PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 30, Fasc. 1 (2010), pp. 141–152

DOMAIN OF ATTRACTION OF GAUSSIAN PROBABILITY OPERATORS IN QUANTUM LIMIT THEORY

BY

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Abstract. We characterise the class of probability operators belonging to the domain of attraction of Gaussian limits in the setup which is a slight generalisation of Urbanik's scheme of noncommutative probability limit theorems.

2000 AMS Mathematics Subject Classification: Primary: 46L53; Secondary: 28D05.

Key words and phrases: Probability operators, quantum limit theorem, domain of attraction.

1. PRELIMINARIES AND NOTATION

In our investigation of the domain of attraction in quantum limit theory we adopt the approach introduced in the fundamental paper [6] which can be briefly described as follows. Let $\mathcal{H}$ be a separable Hilbert space. By a *probability operator* we mean a positive operator on $\mathcal{H}$ of unit trace. It is well known that such operators are in a one-to-one correspondence with normal states ρ on $\mathbb{B}(\mathcal{H})$, and this correspondence is given by the formula

$$\rho(A) = \operatorname{tr} AT, \quad A \in \mathbb{B}(\mathcal{H}).$$

The set of all probability operators on $\mathcal{H}$ will be denoted by $\mathfrak{P}$. By $\mathfrak{L}^1$ we shall denote the set of all trace-class operators on $\mathcal{H}$, and by $\mathfrak{L}^2$ the set of all Hilbert–Schmidt operators.

Let $z \mapsto V(z)$ be an irreducible projective unitary representation of the group $\mathbb{R}^{2d}$ on $\mathcal{H}$, satisfying the Weyl–Segal commutation relations

$$V(z)V(z') = \exp\left(\frac{i}{2}\Delta(z,z')\right)V(z+z'), \tag{1.1}$$

where $z, z' \in \mathbb{R}^{2d}$, $z = (x_1, y_1, \ldots, x_d, y_d)$, $z' = (x_1', y_1', \ldots, x_d', y_d')$, and

$$\Delta(z,z') = \sum_{k=1}^{d} (x_k y_k' - y_k x_k').$$

Fix $z \in \mathbb{R}^{2d}$. It is easily seen that $\{V(tz) : t \in \mathbb{R}\}$ is a one-parameter unitary group, thus by the Stone theorem there is a selfadjoint operator $R(z)$ on $\mathcal{H}$ such that

$$V(tz) = e^{itR(z)}, \qquad t \in \mathbb{R},$$

and, consequently,

$$V(z) = e^{iR(z)}.$$

In [3] the operators $R(z)$ are called *canonical observables*. Let

$$R(z) = \int_{-\infty}^{\infty} \lambda\, E_z(d\lambda)$$

be the spectral representation of $R(z)$. For a probability operator T we define its *mean value* $m_1^T(z)$, *second moment* $m_2^T(z)$ and *variance* $\sigma_T^2(z)$ by the formulae

$$\begin{aligned}
m_1^T(z) &= \int_{-\infty}^{\infty} \lambda \operatorname{tr} TE_z(d\lambda), \\
m_2^T(z) &= \int_{-\infty}^{\infty} \lambda^2 \operatorname{tr} TE_z(d\lambda), \\
\sigma_T^2(z) &= \int_{-\infty}^{\infty} \big(\lambda - m_1^T(z)\big)^2 \operatorname{tr} TE_z(d\lambda) = m_2^T(z) - m_1^T(z)^2
\end{aligned}$$

(cf. [3], Chapter V, Section 4). Note that the notions defined above correspond to the mean value (expectation), second moment and variance, respectively, of the Borel probability measure μ_z determined by the formula

$$\mu_z(\Lambda) = \operatorname{tr} TE_z(\Lambda), \qquad \Lambda \in \mathcal{B}(\mathbb{R}). \tag{1.2}$$

A probability operator T is said to have *finite variance* if for each $z \in \mathbb{R}^{2d}$, $\sigma_T^2(z) < \infty$ (equivalently, $m_2^T(z) < \infty$).

For a probability operator T we define its *characteristic function* $\widehat{T}\colon \mathbb{R}^{2d} \to \mathbb{C}$ as follows:

$$\widehat{T}(z) = \operatorname{tr} TV(z), \qquad z \in \mathbb{R}^{2d}. \tag{1.3}$$

$\widehat{T}$ has the following property called Δ-*positive definiteness*: for arbitrary complex numbers $c_1, \ldots, c_n$ and vectors $z_1, \ldots, z_n \in \mathbb{R}^{2d}$

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \widehat{T}(z_j - z_k) \exp\left(\frac{i}{2}\Delta(z_j, z_k)\right) \geqslant 0.$$

'The quantum Bochner's theorem' states that for a complex-valued function $f\colon \mathbb{R}^{2d} \to \mathbb{C}$ we have $f = \widehat{T}$ for a certain probability operator T if and only if

f is Δ-positive definite, continuous at the origin and $f(0) = 1$ (cf. [3], Chapter V, Section 4).

It is immediately seen that for an arbitrary probability operator T and an arbitrary $z_0 \in \mathbb{R}^{2d}$ the function

$$\mathbb{R}^{2d} \ni z \mapsto \exp(i\langle z_0, z\rangle)\widehat{T}(z)$$

is the characteristic function of some probability operator.

Formula (1.3) for $T \in \mathfrak{L}^1$ defines a map which extends uniquely to a linear isometry from $\mathfrak{L}^2$ onto the space of all complex-valued functions f square integrable with respect to Lebesgue measure and with the norm

$$\|f\|_2 = \left(\frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^{2d}} |f(z)|^2\, dz\right)^{1/2}$$

(cf. [3], Chapter V, Section 3, Theorem 3.2).

Let $\mathfrak{A}$ be the set of all Hilbert–Schmidt operators T for which $\widehat{T}$ vanishes at infinity. We define the convolution $\star$ in $\mathfrak{A}$ by setting

$$\widehat{T_1 \star T_2} = \widehat{T}_1\widehat{T}_2.$$

Moreover, we put $\|T\| = \|\widehat{T}\|$. Then

$$\|T_1 \star T_2\| \leqslant \|T_1\|\|T_2\|,$$

and, consequently, the convolution algebra $\mathfrak{A}$ is a Banach algebra without unit. The following inclusions hold true:

$$\mathfrak{P} \subset \mathfrak{L}^1 \subset \mathfrak{A} \subset \mathfrak{L}^2$$

(cf. [6]).

2. STATEMENT OF THE PROBLEM

The general scheme of quantum limit theorems introduced in [6] is as follows. For a triangular array $\{T_{kn} : k = 1, \ldots, k_n;\ n = 1, 2, \ldots\}$ of probability operators, a norming array $\{a_{kn} : k = 1, \ldots, k_n;\ n = 1, 2, \ldots\}$ of positive numbers, and a sequence $\{z_n\}$ of elements from $\mathbb{R}^{2d}$ we form probability operators S_n defined by the characteristic functions

$$\widehat{S}_n(z) = \exp(i\langle z_n, z\rangle) \prod_{k=1}^{k_n} \widehat{T}_{kn}(a_{kn}z), \qquad z \in \mathbb{R}^{2d}. \tag{2.1}$$

The norming constants a_{kn} should satisfy the assumption of *admissibility*, which means that the maps

$$\mathbb{R}^{2d} \ni z \mapsto \prod_{k=1}^{k_n} \widehat{T}_k(a_{kn}z)$$

are the characteristic functions of some probability operators for each n and *any* probability operators $T_1, \dots, T_n$. Now, if

$$\lim_{n\to\infty} \widehat{S}_n(z) = \widehat{S}(z), \qquad z \in \mathbb{R}^{2d},$$

for some function $\widehat{S}$, then from the quantum Bochner's theorem it follows that $\widehat{S}$ is the characteristic function of some uniquely determined probability operator S. In this case S is called the *limit operator*. In [6] the class of limit operators was described under the assumption of uniform infinitesimality of the operators from $\mathfrak{A}$ given by the functions $\{\widehat{T}_{kn}(a_{kn}\cdot)\colon k = 1, \dots, k_n;\ n = 1, 2, \dots\}$, analogously to the case of the classical infinitely divisible limit laws, while in the paper [5] for the case $d = 1$ norming by arbitrary 2×2 matrices was considered. We shall be concerned with a quantum counterpart of the classical stable limit laws, i.e. we assume that $k_n = n$ and $T_{1n} = \ldots = T_{nn} = T$ for some probability operator T. As for norming we adopt the above-mentioned more general approach and as the norming matrices we take matrices A_n of the form

$$A_n = \begin{bmatrix} a_1^{(n)} & 0 & \dots & 0 & 0 \\ 0 & a_1^{(n)} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & a_d^{(n)} & 0 \\ 0 & 0 & \dots & 0 & a_d^{(n)} \end{bmatrix}. \tag{2.2}$$

As in the scalar case we put the assumption of *admissibility* of the matrices A_n, which means that for each n and *any* probability operators $T_1, \dots, T_n$ the function

$$\mathbb{R}^{2d} \ni z \mapsto \prod_{k=1}^{n} \widehat{T}_k(A_n z) = \prod_{k=1}^{n} \widehat{T}_k(a_1^{(n)} x_1, a_1^{(n)} y_1, \dots, a_d^{(n)} x_d, a_d^{(n)} y_d)$$

is the characteristic function of some probability operator. In this case the limit operator S will be said to belong to the *domain of attraction of the probability operator* T.

To justify this approach let us look at the fundamental notion of the (multidimensional) Schrödinger pair of canonical observables. Define in the Hilbert space $L^2(\mathbb{R}^d)$ operators $p_k^{(0)}$ and $q_k^{(0)}$, $k = 1, \dots, d$ (called *momentum* and *position* operators, respectively) by the formulae

$$\begin{aligned} (p_k^{(0)}\psi)(x_1, \dots, x_d) &= (D_k\psi)(x_1, \dots, x_d), \\ (q_k^{(0)}\psi)(x_1, \dots, x_d) &= -ix_k\psi(x_1, \dots, x_d), \end{aligned}$$

where D_k denotes the k-th partial derivative. The operators $p_k^{(0)}$ and $q_k^{(0)}$ are unbounded densely defined and selfadjoint; moreover, they satisfy the commutation relations

$$[p_k^{(0)}, p_j^{(0)}] = [q_k^{(0)}, q_j^{(0)}] = 0, \qquad [p_k^{(0)}, q_j^{(0)}] = -i\delta_{kj}\mathbf{1}, \tag{2.3}$$

where for operators A, B on $L^2(\mathbb{R}^d)$

$$[A, B] = AB - BA,$$

and $\mathbf{1}$ stands for the identity operator (observe that since $p_k^{(0)}$ and $q_k^{(0)}$ are densely defined, relations (2.3) are assumed to hold only on a dense subspace of $L^2(\mathbb{R}^d)$).

The pair $(p^{(0)}, q^{(0)}) = \big((p_1^{(0)}, q_1^{(0)}), \ldots, (p_d^{(0)}, q_d^{(0)})\big)$ is called the *Schrödinger pair of canonical observables*. Putting, for $(x_1, y_1, \ldots, x_d, y_d) \in \mathbb{R}^{2d}$,

$$V^{(0)}(x_1, y_1, \ldots, x_d, y_d) = \exp\big\{i \sum_{k=1}^{d} (x_k p_k^{(0)} + y_k q_k^{(0)})\big\}, \tag{2.4}$$

we easily see that $z \mapsto V^{(0)}(z)$ is a projective unitary representation of the group $\mathbb{R}^{2d}$ on $\mathcal{H}$, satisfying the Weyl–Segal commutation relations (1.1). Now, if $T^{(0)}$ is a probability operator on $L^2(\mathbb{R}^d)$ (we use a superscript $^{(0)}$ when referring to the space $L^2(\mathbb{R}^d)$), then its characteristic function at the point $A_n z$ for A_n given by the formula (2.2) equals

$$\begin{aligned}\widehat{T}^{(0)}(A_n z) &= \operatorname{tr} T^{(0)} V^{(0)}(A_n z) = \operatorname{tr} T^{(0)} \exp\big\{i \sum_{k=1}^{d} a_k^{(n)} (x_k p_k^{(0)} + y_k q_k^{(0)})\big\} \\ &= \operatorname{tr} T^{(0)} \exp\big\{i \sum_{k=1}^{d} [x_k (a_k^{(n)} p_k^{(0)}) + y_k (a_k^{(n)} q_k^{(0)})]\big\},\end{aligned}$$

which corresponds to the passing from the multidimensional canonical pair $\big((p_1^{(0)}, q_1^{(0)}), \ldots, (p_d^{(0)}, q_d^{(0)})\big)$ to the pair

$$\big((a_1^{(n)} p_1^{(0)}, a_1^{(n)} q_1^{(0)}), \ldots, (a_d^{(n)} p_d^{(0)}, a_d^{(n)} q_d^{(0)})\big),$$

i.e., each of the component pairs $(p_k^{(0)}, q_k^{(0)})$ being normed by possibly different numbers $a_k^{(n)}$, $k = 1, \ldots, d$. It is worth noting that in the pioneering paper [1] on quantum limit theorems, the central limit theorem was formulated just in the language of canonical pairs, though solely in the case $d = 1$ and with the classical scalar norming $a_1^{(n)} = 1/\sqrt{n}$.

Coming back to our setup, we have

$$\begin{aligned}\lim_{n\to\infty} \widehat{S}_n(z) &= \lim_{n\to\infty} \exp(i\langle z_n, z\rangle)[\widehat{T}(a_1^{(n)} x_1, a_1^{(n)} y_1, \ldots, a_d^{(n)} x_d, a_d^{(n)} y_d)]^n \\ &= \widehat{S}(x_1, y_1, \ldots, x_d, y_d), \qquad z = (x_1, y_1, \ldots, x_d, y_d) \in \mathbb{R}^{2d}.\end{aligned}$$

It was proved in [4] that then the limit operator S must be Gaussian, i.e. $\widehat{S}$ is the characteristic function of a Gaussian probability distribution on $\mathbb{R}^{2d}$. In the classical commutative situation various sufficient conditions on belonging to the domain of attraction of a Gaussian law have been obtained – the most celebrated being that of finite variance as in the Lindeberg–Lévy central limit theorem. It turns out that in the quantum case this condition is also necessary. Namely, we shall prove the following

THEOREM 2.1. *Let T be an arbitrary probability operator on $\mathcal{H}$. Then T belongs to the domain of attraction of a Gaussian probability operator if and only if T has finite variance.*

3. PROOFS

We begin with a simple lemma which gives a description of the characteristic function of Gaussian probability operators in a particular case.

LEMMA 3.1. *Let*

$$f(z) = \exp\left(-\frac{a}{2}\|z\|^2\right)$$

for some $a > 0$. Then f is the characteristic function of some Gaussian probability operator if and only if $a \geqslant 1/2$.

P r o o f. Observe that f is the characteristic function of a Gaussian probability measure with the covariance matrix $Q = aI$. From [3], Chapter V, Sections 4 and 5 (see also [6]) it follows that an arbitrary positive-definite $2d \times 2d$ matrix Q is the covariance matrix of a Gaussian probability operator if and only if the following inequality holds:

$$\langle Qz, z\rangle + \langle Qz', z'\rangle \geqslant \Delta(z, z'), \qquad z, z' \in \mathbb{R}^{2d}, \tag{3.1}$$

which in our case amounts to saying that

$$a(\|z\|^2 + \|z'\|^2) \geqslant \Delta(z, z'), \qquad z, z' \in \mathbb{R}^{2d}.$$

The inequality above may be rewritten in the form

$$\sum_{k=1}^{d} (ax_k^2 + ay_k^2 + a{x'_k}^2 + a{y'_k}^2 - x_k y'_k + y_k x'_k) \geqslant 0, \qquad x_k, y_k, x'_k, y'_k \in \mathbb{R}.$$

It is easily seen that this inequality is satisfied if and only if for each $k = 1, \ldots, d$ and arbitrary $x_k, y_k, x'_k, y'_k \in \mathbb{R}$ we have

$$ax_k^2 + ay_k^2 + a{x'_k}^2 + a{y'_k}^2 - x_k y'_k + y_k x'_k \geqslant 0,$$

which, in turn, is equivalent to the positive definiteness of the matrix

$$\begin{bmatrix} a & 0 & 0 & -\frac{1}{2} \\ 0 & a & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & a & 0 \\ -\frac{1}{2} & 0 & 0 & a \end{bmatrix}.$$

Since the eigenvalues of this matrix are equal to $a \pm \frac{1}{2}$, the conclusion follows. ∎

We also have the following simple property of the covariance matrix of a Gaussian probability operator.

LEMMA 3.2. *Let Q be the covariance matrix of a Gaussian probability operator. Then Q is non-singular.*

Proof. Indeed, assume that $Qz' = 0$ for some $0 \neq z' \in \mathbb{R}^{2d}$. Then for each fixed $z \in \mathbb{R}^{2d}$ and an arbitrary $t \in \mathbb{R}$ we have on account of (3.1)

$$\langle Qz, z\rangle = \langle Qz, z\rangle + \langle Q(tz'), (tz')\rangle \geqslant \Delta(z, tz') = t\Delta(z, z'),$$

which is clearly impossible. ∎

The following proposition provides estimation on the coefficients of the norming matrices.

PROPOSITION 3.1. *Let $\{A_n\}$ be an admissible sequence of matrices of the form* (2.2)*. Then*

$$a_k^{(n)} \geqslant \frac{1}{\sqrt{n}} \quad \textit{for each } k = 1, \ldots, d.$$

Proof. Let $T_1 = \ldots = T_n = T$ be Gaussian probability operators with the characteristic function

$$\widehat{T}(z) = \exp\left(-\frac{1}{4}\|z\|^2\right).$$

Then

$$\begin{aligned}\prod_{k=1}^{n} \widehat{T}_k(A_n z) &= \left[\widehat{T}(a_1^{(n)}x_1, a_1^{(n)}y_1, \ldots, a_d^{(n)}x_d, a_d^{(n)}y_d)\right]^n \\ &= \exp\left[-\frac{n}{4}\sum_{k=1}^{d} a_k^{(n)2}(x_k^2 + y_k^2)\right],\end{aligned}$$

which is a Gaussian probability operator with covariance matrix

$$Q = \frac{n}{2}\begin{bmatrix} a_1^{(n)2} & 0 & \ldots & 0 & 0 \\ 0 & a_1^{(n)2} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \ldots & a_d^{(n)2} & 0 \\ 0 & 0 & \ldots & 0 & a_d^{(n)2} \end{bmatrix}.$$

Now the inequality (3.1) takes the form

$$\frac{n}{2}\sum_{k=1}^{d} a_k^{(n)2}(x_k^2 + y_k^2 + {x'_k}^2 + {y'_k}^2) \geqslant \sum_{k=1}^{d}(x_k y'_k - y_k x'_k).$$

Putting $x'_1 = -x_1$, $y'_1 = y_1$, $x_k = y_k = x'_k = y'_k = 0$ for $k = 2, \ldots, d$ we obtain

$$na_1^{(n)2}(x_1^2 + y_1^2) \geqslant 2x_1y_1,$$

which means that the matrix

$$\begin{bmatrix} na_1^{(n)2} & -1 \\ -1 & na_1^{(n)2} \end{bmatrix}$$

is positive definite. Consequently,

$$n^2 a_1^{(n)4} \geqslant 1, \quad \text{i.e.} \quad a_1^{(n)} \geqslant \frac{1}{\sqrt{n}}.$$

By the same token we obtain the required inequalities for $k = 2, \ldots, d$. ■

The next lemma is a known classical result from the theory of domains of attraction (cf. [2], Chapter IX, Section 8).

LEMMA 3.3. *Let ν be a probability measure belonging to the domain of attraction of a Gaussian law, i.e. there are constants $b_n > 0$, $c_n \in \mathbb{R}$ such that*

$$\lim_{n\to\infty} \exp(itc_n)\big[\hat{\nu}(b_n t)\big]^n = \exp\left(itm - \frac{1}{2}\sigma^2 t^2\right), \quad t \in \mathbb{R},$$

for some $m \in \mathbb{R}$, $\sigma > 0$. If $b_n \geqslant 1/\sqrt{n}$, then ν has finite variance, i.e.

$$\int_{-\infty}^{\infty} \lambda^2\, \nu(d\lambda) < \infty.$$

P r o o f. We shall follow [2]. First, note that the theory of limit laws yields $b_n \to 0$. Fix an arbitrary $x > 0$ and write

$$U(x) = \int_{-x}^{x} \lambda^2\, \nu(d\lambda).$$

According to formula (8.12) in [2], Chapter IX, Section 8, Theorem 1a, we have

$$nb_n^2 U\left(\frac{x}{b_n}\right) \to c$$

for some constant c. (We warn the reader that there is a difference in the notation employed in [2] and here, namely, we use b_n for what in [2] is denoted by $1/a_n$ and c_n for what in [2] is denoted by b_n.) Since

$$nb_n^2 \geqslant 1,$$

we get

$$\int_{-\infty}^{\infty} \lambda^2\, \nu(d\lambda) = \lim_{n\to\infty} \int_{-x/b_n}^{x/b_n} \lambda^2\, \nu(d\lambda) = \lim_{n\to\infty} U\left(\frac{x}{b_n}\right) < \infty. \quad ■$$

Now we are in a position to prove our theorem.

P r o o f o f T h e o r e m 2.1. N e c e s s i t y. Assume that for some probability operator T, a sequence $\{z_n\}$ of vectors from $\mathbb{R}^{2d}$ and a sequence $\{A_n\}$ of admissible matrices of form (2.2) we have

$$\lim_{n\to\infty} \exp(i\langle z_n, z\rangle)[\widehat{T}(a_1^{(n)}x_1, a_1^{(n)}y_1, \ldots, a_d^{(n)}x_d, a_d^{(n)}y_d)]^n = \widehat{S}(x_1, y_1, \ldots, x_d, y_d) \tag{3.2}$$

for each $z = (x_1, y_1, \ldots, x_d, y_d) \in \mathbb{R}^{2d}$. Since S is Gaussian, we have

$$\widehat{S}(z) = \exp\left(i\langle z_0, z\rangle - \frac{1}{2}\langle Qz, z\rangle\right) \tag{3.3}$$

for some $z_0 \in \mathbb{R}^{2d}$ and covariance matrix Q. Let μ_z be the probability measure defined by the formula (1.2). Our aim consists in showing that μ_z has finite second moment. We have

$$\widehat{T}(tz) = \operatorname{tr} TV(tz) = \int_{-\infty}^{\infty} e^{it\lambda} \operatorname{tr} TE_z(d\lambda) = \widehat{\mu}_z(t), \quad t \in \mathbb{R}. \tag{3.4}$$

Fix $z = (x_1, y_1, \ldots, x_d, y_d) \in \mathbb{R}^{2d}$, and put

$$\bar{z}_1 = (x_1, y_1, 0, \ldots, 0),\ \bar{z}_2 = (0, 0, x_2, y_2, 0, \ldots, 0), \ldots, \bar{z}_d = (0, \ldots, 0, x_d, y_d).$$

Assume for a while that for each $k = 1, \ldots, d$, $\bar{z}_k \neq 0$. We have on account of (3.2)–(3.4)

$$\begin{aligned}\lim_{n\to\infty}[\widehat{\mu}_{\bar{z}_k}(a_k^{(n)}t)]^n \exp(it\langle z_n, \bar{z}_k\rangle) &= \lim_{n\to\infty}[\widehat{T}(a_k^{(n)}t\bar{z}_k)]^n \exp(i\langle z_n, t\bar{z}_k\rangle)\\ &= \exp\left(it\langle z_0, \bar{z}_k\rangle - \frac{1}{2}t^2\langle Q\bar{z}_k, \bar{z}_k\rangle\right).\end{aligned}$$

From Proposition 3.1 and Lemma 3.3 it follows that all the measures $\mu_{\bar{z}_k}$ for $k = 1, \ldots, d$ have finite second moments,

$$m_2(\mu_{\bar{z}_k}) < \infty.$$

Of course, the same is true if $\bar{z}_k = 0$, because then $\mu_{\bar{z}_k}$ is the Dirac measure concentrated at zero.

By the commutation relations (1.1), the unitary groups $\{V(tz)\colon t \in \mathbb{R}\}$, $\{V(t_1\bar{z}_1)\colon t_1 \in \mathbb{R}\}, \ldots, \{V(t_d\bar{z}_d)\colon t_d \in \mathbb{R}\}$ form a commuting system of operators; moreover,

$$\begin{aligned}\exp\big(itR(z)\big) = V(tz) &= V(t\bar{z}_1)\cdot \ldots \cdot V(t\bar{z}_d)\\ &= \exp\big(itR(\bar{z}_1)\big)\cdot \ldots \cdot \exp\big(itR(\bar{z}_d)\big)\end{aligned} \tag{3.5}$$

for each $t \in \mathbb{R}$. It follows that there is a spectral measure F and Borel functions $f,\ f_k,\ k = 1, \dots, d$, such that

$$R(z) = \int_{-\infty}^{\infty} f(\lambda)\, F(d\lambda), \quad R(\bar{z}_k) = \int_{-\infty}^{\infty} f_k(\lambda)\, F(d\lambda),$$

and the equality (3.5) yields

$$f(\lambda) = f_1(\lambda) + \ldots + f_d(\lambda).$$

Furthermore, substituting $t = f(\lambda)$ we obtain

$$R(z) = \int_{-\infty}^{\infty} f(\lambda)\, F(d\lambda) = \int_{-\infty}^{\infty} t\, (f \circ F)(dt),$$

where

$$(f \circ F)(\Lambda) = F\big(f^{-1}(\Lambda)\big), \quad \Lambda \in \mathcal{B}(\mathbb{R}).$$

On the other hand, we have

$$R(z) = \int_{-\infty}^{\infty} \lambda\, E_z(d\lambda)$$

and the uniqueness of the spectral decomposition yields the equality

$$E_z = f \circ F.$$

By the same token we obtain the equalities

$$E_{\bar{z}_k} = f_k \circ F, \quad k = 1, \dots, d.$$

Consequently, we get

$$\begin{aligned} m_2(\mu_z) &= \int_{-\infty}^{\infty} t^2 \ \mathrm{tr}\, T E_z(dt) = \int_{-\infty}^{\infty} t^2 \ \mathrm{tr}\, T (f \circ F)(dt) \\ &= \int_{-\infty}^{\infty} f^2(\lambda) \ \mathrm{tr}\, T F(d\lambda), \end{aligned}$$

and analogously

$$m_2(\mu_{\bar{z}_k}) = \int_{-\infty}^{\infty} f_k^2(\lambda) \ \mathrm{tr}\, T F(d\lambda), \quad k = 1, \dots, d.$$

Finally, we have

$$f^2(\lambda) = [f_1(\lambda) + \ldots + f_d(\lambda)]^2 \leqslant d[f_1^2(\lambda) + \ldots + f_d^2(\lambda)],$$

yielding

$$m_2(\mu_z) = \int\limits_{-\infty}^{\infty} f^2(\lambda)\ \mathrm{tr}\, TF(d\lambda) \leqslant \int\limits_{-\infty}^{\infty} d \sum_{k=1}^{d} f_k^2(\lambda)\ \mathrm{tr}\, TF(d\lambda)$$
$$= d \sum_{k=1}^{d} \int\limits_{-\infty}^{\infty} f_k^2(\lambda)\ \mathrm{tr}\, TF(d\lambda) = d \sum_{k=1}^{d} m_2(\mu_{\bar{z}_k}) < \infty,$$

which completes the proof of the necessity.

S u f f i c i e n c y. A proof of sufficiency is essentially contained in [1], however, since the setup of [1] is different from the one adopted in our work and since some considerations about centring should be taken into account, we present a short proof. Let T be a probability operator having finite variance. Take

$$a_1^{(n)} = \ldots = a_d^{(n)} = \frac{1}{\sqrt{n}}.$$

Then the sequence of norming matrices $\{A_n\}$ reduces to the sequence of numbers $\{1/\sqrt{n}\}$, and from [6], Proposition 2.5, it follows that this sequence is admissible (this can also be checked straightforwardly, namely, it is to be verified that the function

$$\mathbb{R}^{2d} \ni z \mapsto \prod_{k=1}^{n} \widehat{T}_k \left(\frac{z}{\sqrt{n}} \right)$$

is Δ-positive definite for arbitrary probability operators $T_1, \ldots, T_n$). For an arbitrary $z \in \mathbb{R}^{2d}$, as before, let μ_z be the probability measure defined by the formula (1.2). The mean value of μ_z equals $m_1^T(z)$; moreover, it is pointed out in [3], Chapter V, Section 4, that m_1^T is a linear function of z, which can be checked using the known formula for moments of a probability measure:

$$m_1^T(z) = -i \frac{d}{dt}\, \widehat{\mu}_z(t)\big|_{t=0} = -i \frac{d}{dt}\, \widehat{T}(tz)\big|_{t=0} = -i \frac{d}{dt}\ \mathrm{tr}\, TV(tz)\big|_{t=0}.$$

Consequently, there are vectors $z_n \in \mathbb{R}^{2d}$ such that

$$\langle z_n, z \rangle = -m_1^T(z)\sqrt{n} \quad \text{for each } z \in \mathbb{R}^{2d}.$$

We have

$$\exp(it\langle z_n, z\rangle)\widehat{T}\left(\frac{t}{\sqrt{n}} z \right) = \exp\left(-itm_1^T(z)\sqrt{n} \right) \widehat{\mu}_z \left(\frac{t}{\sqrt{n}} \right).$$

From the classical Lindeberg–Lévy central limit theorem it follows that

$$\lim_{n\to\infty} \exp\left(-itm_1^T(z)\sqrt{n} \right) \left[\widehat{\mu}_z \left(\frac{t}{\sqrt{n}} \right) \right]^n = \exp\left(-\frac{1}{2}\sigma_z^2 t^2 \right)$$

for some $\sigma_z^2 > 0$, which means that

$$\lim_{n\to\infty} \exp(it\langle z_n, z\rangle) \left[\widehat{T}\left(\frac{t}{\sqrt{n}}z\right)\right]^n = \exp\left(-\frac{1}{2}\sigma_z^2 t^2\right).$$

Putting $t = 1$ we get

$$\lim_{n\to\infty} \exp(i\langle z_n, z\rangle) \left[\widehat{T}\left(\frac{z}{\sqrt{n}}\right)\right]^n = \exp\left(-\frac{1}{2}\sigma_z^2\right),$$

and the existence of the limit on the left-hand side means that on the right-hand side we have the characteristic function of a Gaussian probability operator, which completes the proof. ■

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Received on 10.2.2009;
revised version on 6.7.2009