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A FUNCTIONAL EQUATION THAT LEADS TO SEMISTABILITY*

BY

NADJIB BOUZAR (INDIANAPOLIS)

Abstract. Some functional equations related to the notion of semistability of probability distributions on $\mathbf{Z}_+$ and $\mathbf{R}_+$ are studied. The solution sets of these equations are fully described.

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1. INTRODUCTION

Semistable probability distributions (Lévy [12]) have been the object of renewed interest in the last several years. These distributions share many important properties with the stable laws. Most notably, they are infinitely divisible and they arise as solutions to central limit problems. Semistable distributions have also proven to be a richer alternative than stable laws in stochastic modeling. We cite the monograph by Sato [14] for recent theoretical advances and the books by Adler et al. [1] and Kotz et al. [11] for applications.

In this paper we will be mainly interested in the semistable distributions with support in either $\mathbf{Z}_+ := \{0, 1, 2, \ldots\}$ or $\mathbf{R}_+ := [0, \infty)$. The definitions are as follows.

A distribution on $\mathbf{R}_+$ with Laplace–Stieltjes transform (LST) $\phi(\tau)$ is said to be *stable* if for any $\alpha \in (0, 1)$ there exists $\lambda > 0$ such that for all $\tau \geqslant 0$, $\phi(\tau) \neq 0$ and

$$\ln \phi(\tau) = \lambda \ln \phi(\alpha t). \tag{1.1}$$

It is said to be *semistable* if for some $\alpha \in (0, 1)$ there exists $\lambda > 0$ such that (1.1) holds.

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A distribution on $\mathbf{Z}_+$ with probability generating function (pgf) $P(z)$ is said to be *discrete stable* if for any $\alpha \in (0,1)$ there exists $\lambda > 0$ such that for all $z \in [0,1]$, $P(z) \neq 0$ and

$$\ln P(z) = \lambda \ln P(1 - \alpha + \alpha z). \tag{1.2}$$

It is said to be *semistable* if for some $\alpha \in (0,1)$ there exists $\lambda > 0$ such that (1.2) holds.

In equations (1.1) and (1.2), λ and α are related by the equation $\lambda = \alpha^{-\gamma}$ for some $\gamma \in (0,1]$, with γ independent of α for stable distributions (see Huillet et al. [10] and Bouzar [5], or Lemma 2.1 below). We will refer to γ as the exponent of the distribution and α (in the case of semistability) its order.

In a recent article, Ben Alaya and Huillet [3] investigated the solution set of the following functional equation:

$$\ln \phi(\tau) = \sum_{i=1}^{m} \lambda_i \ln \phi(\alpha_i \tau), \quad \tau \geqslant 0, \tag{1.3}$$

where $m \geqslant 1$ is a natural number, $\alpha_i > 0$, $\lambda_i > 0$ ($i = 1, \ldots, m$) are real numbers, and $\phi(\tau)$ is restricted to the set of LST's of infinitely divisible distributions on $\mathbf{R}_+$. The authors offer a full description of the solution set of (1.3) in terms of stable or semistable distributions on $\mathbf{R}_+$. We note that Ben Alaya et al. [2], [4] studied similar equations in the context of max-semistability.

The purpose of this paper is to study the analogue of equation (1.3) for discrete distributions on $\mathbf{Z}_+$. Specifically, we study the solution set of the functional equation

$$\ln P(z) = \sum_{i=1}^{m} \lambda_i \ln P(1 - \alpha_i + \alpha_i z), \quad 0 \leqslant z \leqslant 1, \tag{1.4}$$

where $m \geqslant 1$ is a natural number, $0 < \alpha_i < 1$, $\lambda_i > 0$ ($i = 1, \ldots, m$) are real numbers, and $P(z)$ belongs to the set of pgf's of nondegenerate distributions on $\mathbf{Z}_+$ such that $0 < P(0) < 1$.

We note that the condition that $\alpha_i \in (0,1)$ ($i = 1, \ldots, m$) must be imposed in (1.4) because pgf's are defined on the unit disk ($|z| \leqslant 1$). However, unlike the continuous case, we do not impose the condition of infinite divisibility on the solution to (1.4).

In Section 2, we fully describe the solution set of (1.4). We show that, depending on the commensurability of $\ln \alpha_1, \ln \alpha_2, \ldots, \ln \alpha_m$, or lack thereof, the solution is either discrete semistable or discrete stable. As a corollary, we obtain sufficient conditions that will make a discrete semistable law a discrete stable one. Some examples are discussed. In Section 3, we study the continuous case for distributions with support on $\mathbf{R}_+$. Importantly, we show that the results for the continuous case can be deduced from their discrete counterparts by way of Poisson mixtures. In Section 4, we discuss the solution set of equation (1.4) under the more general condition $\alpha_i > 0$, $i = 1, \dots, m$. As a consequence, we offer a somewhat simplified approach to the proof of the main result in Ben Alaya and Huillet [3].

The following theorems play a fundamental role in establishing the main results. Their proofs can be found in the monograph by Rao and Shanbhag [13].

THEOREM 1.1 (the Lau–Rao theorem). *Let f be an $\mathbf{R}_+$-valued Borel measurable locally integrable function on $\mathbf{R}_+$ such that $l([f > 0]) \neq 0$, where l is the Lebesgue measure. Let μ be a σ-finite measure on the Borel σ-field of $\mathbf{R}_+$ with $\mu(\{0\}) < 1$. Then*

$$f(x) = \int_{\mathbf{R}_+} f(x+y)\,\mu(dy), \tag{1.5}$$

for almost all $x \in \mathbf{R}_+$ with respect to l, if and only if one of the following two conditions, with η such that $\int_{\mathbf{R}_+} \exp\{\eta x\}\,\mu(dx) = 1$, holds:

(i) *μ is arithmetic with some span $\kappa > 0$ and, for almost all $x \in \mathbf{R}_+$ with respect to l,*

$$f(x + n\kappa) = f(x)\exp\{n\kappa\eta\}, \quad n = 0, 1, \dots$$

(ii) *μ is non-arithmetic and, for some constant $c > 0$,*

$$f(x) = c\,\exp\{\eta x\}$$

for almost all $x \in \mathbf{R}_+$ with respect to l.

THEOREM 1.2 (Deny's theorem). *Let f be an $\mathbf{R}_+$-valued Borel measurable locally integrable function on $\mathbf{R}$ such that $l([f > 0]) \neq 0$, where l is the Lebesgue measure. Let μ be a σ-finite measure on the Borel σ-field of $\mathbf{R}$ with $\mu(\{0\}) < 1$. Then*

$$f(x) = \int_{\mathbf{R}} f(x+y)\,\mu(dy), \tag{1.6}$$

for almost all $x \in \mathbf{R}$ with respect to l, if and only if one of the following two conditions, with η_i, $i = 1, 2$, such that $\int_{\mathbf{R}} \exp\{\eta_i x\}\,\mu(dx) = 1$, holds:

(i) *μ is arithmetic with some span $\kappa > 0$ and, for some nonnegative periodic Borel measurable functions g_1 and g_2 with period κ,*

$$f(x) = g_1(x)\exp\{\eta_1 x\} + g_2(x)\exp\{\eta_2 x\} \tag{1.7}$$

for almost all $x \in \mathbf{R}$ with respect to l.

(ii) *μ is non-arithmetic and, for some constants $c_1, c_2 \geqslant 0$,*

$$f(x) = c_1\exp\{\eta_1 x\} + c_2\exp\{\eta_2 x\} \tag{1.8}$$

for almost all $x \in \mathbf{R}$ with respect to l.

2. THE SOLUTION SET OF EQUATION (1.4)

For convenience we denote the solution set of equation (1.4) by $\mathcal{D}(m, \underline{\lambda}, \underline{\alpha})$, where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)$ and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m)$.

We start out with a useful lemma.

LEMMA 2.1. *Assume that* $m \geqslant 1$, $0 < \alpha_i < 1$, $\lambda_i > 0$ $(i = 1, \ldots, m)$.

(i) *If* $\mathcal{D}(m, \underline{\lambda}, \underline{\alpha}) \neq \emptyset$, *then*

$$\sum_{i=1}^{m} \lambda_i \alpha_i \leqslant 1 < \sum_{i=1}^{m} \lambda_i. \tag{2.1}$$

If, in addition, this distribution has finite mean, then

$$\sum_{i=1}^{m} \lambda_i \alpha_i = 1. \tag{2.2}$$

(ii) *A nonempty* $\mathcal{D}(1, \lambda, \alpha)$ (*for some* $\lambda > 0$ *and* $\alpha \in (0, 1)$) *coincides with the set of discrete semistable distributions with exponent* $\gamma = -\ln \lambda / \ln \alpha$ *and order* α. *In this case*, $\lambda\alpha \leqslant 1 < \lambda$.

Proof. (i) Assume that $P(\cdot) \in \mathcal{D}(m, \underline{\lambda}, \underline{\alpha})$. Since $P(0) < P(1 - \alpha_i)$ for each $i = 1, \ldots, m$, it follows by (1.4) that

$$\ln P(0) = \sum_{i=1}^{m} \lambda_i \ln P(1 - \alpha_i) > \big(\sum_{i=1}^{m} \lambda_i\big) \ln P(0),$$

and the second inequality in (2.1) ensues (as $\ln P(0) < 0$). By differentiation,

$$\frac{P'(z)}{P(z)} = \sum_{i=1}^{m} \lambda_i \alpha_i \frac{P'(1 - \alpha_i + \alpha_i z)}{P(1 - \alpha_i + \alpha_i z)}. \tag{2.3}$$

Since $P'(z)$ is increasing over the interval $[0, 1)$ and $1 - \alpha_i + \alpha_i z > z$ for all $i = 1, \ldots, m$ and $z \in (0, 1)$, we have

$$\frac{1}{P(z)} \geqslant \sum_{i=1}^{m} \lambda_i \alpha_i \frac{1}{P(1 - \alpha_i + \alpha_i z)}. \tag{2.4}$$

The first inequality in (2.1) follows by letting $z \uparrow 1$ in (2.4). The additional assumption of finite mean is equivalent to $0 < P'(1) < \infty$ (recall the distribution with pgf $P(z)$ is nondegenerate). Letting $z \uparrow 1$ in (2.3) yields (2.2). To prove (ii), suppose $P(\cdot) \in \mathcal{D}(1, \lambda, \alpha)$, $\lambda > 0$ and $\alpha \in (0, 1)$. Then $\ln P(z) = \lambda \ln P(1 - \alpha + \alpha z)$ for any $z \in [0, 1]$. Note that, by part (i), $\lambda\alpha \leqslant 1 \leqslant \lambda$. Letting $\gamma = -\ln \lambda / \ln \alpha$, we have $0 < \gamma \leqslant 1$. Therefore, $P(z)$ is discrete semistable with exponent γ and order α. ■

The real numbers $a_1, a_2, \ldots, a_m$ are said to be *commensurable* if there exists a real number a such that for every $i \in \{1, \ldots, m\}$, $a_i = r_i a$ for some integer r_i. a is called a *period* of the set $\{a_1, \ldots, a_m\}$.

We now give a full description of $\mathcal{D}(m, \underline{\lambda}, \underline{\alpha})$.

THEOREM 2.1. *Assume that* $m \geqslant 1$, $0 < \alpha_i < 1$, $\lambda_i > 0$ $(i = 1, \ldots, m)$ *satisfy* (2.1). *A pgf* $P(\cdot)$ *belongs to* $\mathcal{D}(m, \underline{\lambda}, \underline{\alpha})$ *if and only if one of the following two conditions holds*, *with* γ *being the unique solution to* $\sum_{i=1}^{m} \lambda_i \alpha_i^{\gamma} = 1$ *and* γ *necessarily in* $(0, 1]$:

(i) $(\ln \alpha_1, \ldots, \ln \alpha_m)$ *are commensurable with period* $\ln \alpha$ *for some* $\alpha \in (0, 1)$ *and* $P(z)$ *is the pgf of a discrete semistable distribution with exponent* γ *and order* α (*and hence of orders* $\alpha_1, \ldots, \alpha_m$). *Moreover*, $P(z)$ *admits the representation*

$$P(z) = \exp\bigl\{-(1 - z)^{\gamma} g\bigl(|\ln(1 - z)|\bigr)\bigr\}, \quad 0 \leqslant z < 1, \tag{2.5}$$

where $g(\cdot)$ *is a nonnegative periodic function defined over* $[0, \infty)$, *with periods* $-\ln \alpha$ *and* $-\ln \alpha_i$, $i = 1, \ldots, m$.

(ii) $(\ln \alpha_1, \ldots, \ln \alpha_m)$ *are noncommensurable and* $P(z)$ *is the pgf of a discrete stable distribution with exponent* γ.

P r o o f. We prove the "if" part first. Assume $\gamma \in (0, 1]$ is a solution to the equation $\sum_{i=1}^{m} \lambda_i \alpha_i^{\gamma} = 1$. Under (i), if $(\ln \alpha_1, \ldots, \ln \alpha_m)$ are commensurable with period $\ln \alpha$ for some $\alpha \in (0, 1)$ and $P(z)$ is the pgf of a discrete semistable distribution with exponent γ and order $\alpha \in (0, 1)$, then, by Proposition 2.3 of Bouzar [5], $P(z)$ admits the representation (2.5) where $g(\cdot)$ is a nonnegative periodic function over $[0, \infty)$ with period $-\ln \alpha$. Since for every $i \in \{1, \ldots, m\}$, $\ln \alpha_i = r_i \ln \alpha$ for some $r_i \in \mathbf{Z}$, it follows that $g(\cdot)$ has periods $-\ln \alpha_i$, $i = 1, \ldots, m$. Therefore,

$$\begin{aligned}\sum_{i=1}^{m} \lambda_i \ln P(1 - \alpha_i + \alpha_i z) &= -(1 - z)^{\gamma} \Bigl(\sum_{i=1}^{m} \lambda_i \alpha_i^{\gamma} g\bigl(|\ln(1 - z)| - \ln a_i\bigr)\Bigr) \\ &= \ln P(z),\end{aligned}$$

and hence $P(\cdot) \in \mathcal{D}(m, \underline{\lambda}, \underline{\alpha})$. Under (ii), $P(z)$ is the pgf of a discrete semistable distribution with exponent γ (we note that the lack of commensurability of the $\ln \alpha_i$'s is not needed at this stage of the proof). Then $\ln P(z) = -c(1 - z)^{\gamma}$ for some $c > 0$ (see Steutel and van Harn [15], Theorem 5.5, Chapter V). It is easily seen that $P(z)$ satisfies (1.4). Therefore, $P(\cdot) \in \mathcal{D}(m, \underline{\lambda}, \underline{\alpha})$. This concludes the proof of the "if" part.

We now prove the "only if" part. We will assume without loss of generality that $\alpha_i \neq \alpha_j$ for all $i, j \in \{1, \ldots, m\}$, $i \neq j$. Let $P(\cdot) \in \mathcal{D}(m, \underline{\lambda}, \underline{\alpha})$ and define $f(x) = -\ln[P(1 - e^{-x})]$, $x \geqslant 0$. f is nonnegative and, by (1.4),

$$f(x) = -\sum_{i=1}^{m} \lambda_i \ln P(1 - \alpha_i e^{-x}) = \sum_{i=1}^{m} \lambda_i f(x - \ln \alpha_i) \tag{2.6}$$

for all $x \geqslant 0$. For $a > 0$, let δ_a be the Dirac point-mass measure on the σ-field of the Borel sets of $\mathbf{R}_+$. Define

$$\mu(\cdot) = \sum_{i=1}^{m} \lambda_i \delta_{a_i}(\cdot), \quad a_i = -\ln \alpha_i. \tag{2.7}$$

Clearly, μ is a finite measure on the σ-field of the Borel sets of $\mathbf{R}_+$ with $\mu(\{0\}) = 0$. It is easily seen that (2.6) can be rewritten in the form of the integral equation (1.5) in Theorem 1.1 (the Lau–Rao theorem) with μ of (2.7). The equation holds for every $x \geqslant 0$. By Theorem 1.1, there exists $\eta \in \mathbf{R}$, necessarily unique, such that $\sum_{i=1}^{m} \lambda_i \alpha_i^{-\eta} = 1$. Setting $\gamma = -\eta$, we have $\sum_{i=1}^{m} \lambda_i \alpha_i^{\gamma} = 1$.

Suppose that μ is arithmetic with some span κ. We can assume without loss of generality that $\kappa > 0$. Now the support of μ is $\{-\ln \alpha_1, \ldots, -\ln \alpha_m\}$ (recall $\mu(\{-\ln \alpha_i\}) = \lambda_i > 0$, $i = 1, \ldots, m$). It follows that $(\ln \alpha_1, \ldots, \ln \alpha_m)$ are commensurable with period κ. Letting $\alpha = e^{-\kappa} \in (0,1)$, we have for every $i \in \{1, \ldots, m\}$, $\ln \alpha_i = r_i \ln \alpha$ for some r_i (necessarily) in $\mathbf{Z}_+$. By Theorem 1.1 (statement (i), $n = 1$),

$$f(x - \ln \alpha) = f(x) e^{\kappa\eta} = \alpha^{\gamma} f(x), \quad x \geqslant 0,$$

or, equivalently, through the change of variable $z = 1 - e^{-x}$,

$$\ln P(1 - \alpha + \alpha z) = \alpha^{\gamma} \ln P(z), \quad 0 \leqslant z < 1.$$

This implies that $P(z)$ satisfies (1.2) with $\lambda = \alpha^{-\gamma}$, and hence $P(z)$ is the pgf of a discrete semistable distribution with exponent γ and order α. The fact that $\gamma \in (0,1]$ follows from Lemma 2.1 (ii). Proposition 2.3 of Bouzar [5] leads to the representation (2.5).

Assume now that μ is not arithmetic. Necessarily, $(\ln \alpha_1, \ldots, \ln \alpha_m)$ are noncommensurable. By Theorem 1.1 (statement (ii)), there exists $c > 0$ such that $f(x) = c e^{\eta x} = c e^{-\gamma x}$, $x \geqslant 0$, or, equivalently, $\ln P(z) = -c(1 - z)^{\gamma}$, $0 \leqslant z < 1$. Thus $P(z)$ is the pgf of a discrete stable distribution with exponent γ, necessarily in $(0, 1]$. ∎

Discrete stable and discrete semistable distributions are infinitely divisible (Steutel and van Harn [15], Theorem 5.6, Chapter V, and Bouzar [5]). Let $\mathcal{I}(\mathbf{Z}_+)$ be the set of pgf's of the infinitely divisible distributions on $\mathbf{Z}_+$. We thus have the following result as an immediate consequence of Theorem 2.1.

Corollary 2.1. *Assume that* $m \geqslant 1$, $0 < \alpha_i < 1$, $\lambda_i > 0$ $(i = 1, \ldots, m)$ *satisfy* (2.1). *Then*

$$\mathcal{D}(m, \underline{\lambda}, \underline{\alpha}) \subset \mathcal{I}(\mathbf{Z}_+).$$

We denote by $DSS(\alpha)$, $\alpha \in (0,1)$, the set of discrete semistable distributions with order α and by DS the set of discrete stable distributions. It is easily seen that

$$DS = \bigcap_{0<\alpha<1} DSS(\alpha).$$

A noncommensurability assumption leads to the following stronger result.

COROLLARY 2.2. *If* α_1 *and* α_2 *in* $(0,1)$ *are such that* $\ln\alpha_1$ *and* $\ln\alpha_2$ *are noncommensurable, then*

$$DS = DSS(\alpha_1) \cap DSS(\alpha_2).$$

P r o o f. Clearly, $DS \subset DSS(\alpha_1) \cap DSS(\alpha_2)$. Conversely, let $P(z)$ be the pgf of a distribution in $DSS(\alpha_1) \cap DSS(\alpha_2)$. There exists $\lambda_i \geqslant 1$, $i = 1, 2$, such that $\ln P(z) = \lambda_i \ln P(1-\alpha_i+\alpha_i z)$, $z \in [0,1]$, which implies

$$\ln P(z) = \frac{\lambda_1}{2}\ln P(1-\alpha_1+\alpha_1 z) + \frac{\lambda_2}{2}\ln P(1-\alpha_2+\alpha_2 z).$$

Therefore, $P(\cdot) \in \mathcal{D}(2, \underline{\lambda}/2, \underline{\alpha})$. Condition (2.1) holds by Lemma 2.1. Since $\ln\alpha_1$ and $\ln\alpha_2$ are noncommensurable, it follows by Theorem 2.1 that $P(z)$ is the pgf of a discrete stable distribution. ■

We note that Rao and Shanbhag [13], Theorem 6.4.6, p. 159, used the Deny–Lau–Rao theorem to obtain characterizations for discrete stability that are similar to the result given in Corollary 2.2 (see also Gupta et al. [8]).

We recall that a function $P(z)$ on $[0,1]$ is the pgf of an infinitely divisible discrete distribution if and only if it admits the representation (see Steutel and van Harn [15], Theorem 4.2, Chapter II)

$$\ln P(z) = -\int_z^1 R(x)\,dx, \quad 0 \leqslant z \leqslant 1, \tag{2.8}$$

where $R(x)=\sum_{n=0}^{\infty} r_n x^n$ with $r_n \geqslant 0$ and, necessarily, $\sum_{n=0}^{\infty} r_n(n+1)^{-1} < \infty$. Following Steutel and van Harn [15], we will refer to $R(z)$ as the R-function of $P(z)$.

We now give a characterization of the solution set $\mathcal{D}(m, \underline{\lambda}, \underline{\alpha})$ in terms of R-functions.

THEOREM 2.2. *Assume that* $m \geqslant 1$, $0 < \alpha_i < 1$, $\lambda_i > 0$ $(i = 1, \ldots, m)$ *satisfy* (2.1) *and* $P(\cdot) \in I(\mathbf{Z}_+)$. *Then* $P(\cdot) \in \mathcal{D}(m, \underline{\lambda}, \underline{\alpha})$ *if and only if one of the following two conditions, with* γ *being the solution to* $\sum_{i=1}^{m} \lambda_i \alpha_i^{\gamma} = 1$ *and* γ *necessarily in* $(0,1]$, *holds*:

(i) $(\ln\alpha_1, \ldots, \ln\alpha_m)$ *are commensurable with period* $\ln\alpha$ *for some* $\alpha \in (0,1)$ *and* $P(z)$ *is the pgf of a discrete semistable distribution with exponent* γ, *order* α (*and hence of orders* $\alpha_1, \ldots, \alpha_m$), *and an R-function with the representation*

$$R(z) = (1-z)^{\gamma-1} r\big(|\ln(1-z)|\big), \quad 0 \leqslant z < 1, \tag{2.9}$$

where $r(\cdot)$ *is a nonnegative periodic function defined over* $[0,\infty)$, *with periods* $-\ln\alpha$ *and* $-\ln\alpha_i$, $i = 1, \ldots, m$.

(ii) $(\ln\alpha_1,\dots,\ln\alpha_m)$ *are noncommensurable and* $P(z)$ *is the pgf of a discrete stable distribution with exponent* γ *and an* R*-function of the form*

$$R(z) = k(1-z)^{\gamma-1}, \quad 0 \leqslant z < 1, \tag{2.10}$$

for some $k > 0$.

P r o o f. By (1.4) and (2.8), $P(\cdot) \in \mathcal{D}(m, \underline{\lambda}, \underline{\alpha})$ if and only if its R-function is a solution to the functional equation

$$R(z) = \sum_{i=1}^{m} \lambda_i \alpha_i R(1-\alpha_i+\alpha_i z), \quad 0 \leqslant z < 1. \tag{2.11}$$

It is easy to see that $R(z)$ of (2.9) (resp. (2.10)), under condition (i) (resp. (ii)), satisfies (2.11). This establishes the "if" part. We now prove the "only if" part. Let $P(\cdot) \in \mathcal{D}(m, \underline{\lambda}, \underline{\alpha})$. We assume first that condition (i) in Theorem 2.1 holds. Noting that $R(z) = (d/dz)[P(z)]$, we deduce the representation (2.9) from (2.5) and differentiation. In this case, $r(x) = \gamma g(x) - g'(x)$, $x \geqslant 0$. The nonnegativity of $r(x)$ follows from that of $R(x)$. Moreover, since $g(x)$ is periodic with periods $-\ln\alpha$ and $-\ln\alpha_i$, $i = 1,\dots,m$, it is easily shown that $r(x)$ enjoys the same property. A similar argument, assuming this time that condition (ii) in Theorem 2.1 holds, leads to the representation (2.10). ∎

We conclude the section by discussing an example.

EXAMPLE 2.1. Let $m = 2$, $\alpha \in (0,1)$ and $0 < \lambda < (1-\alpha)/\alpha^2$. Let also $\lambda_1 = 1$, $\lambda_2 = \lambda$, $\alpha_1 = \alpha$, and $\alpha_2 = \alpha^2$. Clearly, condition (2.1) holds and $\ln\alpha_1$ and $\ln\alpha_2$ are commensurable with period $\ln\alpha$. The assumptions also insure the existence of (a unique) $\gamma \in (0,1)$ such that $\alpha_1^\gamma + \lambda\alpha_2^\gamma = 1$, specifically,

$$\gamma = \ln\left[\frac{-1+\sqrt{1+4\lambda}}{2\lambda}\right] / \ln\alpha.$$

Let $\psi(x)$ be a continuous bounded nonnegative and periodic function, with period $-\ln\alpha$. For example, $\psi(x) = |\sin[(2\pi x)/(-\ln\alpha)]|$. We define the function

$$G(z) = 1 - c\int_0^\infty (1-e^{-(1-z)x})x^{-1-\gamma}\psi(\ln x)\,dx, \quad 0 \leqslant z \leqslant 1,$$

where $c = \left(\int_0^\infty (1-e^{-x})x^{-1-\gamma}\psi(\ln x)\,dx\right)^{-1}$. $G(z)$ is the pgf of $(p_n, n \geqslant 0)$ given by

$$p_0 = 0 \quad \text{and} \quad p_n = \frac{c}{n!}\int_0^\infty x^{n-1-\gamma}e^{-x}\psi(\ln x)\,dx \quad (n \geqslant 1).$$

Let

$$P(z) = \exp\{G(z)-1\}, \quad 0 \leqslant z \leqslant 1. \tag{2.12}$$

$P(z)$ is the pgf of a compound Poisson distribution (see for example Feller [7]). Moreover, a simple change of variable argument leads to

$$\ln P(z) = -(1-z)^{\gamma} g\big(|\ln(1-z)|\big), \quad 0 \leqslant z < 1,$$

with

$$g(\tau) = c\int_0^{\infty}(1-e^{-x})x^{-1-\gamma}\psi(\tau + \ln x)\,dx, \quad \tau \geqslant 0.$$

Since $\psi(x)$ is periodic with period $-\ln\alpha$ and α_1 and α_2 are commensurable with period $\ln\alpha$, it follows that $g(\tau)$ has periods $\ln\alpha_i$, $i = 1, 2$. Therefore, by Theorem 2.1 (or by direct calculation), $P(\cdot) \in \mathcal{D}(2, \underline{\lambda}, \underline{\alpha})$ and $P(\cdot)$ is discrete semistable with exponent γ and orders $\alpha_1 = \alpha$ and $\alpha_2 = \alpha^2$.

REMARK 2.1. A different example can be constructed with γ, α, α_1, α_2 as above and the function $G(z)$ in (2.12) as follows (see Corollary 5.2 of Bouzar [5]):

$$G(z) = 1 - c\int_0^{\infty}(1-e^{-(1-z)f(x)})\,dx, \quad 0 \leqslant z \leqslant 1,$$

where

$$f(x) = x^{-1/\gamma}\big(1 - B\cos(b\ln x)\big), \quad x > 0, \tag{2.13}$$

$b = -2\pi/(\gamma\ln\alpha)$, $0 < B \leqslant (1+b\gamma)^{-1}$, and $c = \big(\int_0^{\infty}(1-e^{-f(x)})\,dx\big)^{-1}$. An argument similar to the one used above shows $P(\cdot) \in \mathcal{D}(2, \underline{\lambda}, \underline{\alpha})$.

3. THE $\mathbf{R}_+$-VALUED CASE VIA POISSON MIXTURES

In this section, we revisit equation (1.3) of Ben Alaya and Huillet [3] by restricting the α_i's to the interval $(0, 1)$ and by dropping the assumption that the LST $\phi(\tau)$ be infinitely divisible. Instead, we simply require that $\phi(\tau)$ belong to the set of LST's of nondegenerate distributions on $\mathbf{R}_+$.

For convenience we denote the solution set of equation (1.3), under the modifications stated above, by $\mathcal{C}(m, \underline{\lambda}, \underline{\alpha})$, where, we recall, $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)$ and $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m)$. Moreover, we note that if $\mathcal{C}(m, \underline{\lambda}, \underline{\alpha}) \neq \emptyset$, then the double inequality (2.1) must hold (the same proof as that of Lemma 2.1).

A similar approach to the one used in the $\mathbf{Z}_+$-valued case, coupled with results in Huillet et al. [10], will lead to a full description of $\mathcal{C}(m, \underline{\lambda}, \underline{\alpha})$. Instead, we will use the Poisson mixtures approach of van Harn and Steutel [9] to extend the results of Section 2 to distributions on $\mathbf{R}_+$.

We recall that if $N_c(\cdot)$ is a Poisson process with intensity $c > 0$ and X is an $\mathbf{R}_+$-valued random variable (rv) independent of $N_c(\cdot)$, then the $\mathbf{Z}_+$-valued rv $N_c(X)$ is called a *Poisson mixture* with mixing rv X. Its pgf $P_c(z)$ is given by

$$P_c(z) = \phi\big(c(1-z)\big) \qquad \big(|z| \leqslant 1\big), \tag{3.1}$$

where ϕ is the LST of X.

THEOREM 3.1. *Assume that* $m \geqslant 1$, $0 < \alpha_i < 1$, $\lambda_i > 0$ $(i = 1, \ldots, m)$ *satisfy* (2.1). *Let* $\phi(\tau)$ *be the LST of a distribution on* $\mathbf{R}_+$. *The following assertions are equivalent.*

(i) $\phi(\cdot)$ *belongs to* $\mathcal{C}(m, \underline{\lambda}, \underline{\alpha})$.

(ii) *For every* $c > 0$, $P_c(\cdot)$ *of* (3.1) *belongs to* $\mathcal{D}(m, \underline{\lambda}, \underline{\alpha})$.

(iii) *One of the following two conditions holds*, *with* γ *being the unique solution to* $\sum_{i=1}^m \lambda_i \alpha_i^\gamma = 1$ *and* γ *necessarily in* $(0, 1]$:

(iii1) $(\ln \alpha_1, \ldots, \ln \alpha_m)$ *are commensurable with period* $\ln \alpha$ *for some* $\alpha \in (0, 1)$ *and* $\phi(\tau)$ *is the LST of a semistable distribution on* $\mathbf{R}_+$ *with exponent* γ *and order* α (*and hence of orders* $\alpha_1, \ldots, \alpha_m$). *Moreover*, $\phi(\tau)$ *admits the representation*

$$\phi(\tau) = \exp\{-\tau^\gamma g(|\ln \tau|)\}, \quad \tau \geqslant 0, \tag{3.2}$$

where $g(\cdot)$ *is a nonnegative periodic function defined over the real line*, *with periods* $-\ln \alpha$ *and* $-\ln \alpha_i$, $i = 1, \ldots, m$.

(iii2) $(\ln \alpha_1, \ldots, \ln \alpha_m)$ *are noncommensurable and* $\phi(\tau)$ *is the LST of a stable distribution on* $\mathbf{R}_+$ *with exponent* γ.

Proof. (i) ⇒ (ii). By (3.1), if $\phi(\tau)$ is a solution to (1.3), then for any $c > 0$, $P_c(z)$ is a solution to (1.4).

(ii) ⇒ (iii). By Theorem 2.1, there exists $\gamma \in (0, 1]$, a unique solution to

$$\sum_{i=1}^m \lambda_i \alpha_i^\gamma = 1,$$

such that for any $c > 0$, $P_c(z)$ is either discrete semistable or discrete stable. More specifically, if $(\ln \alpha_1, \ldots, \ln \alpha_m)$ are commensurable with period $\ln \alpha$ for some $\alpha \in (0, 1)$, then

$$P_c(z) = \exp\bigl\{-(1 - z)^\gamma g_c\bigl(|\ln(1 - z)|\bigr)\bigr\}, \quad 0 \leqslant z < 1, \tag{3.3}$$

where $g_c(\cdot)$ is a nonnegative periodic function defined over $[0, \infty)$, with periods $-\ln \alpha$ and $-\ln \alpha_i$, $i = 1, \ldots, m$. If $0 \leqslant \tau < c < c'$, then by (3.1)

$$\phi(\tau) = P_c(1 - \tau/c) = P_{c'}(1 - \tau/c'). \tag{3.4}$$

Therefore, by (3.3),

$$c^{-\gamma} g_c\bigl(|\ln(\tau/c)|\bigr) = c'^{-\gamma} g_{c'}\bigl(|\ln(\tau/c')|\bigr)$$

for any $0 < \tau < c < c'$. Let

$$g(x) = \lim_{c \to \infty} c^{-\gamma} g_c(|x - \ln c|), \quad x \in \mathbf{R}.$$

It is easy to see that $g(x)$ is periodic with periods $-\ln \alpha$, $-\ln \alpha_1, \ldots, -\ln \alpha_m$. The representation (3.2) follows then from (3.3) and (3.4). Hence $\phi(\tau)$ is the LST of a semistable distribution on $\mathbf{R}_+$. Now, if $(\ln \alpha_1, \dots, \ln \alpha_m)$ are noncommensurable, then, by Theorem 2.1, $P_c(z)$ is discrete stable with exponent γ for every $c > 0$ (recall that γ is independent of c). Therefore,

$$\ln P_c(z) = -a_c(1-z)^\gamma, \quad 0 \leqslant z < 1, \tag{3.5}$$

for some $a_c > 0$. Combining (3.4) and (3.5), we obtain $a_c = a_{c'} (= a)$ for any $c, c' > 0$ and $\ln \phi(\tau) = -a\tau^\gamma$, $\tau \geqslant 0$. Therefore, $\phi(\tau)$ is the LST of a stable distribution on $\mathbf{R}_+$.

(iii) $\Rightarrow$ (i). The proof in the discrete case (Theorem 2.1) carries over verbatim. The details are skipped. ∎

We present two corollaries next. Their proofs are omitted.

We will denote by $\mathcal{I}(\mathbf{R}_+)$ the set of LST's of the infinitely divisible distributions on $\mathbf{R}_+$.

COROLLARY 3.1. *Assume that* $m \geqslant 1$, $0 < \alpha_i < 1$, $\lambda_i > 0$ $(i = 1, \dots, m)$ *satisfy* (2.1). *Then*

$$\mathcal{C}(m, \underline{\lambda}, \underline{\alpha}) \subset \mathcal{I}(\mathbf{R}_+).$$

We denote by $SS_+(\alpha)$, $\alpha \in (0,1)$, the set of semistable distributions on $\mathbf{R}_+$ with order α and by S_+ the set of stable distributions on $\mathbf{R}_+$. We have

$$S_+ = \bigcap_{0<\alpha<1} SS_+(\alpha).$$

COROLLARY 3.2. *If* α_1 *and* α_2 *in* $(0,1)$ *are such that* $\ln \alpha_1$ *and* $\ln \alpha_2$ *are noncommensurable*, *then*

$$S_+ = SS_+(\alpha_1) \cap SS_+(\alpha_2).$$

We conclude with an example.

EXAMPLE 3.1. Let $m, \alpha, \lambda_1, \lambda_2, \alpha_1, \alpha_2, \gamma$, and $\psi(x)$ be as in Example 2.1. We define the function

$$\phi(\tau) = \exp\Big\{-\int_0^\infty (1-e^{-\tau x}) x^{-1-\gamma} \psi(\ln x)\, dx\Big\}, \quad \tau \geqslant 0.$$

Since $\int_1^\infty x^{-1-\gamma}\psi(\ln x)\, dx < \infty$, it follows by Theorem 4.3, Chapter III, in Steutel and van Harn [15] that $\phi(\tau)$ is the LST of an infinitely divisible distribution on $\mathbf{R}_+$. Moreover, a simple change of variable argument leads to

$$\ln \phi(\tau) = -\tau^\gamma g(\ln \tau), \quad \tau > 0,$$

with

$$g(\tau) = \int_0^\infty (1-e^{-x}) x^{-1-\gamma} \psi(\ln x - \tau)\, dx, \quad \tau \geqslant 0.$$

Since $\psi(x)$ is periodic with period $-\ln\alpha$ and $\ln\alpha_1$ and $\ln\alpha_2$ are commensurable with period $\ln\alpha$, it follows that $g(\tau)$ has periods $\ln\alpha_i$, $i=1,2$. Therefore, by Theorem 3.1 (or by direct calculation), $\phi(\cdot)\in\mathcal{C}(2,\underline{\lambda},\underline{\alpha})$ and $\phi(\cdot)$ is semistable with exponent γ and orders $\alpha_1=\alpha$ and $\alpha_2=\alpha^2$.

REMARK 3.1. As in the discrete case (see Remark 2.1), a different example can be constructed with γ, α, α_1, α_2 as above and

$$\phi(\tau)=\exp\Big\{-\int_0^\infty(1-e^{-\tau f(x)})\,dx\Big\},\quad \tau\geqslant 0,$$

with $f(x)$ given by (2.13). $\phi(\tau)$ is the LST of a semistable distribution on $\mathbf{R}_+$ with exponent γ and order α (Proposition 5.1 of Bouzar [5]), from which it follows that $\phi(\cdot)\in\mathcal{C}(2,\underline{\lambda},\underline{\alpha})$.

4. THE SOLUTION SET OF EQUATION (1.4) WITH $\alpha_i>0,\ i=1,\dots,m$

In this section, we discuss the solution set of equation (1.4) assuming $\alpha_i>0$, $i=1,\dots,m$ (instead of $\alpha_i\in(0,1)$). We restrict the solution set to those pgf's $P(z)$ of distributions of infinitely divisible Poisson mixtures that satisfy (1.4) for every $z\leqslant 1$. Such pgf's are defined over $(-\infty,1]$ (see (3.1)) and have no zeros over $(-\infty,1)$. As a result, $P(1-\alpha_i+\alpha_i z)$ is well defined for all $\alpha_i>0$, $i=1,\dots,m$, and $z\leqslant 1$. Finally, we will exclude the trivial case $\underline{\alpha}=\underline{1}$, where $\underline{1}=(1,1,\dots,1)$.

We denote by $\mathcal{IPM}(m,\underline{\lambda},\underline{\alpha})$ the solution set described above.

LEMMA 4.1. *Assume* $\alpha_i,\ \lambda_i>0,\ i=1,\dots,m$, *and* $\underline{\alpha}\neq\underline{1}$. *If*

$$\mathcal{IPM}(m,\underline{\lambda},\underline{\alpha})\neq\emptyset,$$

then

$$\sum_{\{i:\,\alpha_i=1\}}\lambda_i<1. \tag{4.1}$$

Moreover, *if* $\{i:\ \alpha_i=1\}=\emptyset$, *we set* $\sum_{\{i:\,\alpha_i=1\}}\lambda_i=0$.

Proof. Assume that $P(\cdot)\in\mathcal{IPM}(m,\underline{\lambda},\underline{\alpha})$. It follows from (1.4) that

$$\Big(1-\sum_{\{i:\,\alpha_i=1\}}\lambda_i\Big)\ln P(z)=\sum_{\{i:\,\alpha_i\neq 1\}}\lambda_i\ln P(1-\alpha_i+\alpha_i z),\quad z\leqslant 1, \tag{4.2}$$

with $\sum_{\{i:\,\alpha_i=1\}}\lambda_i=0$ if $\{i:\ \alpha_i=1\}=\emptyset$. Since $\underline{\alpha}\neq\underline{1}$, there exists $1\leqslant i_0\leqslant m$ such that $\alpha_{i_0}\neq 1$. Suppose $\alpha_{i_0}<1$. Then $P(1-\alpha_{i_0})<1$ (otherwise, $P(\cdot)\equiv 1$ over $[1-\alpha_{i_0},1]$, which would imply the corresponding distribution is degenerate). Setting $z=0$ in (4.2), and noting that $\ln P(s)\leqslant 0$ for every $s\leqslant 1$, we have

$$\Big(1-\sum_{\{i:\,\alpha_i=1\}}\lambda_i\Big)\ln P(0)\leqslant\lambda_{i_0}\ln P(1-\alpha_{i_0})<0,$$

which implies (4.1). If $\alpha_{i_0} > 1$, then letting $z = 1 - 1/(2\alpha_{i_0})$ in (4.2), it follows (as in the case $\alpha_{i_0} < 1$) that

$$\Big(1 - \sum_{\{i:\, \alpha_i = 1\}} \lambda_i\Big) \ln P\big(1 - 1/(2\alpha_{i_0})\big) \leqslant \lambda_{i_0} \ln P(1/2) < 0,$$

which again implies (4.1). ■

We now state and prove the main result of the section.

THEOREM 4.1. *Assume* $\alpha_i > 0, i = 1, \ldots, m$, *and* $\underline{\alpha} \neq \underline{1}$. *A pgf* $P(\cdot)$ *belongs to* $\mathcal{IPM}(m, \underline{\lambda}, \underline{\alpha})$ *if and only if one of the following two conditions holds, with* γ_j, $j = 1, 2$, *being the solutions to* $\sum_{i=1}^{m} \lambda_i \alpha_i^{\gamma} = 1$, *and* γ_j *necessarily in* $(0, 1]$:

(i) $(\ln \alpha_1, \ldots, \ln \alpha_m)$ *are commensurable with period* $\ln \alpha$ *for some* $\alpha \in (0, 1)$ *and either*

(i1) $\gamma_1 = \gamma_2$ $(= \gamma)$, *and in this case* $P(z)$ *is the pgf of a discrete semistable distribution with exponent* γ *and order* α, *or*

(i2) $\gamma_1 \neq \gamma_2$, *and in this case* $P(z)$ *is the pgf of the convolution of two discrete semistable distributions with respective exponents* γ_1, γ_2 *and common order* α.

(ii) $(\ln \alpha_1, \ldots, \ln \alpha_m)$ *are noncommensurable and either*

(ii1) $\gamma_1 = \gamma_2$ $(= \gamma)$, *and in this case* $P(z)$ *is the pgf of a discrete stable distribution with exponent* γ, *or*

(ii2) $\gamma_1 \neq \gamma_2$, *and in this case* $P(z)$ *is the pgf of the convolution of two discrete stable distributions with respective exponents* γ_1, γ_2.

Proof. We prove the "if" part first. Assume (i) holds. Under (i1), the proof that $P(z)$ is a solution to (1.4) is identical to the one given for the "if" part of Theorem 2.1. Under (i2), $P(z)$ admits the representation (see (2.5))

$$P(z) = \exp\Big\{-\sum_{j=1}^{2} (1 - z)^{\gamma_j} g_j\big(|\ln(1 - z)|\big)\Big\}, \quad 0 \leqslant z < 1, \tag{4.3}$$

where $g_j(\cdot)$ $(j = 1, 2)$ is a nonnegative periodic function defined over $\mathbf{R}$, with period $-\ln \alpha$. Commensurability of the $\ln \alpha_i$'s (with period $\ln \alpha$) implies that the functions $g_j(\cdot)$, $j = 1, 2$, in (4.3) admit $-\ln \alpha_i$, $i = 1, \ldots, m$, as periods as well. Straightforward calculations imply that $P(z)$ of (4.3) satisfies equation (1.4). By Corollary 2 of Bouzar [6], any discrete semistable distribution is a Poisson mixture where the mixing distribution is semistable, and thus infinitely divisible, on $\mathbf{R}_+$. Hence

$$P(\cdot) \in \mathcal{IPM}(m, \underline{\lambda}, \underline{\alpha})$$

under (i1). The same conclusion is reached under (i2) as convolution preserves the Poisson mixture representation and the infinite divisibility property. The exact same argument (with discrete stability replacing discrete semistability) carries over if we assume (ii) holds. We note that in this case the functions g, g_1, g_2 in (2.5) and (4.3) become constants. We omit the details.

We now prove the "only if" part. We will assume without loss of generality that $\alpha_i \neq \alpha_j$ for all $i, j \in \{1, \dots, m\}$, $i \neq j$. Let $P(\cdot) \in \mathcal{IPM}(m, \underline{\lambda}, \underline{\alpha})$ and define $f(x) = -\ln[P(1 - e^{-x})]$, $x \in \mathbf{R}$. f is nonnegative and, by (1.4) (recall the latter holds for all $z \leqslant 1$), $f(x)$ satisfies (2.6) for all $x \in \mathbf{R}$. Define the set function μ by (2.7), where now δ_{a_i} $(i = 1, \dots, m)$ is the Dirac point-mass measure on the σ-field of the Borel sets of $\mathbf{R}$. μ is a σ-finite measure on the σ-field of the Borel sets of $\mathbf{R}$. Note that $\mu(\{0\}) = \lambda_{i_0}$ if there exists $1 \leqslant i_0 \leqslant m$ such that $\alpha_{i_0} = 1$. Otherwise, $\mu(\{0\}) = 0$. It follows by Lemma 4.1 that $\mu(\{0\}) < 1$. It is easily seen that (2.6) can be rewritten in the form of the integral equation (1.6) in Theorem 1.2 (Deny's theorem). The equation holds for every $x \in \mathbf{R}$. By Theorem 1.2, there exist exactly two solutions $\eta_j \in \mathbf{R}$, $j = 1, 2$, of the equation $\sum_{i=1}^{m} \lambda_i \alpha_i^{-\eta_j} = 1$, $j = 1, 2$. Let $\gamma_j = -\eta_j$, $j = 1, 2$. Note $\sum_{i=1}^{m} \lambda_i \alpha_i^{\gamma_j} = 1$, $j = 1, 2$. Proceeding as in the proof of Theorem 2.1, if μ is arithmetic with some span $\kappa > 0$, then $(\ln \alpha_1, \dots, \ln \alpha_m)$ are commensurable with period κ. Letting $\alpha = e^{-\kappa} \in (0, 1)$, we have for every $i \in \{1, \dots, m\}$, $\ln \alpha_i = r_i \ln \alpha$ for some r_i (necessarily) in $\mathbf{Z}$. By Theorem 1.2 (statement (i)), $f(x)$ takes on the form (1.7) for some nonnegative periodic Borel measurable functions g_1 and g_2 with period $\kappa = -\ln \alpha$. The representation (4.3) for $P(z)$ ensues from the change of variable $z = 1 - e^{-x}$ (we are restricting (1.7) to $x \geqslant 0$). If $\gamma_1 = \gamma_2$ $(= \gamma)$, then equation (4.3) reduces to (2.5) with $g(\cdot) = g_1(\cdot) + g_2(\cdot)$. This implies that $P(z)$ satisfies (1.2) with $\lambda = \alpha^{-\gamma}$, and hence $P(z)$ is the pgf of a discrete semistable distribution with exponent γ and order α. The fact that $\gamma \in (0, 1]$ follows from Lemma 2.1 (ii). If $\gamma_1 \neq \gamma_2$, we let

$$P_j(z) = \begin{cases} \exp\bigl\{-(1-z)^{\gamma_j} g_j\bigl(|\ln(1-z)|\bigr)\bigr\} & \text{if } 0 \leqslant z < 1, \\ 1 & \text{if } z = 1 \end{cases}$$

for $j = 1, 2$. We show that $P_j(z)$, $j = 1, 2$, is the pgf of a discrete semistable distribution. Since $g_j(\cdot)$ is nonnegative, we see by (4.3) that

$$0 \leqslant (1-z)^{\gamma_j} g_j\bigl(|\ln(1-z)|\bigr) \leqslant -\ln P(z), \quad 0 \leqslant z < 1 \text{ and } j = 1, 2.$$

Therefore, $\lim_{z \uparrow 1} P_j(z) = 1$, which implies that $P_j(z)$ is left-continuous at $z = 1$. We will assume without loss of generality that $\gamma_1 < \gamma_2$. By applying (4.3), we have for any $n \geqslant 1$ and $0 \leqslant z < 1$

$$\ln P(1 - \alpha^{-n} + \alpha^{-n} z) = \alpha^{-n\gamma_2} \left[1 + \frac{g_1\bigl(|\ln(1-z)|\bigr)}{g_2\bigl(|\ln(1-z)|\bigr)} (1-z)^{\gamma_1 - \gamma_2} \alpha^{n(\gamma_2 - \gamma_1)}\right] \ln P_2(z),$$

which implies that

(4.4) $$P_2(z) = \lim_{n\to\infty} [P(1 - \alpha^{-n} + \alpha^{-n} z)]^{\alpha^{n\gamma_2}}, \quad 0 \leqslant z \leqslant 1.$$

Since $P(z)$ is the pgf of an infinitely Poisson mixture ($P(z) = \phi(1 - z)$ for some infinitely divisible LST $\phi(\cdot)$), so is $P(1 - \alpha^{-n} + \alpha^{-n} z)$ for every $n \geqslant 1$. This implies that $[P(1 - \alpha^{-n} + \alpha^{-n} z)]^{\alpha^{n\gamma_2}}$ is infinitely divisible for every $n \geqslant 1$. We conclude from (4.4), and the left-continuity of $P_2(z)$ at $z = 1$, that $P_2(z)$ is an (infinitely divisible) pgf (see Steutel and van Harn [15], Theorem 4.1, Appendix A). Since $P_2(z)$ satisfies (1.2) (with $\lambda = \alpha^{\gamma_2}$), it has to be the pgf of a discrete semistable distribution with exponent γ_2, necessarily in $(0, 1]$ (again, by Lemma 2.1), and order α. A similar argument leads to the limiting result

$$P_1(z) = \lim_{n\to\infty} [P(1 - \alpha^{n} + \alpha^{n} z)]^{\alpha^{-n\gamma_1}}, \quad 0 \leqslant z \leqslant 1,$$

which, combined with the left-continuity of $P_1(z)$, in turn leads to $P_1(z)$ being the pgf of discrete semistable distribution with exponent γ_1, necessarily in $(0, 1]$, and order α.

Finally, if μ is not arithmetic, then, necessarily, $(\ln \alpha_1, \ldots, \ln \alpha_m)$ are noncommensurable. By Theorem 1.2 (statement (ii)), there exists $c_1, c_2 \geqslant 0$ such that

$$f(x) = c_1 \exp\{-\gamma_1 x\} + c_2 \exp\{-\gamma_2 x\}, \quad x \in \mathbf{R},$$

or, equivalently,

$$\ln P(z) = -c_1(1 - z)^{\gamma_1} - c_2(1 - z)^{\gamma_2}, \quad 0 \leqslant z \leqslant 1.$$

A similar, but much simpler, argument as the one that established (i) above leads to (ii1) and (ii2). The details are omitted. ■

The main result in Ben Alaya and Huillet [3], Theorem 3.1 therein, can be deduced from Theorem 4.1. We state the result below. The proof is not provided as it is similar to that of our Theorem 3.1.

We will denote by $\mathcal{IC}(m, \underline{\lambda}, \underline{\alpha})$ the set of LST's of infinitely divisible distributions on **R** that satisfy (1.3) for some α_i, $\lambda_i > 0$, $i = 1, \ldots, m$, and $\underline{\alpha} \neq \underline{1}$.

THEOREM 4.2. *Assume* α_i, $\lambda_i > 0$, $i = 1, \ldots, m$, *and* $\underline{\alpha} \neq \underline{1}$. *Let* $\phi(\tau)$ *be the LST of an infinitely divisible distribution on* $\mathbf{R}_+$. *The following assertions are equivalent.*

(i) $\phi(\tau)$ *belongs to* $\mathcal{IC}(m, \underline{\lambda}, \underline{\alpha})$.

(ii) *For every* $c > 0$, $P_c(z)$ *of* (3.1) *belongs to* $\mathcal{IPM}(m, \underline{\lambda}, \underline{\alpha})$.

(iii) *One of the following two conditions holds*, *with* γ_j, $j = 1, 2$, *being the solutions to* $\sum_{i=1}^m \lambda_i \alpha_i^{\gamma} = 1$, *and* γ_j *necessarily in* $(0, 1]$:

(iii1) $(\ln \alpha_1, \ldots, \ln \alpha_m)$ *are commensurable with period* $\ln \alpha$ *for some* $\alpha \in (0, 1)$ *and either*

(iii1a) $\gamma_1 = \gamma_2$ $(= \gamma)$, *and in this case* $\phi(\tau)$ *is the LST of a semistable distribution on* $\mathbf{R}_+$ *with exponent* γ *and order* α, *or* (iii1b) $\gamma_1 \neq \gamma_2$, *and in this case* $\phi(\tau)$ *is the LST of the convolution of two semistable distributions on* $\mathbf{R}_+$ *with respective exponents* γ_1 *and* γ_2, *and common order* α.

(iii2) $(\ln \alpha_1, \ldots, \ln \alpha_m)$ *are noncommensurable and either*

(iii2a) $\gamma_1 = \gamma_2\ (= \gamma)$, *and in this case* $\phi(\tau)$ *is the LST of stable distribution on* $\mathbf{R}_+$ *with exponent* γ, *or*

(iii2b) $\gamma_1 \neq \gamma_2$, *and in this case* $\phi(\tau)$ *is the LST of the convolution of two stable distributions on* $\mathbf{R}_+$ *with respective exponents* γ_1 *and* γ_2.

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University of Indianapolis
Department of Mathematics and Computer Science
1400 E. Hanna Ave.
Indianapolis, IN 46227, USA
E-mail: nbouzar@uindy.edu

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