)

$$EZ_n = 0, \quad n \geqslant 1,$$

$$\mathrm{Var}(Z_n) = \gamma_n^2 \sum_{k=1}^{n} (A_k^n)^2 \sigma_k^2 \to 1 \quad \text{as } n \to \infty,$$

and in order to get (3.3) it is enough to prove the Lindeberg condition, but it follows from (2.4). Thus we get

$$\lim_{n\to\infty} \left| P\left[\gamma_n \sum_{k=1}^{n} \log(C_k) \leqslant x\right] - \Phi(x) \right| \leqslant 2\delta_m,$$

and (2.5) holds when we take limit as m tends to infinity.

For the proof of (2.6), by Lemma 3.1, it is enough to show that

$$\lambda_n/\gamma_n \xrightarrow{\mathcal{P}} 1 \quad \text{as } n \to \infty, \tag{3.4}$$

which is equivalent to

$$I_n = \gamma_n^2 \Big(\sum_{k=1}^{n} (A_k^n)^2 X_k^2 - \sum_{k=1}^{n} (A_k^n)^2 \sigma_k^2 \Big) \xrightarrow{\mathcal{P}} 0 \quad \text{as } n \to \infty. \tag{3.5}$$

From (2.4), for every $\varepsilon > 0$,

$$B_n(\varepsilon) = \gamma_n^2 \sum_{k=1}^{n} (A_k^n)^2 E X_k^2 I[\gamma_n A_k^n |X_k| > \varepsilon] \to 0 \quad \text{as } n \to \infty.$$

Thus, there exists a sequence $\{\varepsilon_n,\ n \geqslant 1\}$ such that $\varepsilon_n \to 0$ and $B_n(\varepsilon_n) \to 0$. Then

$$\begin{aligned} I_n &= \gamma_n^2 \Big[\sum_{k=1}^{n} (A_k^n)^2 X_k^2 I[\gamma_n A_k^n |X_k| \leqslant \varepsilon_n] - \sum_{k=1}^{n} (A_k^n)^2 E X_k^2 I[\gamma_n A_k^n |X_k| \leqslant \varepsilon_n] \Big] \\ &\quad + \gamma_n^2 \sum_{k=1}^{n} (A_k^n)^2 X_k^2 I[\gamma_n A_k^n |X_k| > \varepsilon_n] - \gamma_n^2 \sum_{k=1}^{n} (A_k^n)^2 E X_k^2 I[\gamma_n A_k^n |X_k| > \varepsilon_n] \\ &= P_{n,1} + P_{n,2} - B_n(\varepsilon_n). \end{aligned}$$

From Markov's and Hölder's inequalities, for arbitrary $\delta > 0$, we have

$$\begin{aligned} P[|P_{n,1}| > \delta] &\leqslant \frac{\gamma_n^4 \sum_{k=1}^n (A_k^n)^4 E X_k^4 I[\gamma_n A_k^n |X_k| \leqslant \varepsilon_n]}{\delta^2} \\ &\leqslant \frac{\varepsilon_n^2 \gamma_n^2 \sum_{k=1}^n (A_k^n)^2 \sigma_k^2}{\delta^2} \sim \frac{\varepsilon_n^2}{\delta^2} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

On the other hand, we get

$$P[|P_{n,2}| > \delta] \leqslant \frac{B_n(\varepsilon_n)}{\delta} \to 0 \quad \text{as } n \to \infty.$$

Thus the proof of Theorem 2.1 is completed. ∎

Proof of Corollary 2.2. We have

$$\begin{aligned} \gamma_n^2 \sum_{k=1}^n (A_k^n)^2 E\big[X_k^2 I[\gamma_n A_k^n |X_k| > \varepsilon]\big] &\leqslant \gamma_n^2 \sum_{k=1}^n (A_k^n)^2 E X_k^2 |X_k|^\delta \gamma_n^\delta (A_k^n)^\delta / \varepsilon^\delta \\ &= \gamma_n^{2+\delta} \sum_{k=1}^n (A_k^n)^{2+\delta} E|X_k|^{2+\delta} / \varepsilon^\delta. \end{aligned}$$ ∎

Proof of Corollary 2.3. We have $s_n^2 = n\sigma^2$, and $\gamma_n = \mu/(\sigma\sqrt{2n}) \to 0$ as $n \to \infty$. Here and in the sequel we put

$$H_n = \sum_{k=1}^n \frac{1}{k},$$

the harmonic numbers. Then

$$\lim_{n\to\infty} H_n - \ln(n) = \gamma,$$

where γ is Euler's constant, and consequently

$$\gamma_n \sum_{k=1}^n \frac{s_k^2}{a_k^2} = \frac{\mu}{\sigma\sqrt{2n}} H_n \to 0 \quad \text{as } n \to \infty.$$

By induction on n we have:

$$\sum_{k=1}^n (H_n - H_{k-1})^2 = 2n - H_n, \qquad H_0 = 0.$$

Thus

$$\gamma_n^2 \sum_{k=1}^n (A_k^n)^2 \sigma^2 = \frac{1}{2n} \sum_{k=1}^n (H_n - H_{k-1})^2 = 1 - \frac{H_n}{2n} \to 1 \quad \text{as } n \to \infty,$$

and

$$\gamma_n^2 \sum_{k=1}^{n} (A_k^n)^2 E(X_k-\mu)^2 I[\gamma_n A_k^n |X_k-\mu| > \varepsilon]$$
$$\leqslant CE(X_1-\mu)^2 I[\ln(n)|X_1-\mu| > \varepsilon\sigma\sqrt{2n}] \to 0 \quad \text{as } n\to\infty. \quad \blacksquare$$

Proof of Theorem 2.2. Firstly we will prove

$$\sum_{k=1}^{n}(X_k/s_n) - \sum_{k=1}^{n} A_k^n (X_k/s_n) \xrightarrow{\mathcal{P}} 0 \quad \text{as } n\to\infty. \tag{3.6}$$

From (2.8) and Markov's inequality we have

$$P\Big[\sum_{k=1}^{n} \frac{A_k^n - 1}{s_n} X_k > \varepsilon\Big] \leqslant \frac{\sum_{k=1}^{n}(A_k^n-1)^2\sigma_k^2}{\varepsilon^2 s_n^2} \to 0 \quad \text{as } n\to\infty.$$

Noting

$$C_n = S_n/a_n + 1, \quad n \geqslant 1,$$

similarly to the proof of Theorem 2.1 we must show (3.1)–(3.3). But (3.1) follows from the fact that $s_n \to \infty$, (3.2) from Markov's inequality and (2.7), whereas (3.3) follows from (3.6) and (2.9). $\blacksquare$

Proof of Lemma 2.1. From (2.3), for large k and l, we have

$$\left|\frac{\gamma_k^2}{\gamma_l^2} - 1\right| \approx \gamma_k^2\Big(\sum_{i=1}^{k\wedge l}(A_i^l + A_i^k)\sigma_i^2 A_{k\wedge l+1}^{k\vee l} + \sum_{i=k\wedge l+1}^{k\vee l}(A_i^{k\vee l})^2\sigma_i^2\Big), \tag{3.7}$$

but

$$\gamma_k^2 (A_{k\wedge l+1}^{k\vee l})^2 s_{k\wedge l}^2 \leqslant \gamma_k^2 A_{k\wedge l+1}^{k\vee l}\sum_{i=1}^{k\wedge l}(A_i^l + A_i^k)\sigma_i^2 \leqslant \gamma_k A_{k\wedge l+1}^{k\vee l}\Big[\frac{\gamma_k}{\gamma_l}+1\Big] s_{k\wedge l}. \tag{3.8}$$

The lemma follows from (3.7) and (3.8). $\blacksquare$

It is easy to check that the assumptions of Theorem 2.3 imply (A), (B), (C) and (D). This fact is a consequence of Lemma 2.1 and the obvious implication (D) $\Rightarrow$ (B).

Proof of Theorem 2.3. We have

$$\gamma_{N_n}\sum_{k=1}^{N_n}\log(S_k/a_k+1) = \frac{\gamma_{N_n}}{\gamma_{k_n}}\gamma_{k_n}\sum_{k=1}^{k_n}\log(S_k/a_k+1)$$
$$+ (-1)^{I[N_n<k_n]}\frac{\gamma_{N_n}}{\gamma_{k_n}}\gamma_{k_n}\sum_{k=m(n)+1}^{M(n)}\log(S_k/a_k+1),$$

and by (A), Lemma 3.1 and Theorem 2.1 it is enough to prove that

$$\gamma_{k_n} \sum_{k=m(n)+1}^{M(n)} \log(S_k/a_k + 1) \xrightarrow{\mathcal{P}} 0 \quad \text{as } n \to \infty.$$

Taking into account that $k_n \to \infty$, $N_n \xrightarrow{\mathcal{P}} \infty$, using the notation $\{R_m, \delta_m, m \geqslant 1\}$ for arbitrary $\varepsilon > 0$, we have

$$\begin{aligned}
&P\big[\big|\gamma_{k_n} \sum_{k=m(n)+1}^{M(n)} \log(S_k/a_k+1)\big| > \varepsilon, m(n) > R_m\big] \\
&\leqslant P\left[\left|\gamma_{k_n} \sum_{k=m(n)+1}^{M(n)} \left((S_k/a_k) + \frac{(S_k/a_k)^2}{(1+\theta(S_k/a_k))^2}\right)\right| > \varepsilon, \sup_{k>m(n)} \left|\frac{S_k}{a_k}\right| < \delta_m\right] \\
&\leqslant P\left[\gamma_{k_n}(1+4\delta_1) \sum_{k=m(n)+1}^{M(n)} \left|\frac{S_k}{a_k}\right| > \varepsilon\right] \\
&\leqslant P\big[\gamma_{k_n}(1+4\delta_1)\big(|S_{m(n)}| A_{m(n)+1}^{M(n)} + \sum_{k=m(n)+1}^{M(n)} A_k^{M(n)} |X_k|\big) > \varepsilon\big] \\
&\leqslant P[2(1+4\delta_1) \sup_{k \leqslant k_n} |S_k| > \sqrt{\varepsilon} s_{k_n}, \gamma_{k_n} s_{k_n} A_{m(n)+1}^{M(n)} < \sqrt{\varepsilon/2}] \\
&\quad + P[\gamma_{k_n} s_{k_n} A_{m(n)+1}^{M(n)} \geqslant \sqrt{\varepsilon}] \\
&\quad + P\big[2(1+4\delta_1)\gamma_{k_n} A_{m(n)+1}^{M(n)} \sum_{k=m(n)+1}^{M(n)} |X_k| > \varepsilon\big].
\end{aligned}$$

Now Theorem 2.3 follows from Kolmogorov's maximal inequality (cf. [4], (2.17), p. 54), (D) and (2.12). ∎

4. EXAMPLES AND APPLICATIONS

EXAMPLE 4.1. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with the law $P[X_n = \pm\sqrt{n(n+1)}] = 1/2$, $n \geqslant 1$. Then

$$\left(\prod_{k=1}^{n} \left(\frac{S_k}{k(k+1)} + 1\right)\right)^{\sqrt{3/n}} \xrightarrow{\mathcal{D}} e^N \quad \text{as } n \to \infty. \tag{4.1}$$

Furthermore, let $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued random variables with the law $N_n = \inf_{k > n - \sqrt{n}}[X_k < 0]$, $n \geqslant 1$. Then

$$\left(\prod_{k=1}^{N_n} \left(\frac{S_k}{k(k+1)} + 1\right)\right)^{\sqrt{3/N_n}} \xrightarrow{\mathcal{D}} e^N \quad \text{as } n \to \infty \tag{4.2}$$

and

$$\left(\prod_{k=1}^{N_n}\left(\frac{S_k}{k(k+1)}+1\right)\right)^{\sqrt{3/n}} \xrightarrow{\mathcal{D}} e^N \quad \text{as } n\to\infty. \tag{4.3}$$

P r o o f o f E x a m p l e 4.1. We put $a_n = n(n+1), n \geqslant 1$, and observe that $A_k^n = 1/k - 1/(n+1)$. Furthermore $EX_n = 0, \sigma^2 X_n = EX_n^2 = n(n+1)$, and

$$s_n^2 = \sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}, \quad n \geqslant 1.$$

It is obvious that for $p = 2$

$$\sum_{n=1}^{\infty} \frac{E|X_n|^p}{a_n^p} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1.$$

We have

$$\sum_{k=1}^{n} (A_k^n)^2 \sigma_k^2 = \sum_{k=1}^{n}\left(\frac{k+1}{k} - 2\frac{k+1}{n+1} + \frac{k(k+1)}{(n+1)^2}\right) = H_n - \frac{8n}{6(n+1)} + \frac{2n^2}{6(n+1)}.$$

Hence $\gamma_n = \sqrt{3/n}$, and

$$\sqrt{\frac{3}{n}} \sum_{k=1}^{n} \frac{k(k+1)(k+2)}{3k^2(k+1)^2} = \sqrt{\frac{1}{3n}}\left(H_{n+1} + 1 - \frac{2}{n+1}\right) \to 0 \quad \text{as } n\to\infty.$$

We also have

$$\begin{aligned} I[\gamma_n A_k^n |X_k| > \varepsilon] &= I\left[\sqrt{k(k+1)} > \frac{\varepsilon}{\sqrt{3/n}\big(1/k - 1/(n+1)\big)}\right] \\ &= I\left[\sqrt{\frac{k+1}{k}} - \frac{\sqrt{k(k+1)}}{n+1} > \frac{\sqrt{n}\varepsilon}{\sqrt{3}}\right] = 0 \end{aligned}$$

for $n > 6/\varepsilon^2$, because

$$\sqrt{2} > \sqrt{\frac{k+1}{k}} > \sqrt{\frac{k+1}{k}} - \frac{\sqrt{k(k+1)}}{n+1}.$$

Since for any fixed n_0 we have

$$\begin{aligned} \gamma_n^2 \sum_{k=1}^{n_0} (A_k^n)^2 E\big[{X_k}^2 I[\gamma_n A_k^n |X_k| > \varepsilon]\big] &\leqslant \gamma_n^2 \sum_{k=1}^{n_0}\left(\frac{1}{k} - \frac{1}{n}\right)^2 k(k+1) \\ &\leqslant \gamma_n^2 (n_0 + H_{n_0}) = \frac{3}{n}(n_0 + H_{n_0}) \to 0 \quad \text{as } n\to\infty, \end{aligned}$$

so (2.4) holds.

Let us put $N_n' = N_n \wedge (n + \lfloor\sqrt{n}\rfloor), n \geqslant 1$. Because

$$P[N_n \neq N_n'] \leqslant \left(\frac{1}{2}\right)^{2\sqrt{n}} \to 0 \quad \text{as } n \to \infty,$$

it is enough to use N_n' instead of N_n in Theorem 2.3. Furthermore, conditions (A) and (D) are reduced to

$$\frac{N_n'}{n} \xrightarrow{\mathcal{P}} 0 \quad \text{as } n \to \infty,$$

whereas (2.12) may be expressed as

$$\frac{|N_n'^2 - n^2|(N_n' + n)}{N_n' n\sqrt{n}} \xrightarrow{\mathcal{P}} 0 \quad \text{as } n \to \infty,$$

which holds for the above-defined sequence of random indices $\{N_n', n \geqslant 1\}$.

EXAMPLE 4.2. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with the Poisson law and parameter n, i.e.

$$P[X_n = k] = \frac{e^{-n}n^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Then

$$\left(\prod_{k=1}^{n} \frac{2S_k}{k(k+1)}\right)^{1/(2\sqrt{\ln(n)})} \xrightarrow{\mathcal{D}} e^N \quad \text{as } n \to \infty. \tag{4.4}$$

Proof of Example 4.2. Putting $a_n = n(n+1)/2, n \geqslant 1$, we have $A_k^n = 2\big(1/k - 1/(n+1)\big)$. Since the moments of Poisson's random variable are

$$EX_n = \mu_n = n, \quad \sigma_n^2 = n, \quad E(X_n - EX_n)^4 = 3n^2 + n, \quad n \geqslant 1,$$

we have

$$s_n^2 = \sum_{i=0}^{n} i = \frac{n(n+1)}{2}, \quad n \geqslant 1.$$

It is easy to check that (2.1) holds with $p = 2$,

$$\sum_{n=1}^{\infty} \frac{E|X_n - \mu_n|^p}{a_n^p} = \sum_{n=1}^{\infty} \frac{4}{n(n+1)^2} < \infty.$$

Furthermore,

$$\sum_{k=1}^{n} (A_k^n)^2 \sigma_k^2 = 4\left(H_n - \frac{3n}{2n+2}\right);$$

hence $\gamma_n = 1/\big(2\sqrt{\ln(n)}\big)$, and

$$\gamma_n \sum_{k=1}^{n} s_k^2/a_k^2 = \frac{1}{\sqrt{\ln(n)}} \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{\sqrt{\ln(n)}}\left(1 - \frac{1}{n+1}\right) \to 0 \quad \text{as } n \to \infty.$$

Moreover,

$$\gamma_n^4 \sum_{k=1}^{n} (A_k^n)^4 E(X_k - \mu_k)^4 \leqslant \frac{16}{\ln^2(n)} \sum_{k=1}^{n} \frac{4}{k^2} \to 0 \quad \text{as } n \to \infty,$$

and (4.2) follows from Corollary 2.2 with $\delta = 2$.

Acknowledgments. The authors would like to express their gratitude to a referee for his/her constructive comments which led to an improved presentation of the paper.

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Received on 19.6.2008;
revised version on 13.6.2009