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THE KARLIN–McGREGOR FORMULA FOR PATHS CONNECTED WITH A CLIQUE

BY

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Abstract. The Karlin–McGregor formula, a well-known integral expression of the m-step transition probability for a nearest-neighbor random walk on the non-negative integers (an infinite path graph), is reformulated in terms of one-mode interacting Fock spaces. A truncated direct sum of one-mode interacting Fock spaces is newly introduced and an integral expression for the m-th moment of the associated operator is derived. This integral expression gives rise to an extension of the Karlin–McGregor formula to the graph of paths connected with a clique.

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1. INTRODUCTION

The Karlin–McGregor formula [13] is a well-known integral expression of the m-step transition probability for a nearest-neighbor random walk on the non-negative integers (or a birth-and-death process). In accordance with our notation the formula is of the form

$$(P^m)_{ij} = \frac{1}{\pi(j)} \int_{-1}^{+1} x^m Q_i(x) Q_j(x) \mu(dx), \qquad i, j = 0, 1, 2, \ldots,$$

where $(P^m)_{ij}$ is the m-step transition probability from j to i, $\pi(j)$ is a positive constant, μ is a probability distribution on $[-1, 1]$, and $\{Q_j(x)\}$ are the orthogonal polynomials with respect to μ. Various generalizations of the Karlin–McGregor formula have been investigated to cover a wider class of Markov chains, see, e.g., [8], [9], [15] and references therein. While, it is almost apparent that a similar integral expression is possible for a general real, tridiagonal matrix T.

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Over the last decade the notion of an interacting Fock space has attracted wide attention because of its key role in quantum probability and its applications. In fact, an interacting Fock space was grasped first in the study of quantum field theory [4], where infinite-degree of freedom is essential. Later, although the simplest case, an interacting Fock space of one-degree of freedom, also called a *one-mode interacting Fock space*, became one of the central concepts in quantum probability for its various applications to orthogonal polynomials, spectral analysis of graphs, classical probability and so on, see, e.g., [2], [3], [5], [12].

Structural similarity between one-mode interacting Fock spaces and birth-and-death processes naturally leads us to a reformulation of the Karlin–McGregor formula. In fact, in the previous paper [16], slightly generalizing the existing framework of one-mode interacting Fock spaces, we derived the Karlin–McGregor formula without using the spectral theory for self-adjoint operators. The argument will be reviewed in Section 2.

It seems interesting to explore further similarity through the Karlin–McGregor formula between interacting-Fock-space like structure and random walks on graphs. In this paper we introduce a new variant of direct sum of one-mode interacting Fock spaces, called the *truncated direct sum*. The underlying graph structure of a one-mode interacting Fock space is just an infinite path P_∞ consisting of non-negative integers. While, the underlying graph of the truncated direct sum is a graph obtained from paths by connecting their endpoints with a complete graph (clique), see Figure 1. This graph is also known to be the comb product $K_N \triangleright P_\infty$ (see [1] and [12]).

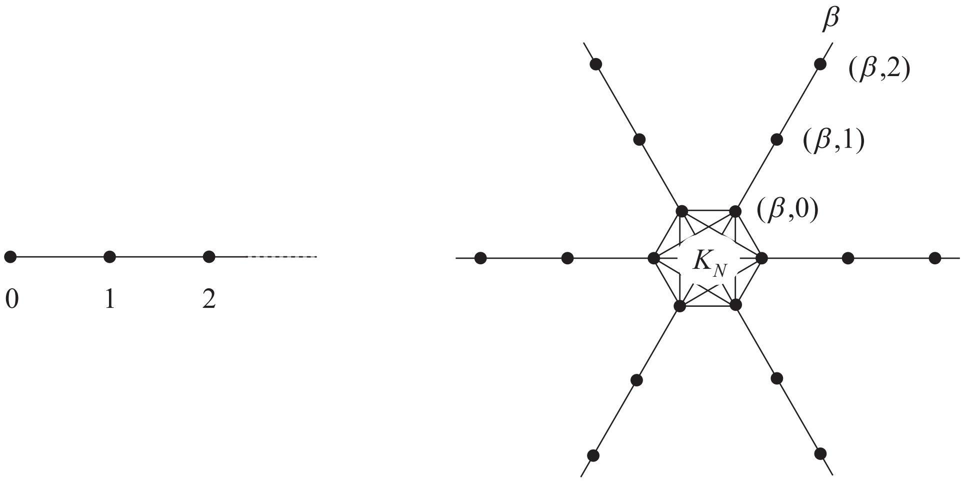

FIGURE 1. Paths connected with a clique

In the original Karlin–McGregor formula the polynomials $Q_j(x)$ are essential, where j runs over the non-negative integers, i.e., the state space of the birth-and-death process. It is our key observation (Section 3.2) that we can construct a family of polynomials $Q^{(\alpha)}_{\beta,j}(x)$, where (β, j) runs over the "state space" of the truncated direct sum. These polynomials are essential to our main integral expression (Theorem 3.3). Moreover, we discuss how to obtain the measures therein by means of generating functions (Section 3.4) and show an example of concrete computation by a free Fock space (Section 3.5).

2. THE KARLIN–MCGREGOR FORMULA IN TERMS OF ONE-MODE INTERACTING FOCK SPACES (REVISITED)

We assemble some general results on one-mode interacting Fock spaces from the point of view of tridiagonal matrices, for details see [16].

2.1. Tridiagonal matrices of infinite type. Let $\mathfrak{T}_\infty$ denote the set of real infinite tridiagonal matrices with non-zero off-diagonal entries of the form

$$T = \begin{bmatrix} b_0 & a_0 & & & & & \\ c_1 & b_1 & a_1 & & & & \\ & c_2 & b_2 & a_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & c_n & b_n & a_n & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}, \qquad \begin{matrix} a_n, b_n, c_n \in \mathbb{R}, \\ a_n \neq 0,\ c_n \neq 0. \end{matrix}$$

A tridiagonal matrix $T \in \mathfrak{T}_\infty$ is called a *Jacobi matrix* (*of infinite type*) if it is symmetric with positive off-diagonal entries, i.e., if

$$a_{n-1} = c_n \equiv \sqrt{\omega_n}\,, \quad b_{n-1} \equiv \alpha_n\,, \quad n = 1, 2, \dots,$$

with some $\omega_n > 0$ and $\alpha_n \in \mathbb{R}$. In this case the pair of two sequences $(\{\omega_n\}, \{\alpha_n\})$ is called the *Jacobi parameters*.

2.2. One-mode interacting Fock spaces. Each $T \in \mathfrak{T}_\infty$ is considered naturally as the matrix representation of a linear operator acting on a Hilbert space. To fix our notation, take a Hilbert space Γ equipped with a complete orthonormal basis $\{\Phi_n\,;\, n = 0, 1, 2, \dots\}$. Let $\Gamma_0 \subset \Gamma$ denote the subspace spanned by this basis, and $\mathcal{L}(\Gamma_0)$ the space of linear operators from Γ_0 into itself. We define linear operators B^+, B°, B^- in $\mathcal{L}(\Gamma_0)$ by

$$\begin{aligned} B^+\Phi_n &= c_{n+1}\Phi_{n+1}, \quad n = 0, 1, 2, \dots, \\ B^\circ\Phi_n &= b_n\Phi_n, \quad n = 0, 1, 2, \dots, \\ B^-\Phi_0 &= 0, \quad B^-\Phi_n = a_{n-1}\Phi_{n-1}, \quad n = 1, 2, \dots \end{aligned}$$

The above actions are illustrated in Figure 2. Then T is the matrix representation of $B^+ + B^- + B^\circ$ with respect to $\{\Phi_n\}$. In this sense we write

$$T = B^+ + B^\circ + B^-.$$

In particular, $T \in \mathcal{L}(\Gamma_0)$. The system $(\Gamma, \{\Phi_n\}, B^+, B^-, B^\circ)$, denoted also by $(\Gamma, \{\Phi_n\}, T)$ or Γ for brevity, is called a *one-mode interacting Fock space* associated with $T \in \mathfrak{T}_\infty$. This terminology is slightly abusive, as a one-mode interacting Fock space originally corresponds to a Jacobi matrix. However, many combinatorial properties are maintained.

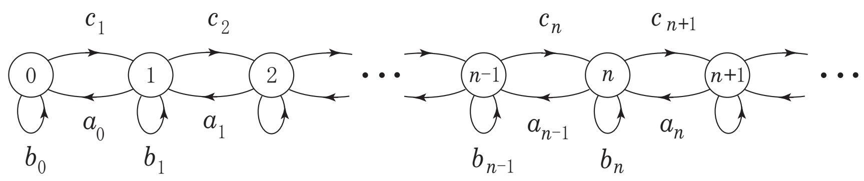

FIGURE 2. Interacting Fock space $(\Gamma, \{\Phi_n\}, B^+, B^-, B^\circ)$

In general, for $S \in \mathcal{L}(\Gamma_0)$ the matrix elements are defined by

$$(S)_{ij} = \langle \Phi_i, S\Phi_j \rangle, \qquad i, j = 0, 1, 2, \dots,$$

where $\langle \cdot, \cdot \rangle$ stands for the inner product of Γ satisfying $\langle \Phi_i, \Phi_j \rangle = \delta_{ij}$. It is sometimes convenient to consider the underlying digraph with vertex set $\{0, 1, 2, \dots\}$, where each directed arc $j \to i$ is given a weight $(S)_{ij}$.

REMARK 2.1. In this paper, for technical simplicity of presentation, we restrict ourselves to tridiagonal matrices of infinite type. We may include those of finite type by repeating parallel arguments, see also [12] and [16].

2.3. Generating functions. For a general $S \in \mathcal{L}(\Gamma_0)$ we define the generating functions by

$$g_{ij}(S; z) = \sum_{m=0}^{\infty} (S^m)_{ij} z^m = \sum_{m=0}^{\infty} \langle \Phi_i, S^m \Phi_j \rangle \, z^m, \tag{2.1}$$

$$G_{ij}(S; z) = z^{-1} g_{ij}(S; z^{-1}) = \sum_{m=0}^{\infty} (S^m)_{ij} \, z^{-m-1}, \tag{2.2}$$

where $(S^0)_{ij} = \delta_{ij}$ as usual. The latter is often referred to as the *Green function*.

For $S \in \mathcal{L}(\Gamma_0)$ and $m \geqslant 1$ we have, by definition,

$$(S^m)_{ij} = \sum_{k_1, \dots, k_{m-1}} (S)_{ik_{m-1}} \cdots (S)_{k_2 k_1} (S)_{k_1 j}, \tag{2.3}$$

where the sum is taken over all possible $k_1, \dots, k_{m-1} \in \{0, 1, 2, \dots\}$. Hence the right-hand side of (2.3) is interpreted as the sum of weights of m-step directed walks from j to i. It is natural to consider the partial sum corresponding to m-step directed walks that start from j, terminate at i but do not visit i during the journey.

We set

$$
F_{ij}(S;m) = \begin{cases} (S)_{ij}, & m=1, \\ \displaystyle\sum_{k_1,\dots,k_{m-1}\neq i} (S)_{ik_{m-1}} \dots (S)_{k_2k_1}(S)_{k_1j}, & m \geqslant 2. \end{cases} \tag{2.4}
$$

The generating function is defined by

$$
f_{ij}(S;z) = \sum_{m=1}^{\infty} F_{ij}(S;m)\, z^m.
$$

Then we have

$$
g_{ij}(S;z) - \delta_{ij} = g_{ii}(S;z) f_{ij}(S;z). \tag{2.5}
$$

The above relation is fundamental in the study of Markov chains and is proved in a similar manner.

2.4. Associated polynomials and the Karlin–McGregor formula. With each $T \in \mathfrak{T}_\infty$ we associate a sequence of polynomials $Q_0(x), Q_1(x), Q_2(x), \dots$ defined by

$$
\begin{aligned} &xQ_n(x) = a_nQ_{n+1}(x) + b_nQ_n(x) + c_nQ_{n-1}(x), \\ &Q_0(x) = 1, \quad Q_{-1}(x) = 0. \end{aligned} \tag{2.6}
$$

The recurrence relation (2.6) is extensively studied in connection with orthogonal polynomials, see, e.g., [6] and [7].

We say that $T \in \mathfrak{T}_\infty$ satisfies the *positivity condition* if

$$
\omega_n \equiv a_{n-1}c_n > 0, \quad \alpha_n \equiv b_{n-1} \in \mathbb{R}, \quad n = 1, 2, \dots \tag{2.7}
$$

In this case, $(\{\omega_n\}, \{\alpha_n\})$ becomes Jacobi parameters. Then, there exists a probability distribution μ on $\mathbb{R}$ whose Jacobi parameters coincide with $(\{\omega_n\}, \{\alpha_n\})$. Namely, the orthogonal polynomials $\{P_n(x)\}$ associated with μ, normalized as $P_n(x) = x^n + \dots$, fulfill the recurrence relation

$$
\begin{aligned} &xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_nP_{n-1}(x), \\ &P_0(x) = 1, \quad P_{-1}(x) = 0. \end{aligned} \tag{2.8}
$$

The probability distribution μ is uniquely specified if and only if it is the solution to a determinate moment problem. For more details see [12], Chapter 1.

We now come to the Karlin–McGregor formula for a one-mode interacting Fock space.

THEOREM 2.1. *Let $T \in \mathfrak{T}_\infty$ and let $\{Q_n(x)\}$ be the associated polynomials. Assume that T satisfies the positivity condition and take a probability distribution μ on $\mathbb{R}$ with Jacobi parameters $(\{\omega_n\}, \{\alpha_n\})$ in* (2.7). *Then we have*

$$(T^m)_{ij} = \frac{1}{\pi(j)} \int_{-\infty}^{+\infty} x^m Q_i(x) Q_j(x) \mu(dx), \tag{2.9}$$

for $i, j = 0, 1, \dots$ and $m = 1, 2, \dots$, where

$$\pi(0) = 1, \quad \pi(j) = \frac{c_1 c_2 \dots c_j}{a_0 a_1 \dots a_{j-1}}, \quad j = 1, 2, \dots \tag{2.10}$$

In particular,

$$(T^m)_{00} = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \dots \tag{2.11}$$

For the proof see [16]. We also note the following result, the verification of which is straightforward and is omitted.

THEOREM 2.2 (Detailed balance condition). *The notation and assumptions being as in Theorem* 2.1, *we have*

$$(T)_{ij} \pi(j) = (T)_{ji} \pi(i), \quad i, j = 0, 1, 2, \dots \tag{2.12}$$

Under the positivity condition we can define a new inner product on Γ_0 by

$$\langle \Phi_i, \Phi_j \rangle_\pi = \frac{1}{\pi(i)} \langle \Phi_i, \Phi_j \rangle, \quad i, j = 0, 1, 2, \dots,$$

since $\pi(i) > 0$ for all i. Let Γ_π be the Hilbert space obtained by completing Γ_0 with respect to this new inner product. We easily see that (2.12) is equivalent to

$$\langle \Phi_i, T\Phi_j \rangle_\pi = \langle T\Phi_i, \Phi_j \rangle_\pi, \quad i, j = 0, 1, 2, \dots$$

In other words, T becomes a symmetric operator in Γ_π (but not necessarily self-adjoint, of course).

REMARK 2.2. *Theorem* 2.1 *extends the famous Karlin and McGregor formula* [13] *originally stated for T being a transition probability matrix* (*i.e.*, *there is a random walk on $\{0, 1, 2, \dots\}$ behind*). *Moreover*, *we note that Theorem* 2.1 *holds without assuming that T is self-adjoint and our proof* [16] *is free from the spectral theory. As a result*, *although the existence of a probability distribution μ in* (2.9) *is always guaranteed*, *the uniqueness does not hold in general.*

3. TRUNCATED DIRECT SUM OF ONE-MODE INTERACTING FOCK SPACES

3.1. Definition. For $\alpha = 1, 2, \dots, N$ let $T^{(\alpha)} \in \mathfrak{T}_\infty$ and denote their matrix elements by $a_n^{(\alpha)}, b_n^{(\alpha)}, c_n^{(\alpha)}$. Let $(\Gamma^{(\alpha)}, \{\Phi_n^{(\alpha)}\}, T^{(\alpha)})$ be the one-mode interacting Fock space associated with $T^{(\alpha)}$. The direct sum

$$\Gamma = \Gamma^{(1)} \oplus \Gamma^{(2)} \oplus \ldots \oplus \Gamma^{(N)}$$

is naturally equipped with a complete orthonormal basis

$$\{\Phi_n^{(\alpha)}\} = \{\Phi_n^{(1)}\} \cup \{\Phi_n^{(2)}\} \cup \ldots \cup \{\Phi_n^{(N)}\}.$$

Let $\Gamma_0 \subset \Gamma$ be the subspace spanned by this basis. Given $p_1, p_2, \dots, p_N \in \mathbb{R}$, we define $T \in \mathcal{L}(\Gamma_0)$ by

$$T\Phi_n^{(\alpha)} = \begin{cases} T^{(\alpha)}\Phi_0^{(\alpha)} + \sum\limits_{\beta \neq \alpha} p_\beta \Phi_0^{(\beta)}, & n = 0, \\ T^{(\alpha)}\Phi_n^{(\alpha)}, & n \geqslant 1. \end{cases} \tag{3.1}$$

These actions are illustrated in Figure 3. The triple $(\Gamma, \{\Phi_n^{(\alpha)}\}, T)$ is called the *truncated direct sum* of the interacting Fock spaces $\Gamma^{(1)}, \dots, \Gamma^{(N)}$. The constant numbers $p_1, p_2, \dots, p_N$ are called *joint parameters*. It is noted that the truncated direct sum is no longer an interacting Fock space.

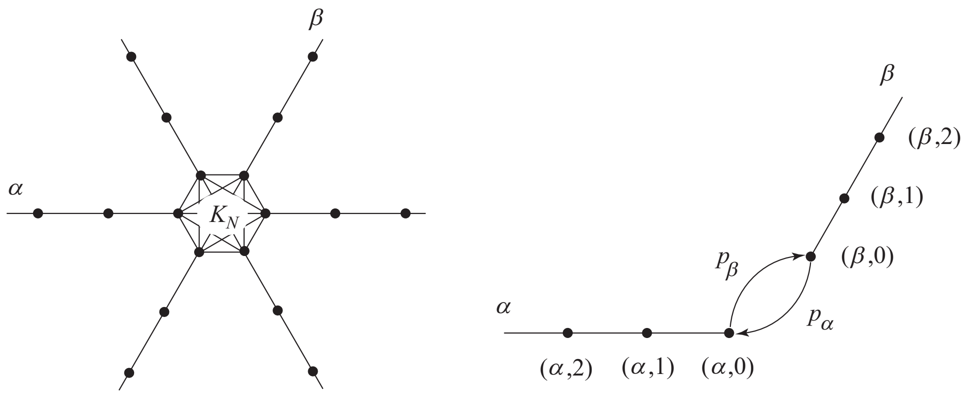

FIGURE 3. Truncated direct sum of interacting Fock spaces

The matrix elements are defined by

$$(T)_{(\alpha,i),(\beta,j)} = \langle \Phi_i^{(\alpha)}, T\Phi_j^{(\beta)} \rangle$$

as usual. Our main goal is to obtain an integral expression for $(T^m)_{(\alpha,i),(\beta,j)}$.

REMARK 3.1. *We see naturally that the underlying graph structure of a one-mode interacting Fock space* $(\Gamma, \{\Phi_n\}, T)$ *with* $T \in \mathfrak{T}_\infty$ *is the infinite path* P_∞. *In a similar sense the underlying graph of the truncated direct sum of* N *one-mode interacting Fock spaces is a graph of* N *copies of infinite paths, of which endpoints are connected with a complete graph* K_N (see Figure 3). *In this case the complete graph is also called a clique. It is noteworthy that this graph admits a comb product structure* $K_N \triangleright P_\infty$ (see also [1] and [12]).

3.2. Associated polynomials. It is convenient to consider

$$(\beta, n) \in \{1, 2, \dots, N\} \times \{0, 1, 2, \dots\}$$

as a natural label of a vertex of the graph $K_N \triangleright P_\infty$ introduced at the end of the previous section, see also Figure 3. For each $\alpha \in \{1, 2, \dots, N\}$ we will introduce a system of polynomials indexed by (β, n):

$$Q^{(\alpha)}_{\beta,n}(x), \quad \beta = 1, 2, \dots, N, \quad n = 0, 1, 2, \dots \tag{3.2}$$

For $\beta = \alpha$ we define $\{Q^{(\alpha)}_{\alpha,n}(x)\,;\, n = 0, 1, 2, \dots\}$ by the standard recurrence relation:

$$xQ^{(\alpha)}_{\alpha,n}(x) = a^{(\alpha)}_n Q^{(\alpha)}_{\alpha,n+1}(x) + b^{(\alpha)}_n Q^{(\alpha)}_{\alpha,n}(x) + c^{(\alpha)}_n Q^{(\alpha)}_{\alpha,n-1}(x), \tag{3.3}$$

$$Q^{(\alpha)}_{\alpha,0}(x) = 1, \quad Q^{(\alpha)}_{\alpha,-1}(x) = 0, \tag{3.4}$$

see Section 2.4. In other words, $\{Q^{(\alpha)}_{\alpha,n}(x)\,;\, n = 0, 1, 2, \dots\}$ is nothing else but the polynomials associated with $T^{(\alpha)}$. These are extended to a system of polynomials indexed by all the vertices (β, n) as follows: For $\beta \neq \alpha$ we define

$$xQ^{(\alpha)}_{\beta,n}(x) = \begin{cases} a^{(\beta)}_n Q^{(\alpha)}_{\beta,n+1}(x) + b^{(\beta)}_n Q^{(\alpha)}_{\beta,n}(x) + c^{(\beta)}_n Q^{(\alpha)}_{\beta,n-1}(x), & n \geqslant 1, \\ a^{(\beta)}_0 Q^{(\alpha)}_{\beta,1}(x) + b^{(\beta)}_0 Q^{(\alpha)}_{\beta,0}(x) + p_\beta Q^{(\alpha)}_{\beta,-1}(x), & n = 0, \end{cases} \tag{3.5}$$

$$Q^{(\alpha)}_{\beta,0}(x) = 0, \quad Q^{(\alpha)}_{\beta,-1}(x) = 1. \tag{3.6}$$

The coefficients are obtained naturally from the diagram in Figure 3.

We now state an important result, which is motivated by the argument in [13].

THEOREM 3.1. *Let* $\{Q^{(\alpha)}_{\beta,n}(x)\}$ *be the polynomials associated with the truncated direct sum* T *as above. Then, for* $\beta = 1, 2, \dots, N$ *and* $j = 0, 1, 2, \dots,$ *we have*

$$\sum_{\alpha=1}^{N} p_\alpha Q^{(\alpha)}_{\beta,j}(T)\Phi^{(\alpha)}_0 = \pi(\beta, j)\Phi^{(\beta)}_j,$$

where

$$\pi(\beta, j) = p_\beta \, \frac{c^{(\beta)}_1 \dots c^{(\beta)}_j}{a^{(\beta)}_0 \dots a^{(\beta)}_{j-1}}\,. \tag{3.7}$$

P r o o f. It is sufficient to prove the assertion for $\beta = 1$, i.e.,

$$\sum_{\alpha=1}^{N} p_\alpha Q_{1,j}^{(\alpha)}(T)\Phi_0^{(\alpha)} = \pi(1,j)\Phi_j^{(1)}. \tag{3.8}$$

We employ induction on j. For $j = 0$, in view of (3.4) and (3.6) we obtain

$$\sum_{\alpha=1}^{N} p_\alpha Q_{1,0}^{(\alpha)}(T)\Phi_0^{(\alpha)} = p_1\Phi_0^{(1)} = \pi(1,0)\Phi_0^{(1)}.$$

Next, let $j = 1$. By simple calculation we obtain

$$Q_{11}^{(\alpha)}(x) = \begin{cases} (x - b_0^{(1)})/a_0^{(1)}, & \alpha = 1, \\ -p_1/a_0^{(1)}, & \alpha \neq 1. \end{cases}$$

In order to compute the left-hand side of (3.8), we first observe that

$$\begin{aligned} p_1 Q_{1,1}^{(1)}(T)\Phi_0^{(1)} &= \frac{p_1}{a_0^{(1)}}\,(T - b_0)\Phi_0^{(1)} \\ &= \frac{p_1}{a_0^{(1)}}\big(c_1^{(1)}\Phi_1^{(1)} + b_0^{(1)}\Phi_0^{(1)} + \sum_{\beta=2}^{N} p_\beta \Phi_0^{(\beta)}\big) - \frac{p_1 b_0^{(1)}}{a_0^{(1)}}\,\Phi_0^{(1)} \\ &= p_1\,\frac{c_1^{(1)}}{a_0^{(1)}}\,\Phi_1^{(1)} + \frac{p_1}{a_0^{(1)}}\sum_{\beta=2}^{N} p_\beta \Phi_0^{(\beta)}, \end{aligned} \tag{3.9}$$

where (3.1) is taken into account. For $\alpha \neq 1$ we have

$$p_\alpha Q_{1,1}^{(\alpha)}(T)\Phi_0^{(\alpha)} = p_\alpha\bigg(-\frac{p_1}{a_0^{(1)}}\bigg)\Phi_0^{(\alpha)}. \tag{3.10}$$

It then follows from (3.9) and (3.10) that

$$\begin{aligned} \sum_{\alpha=1}^{N} p_\alpha Q_{1,1}^{(\alpha)}(T)\Phi_0^{(\alpha)} &= p_1\,\frac{c_1^{(1)}}{a_0^{(1)}}\,\Phi_1^{(1)} + \frac{p_1}{a_0^{(1)}}\sum_{\beta=2}^{N} p_\beta\Phi_0^{(\beta)} + \sum_{\alpha=2}^{N} p_\alpha\bigg(-\frac{p_1}{a_0^{(1)}}\bigg)\Phi_0^{(\alpha)} \\ &= p_1\,\frac{c_1^{(1)}}{a_0^{(1)}}\,\Phi_1^{(1)} \\ &= \pi(1,1)\Phi_1^{(1)}, \end{aligned}$$

which shows that (3.8) is valid for $j = 1$. Now, let $j \geqslant 1$ and suppose that the assertion is valid up to j, i.e., (3.8) holds. Using the identities

$$\begin{aligned} TQ_{1,j}^{(\alpha)}(T) &= a_j^{(1)}Q_{1,j+1}^{(\alpha)}(T) + b_j^{(1)}Q_{1,j}^{(\alpha)}(T) + c_j^{(1)}Q_{1,j-1}^{(\alpha)}(T), \\ T\Phi_j^{(\alpha)} &= c_{j+1}^{(\alpha)}\Phi_{j+1}^{(\alpha)} + b_j^{(\alpha)}\Phi_j^{(\alpha)} + a_{j-1}^{(\alpha)}\Phi_{j-1}^{(\alpha)}, \end{aligned}$$

we compute the left-hand side of (3.8) multiplied by T as follows:

$$
\begin{aligned}
(3.11)\quad & T\sum_{\alpha=1}^{N} p_\alpha Q_{1,j}^{(\alpha)}(T)\Phi_0^{(\alpha)} \\
&= \sum_{\alpha=1}^{N} p_\alpha \big(a_j^{(1)} Q_{1,j+1}^{(\alpha)}(T) + b_j^{(1)} Q_{1,j}^{(\alpha)}(T) + c_j^{(1)} Q_{1,j-1}^{(\alpha)}(T)\big)\Phi_0^{(\alpha)} \\
&= a_j^{(1)} \sum_{\alpha=1}^{N} p_\alpha Q_{1,j+1}^{(\alpha)}(T)\Phi_0^{(\alpha)} + b_j^{(1)}\pi(1,j)\Phi_j^{(1)} + c_j^{(1)}\pi(1,j-1)\Phi_{j-1}^{(1)} \\
&= a_j^{(1)} \sum_{\alpha=1}^{N} p_\alpha Q_{1,j+1}^{(\alpha)}(T)\Phi_0^{(\alpha)} + b_j^{(1)}\pi(1,j)\Phi_j^{(1)} + a_{j-1}^{(1)}\pi(1,j)\Phi_{j-1}^{(1)}.
\end{aligned}
$$

On the other hand, the right-hand side of (3.8) becomes

(3.12)

$$
\begin{aligned}
T\pi(1,j)\Phi_j^{(1)} &= c_{j+1}^{(1)}\pi(1,j)\Phi_{j+1}^{(1)} + b_j^{(1)}\pi(1,j)\Phi_j^{(1)} + a_{j-1}^{(1)}\pi(1,j)\Phi_{j-1}^{(1)} \\
&= a_j^{(1)}\pi(1,j+1)\Phi_{j+1}^{(1)} + b_j^{(1)}\pi(1,j)\Phi_j^{(1)} + a_{j-1}^{(1)}\pi(1,j)\Phi_{j-1}^{(1)}.
\end{aligned}
$$

Since (3.11) and (3.12) coincide, we obtain

$$
a_j^{(1)} \sum_{\alpha=1}^{N} p_\alpha Q_{1,j+1}^{(\alpha)}(T)\Phi_0^{(\alpha)} = a_j^{(1)}\pi(1,j+1)\Phi_{j+1}^{(1)},
$$

which implies that (3.8) remains valid with j being replaced with $j+1$. ■

Moreover, the "detailed balance condition" mentioned in Theorem 2.2 remains valid. The proof is straightforward computation and is omitted.

THEOREM 3.2. *Let $\pi(\beta, j)$ be defined as in* (3.7). *Then we have*

$$
(3.13)\qquad (T)_{(\alpha,i)(\beta,j)}\pi(\beta,j) = (T)_{(\beta,j)(\alpha,i)}\pi(\alpha,i)
$$

for all (α, i) and (β, j).

3.3. Integral formula. We will continue to consider the truncated direct sum $(\Gamma, \{\Phi_n^{(\alpha)}\}, T)$ of one-mode interacting Fock spaces $(\Gamma^{(\alpha)}, \{\Phi_n^{(\alpha)}\}, T^{(\alpha)})$, $\alpha = 1, 2, \dots, N$, with joint parameters $p_1, \dots, p_N$. From now on, we assume that each $T^{(\alpha)}$ fulfills the positivity condition and that the joint parameters $p_1, \dots, p_N$ are all positive. The main purpose is to derive an integral expression for the matrix elements of T^m.

By assumption we see that $\pi(\beta, j) > 0$ for all (β, j), see (3.7). Then

$$
\langle \Phi_i^{(\alpha)}, \Phi_j^{(\beta)} \rangle_\pi = \frac{1}{\pi(\beta, j)} \langle \Phi_i^{(\alpha)}, \Phi_j^{(\beta)} \rangle
$$

becomes an inner product on $\Gamma_0 \subset \Gamma$, the subspace spanned by $\{\Phi_n^{(\alpha)}\}$. Let Γ_π be the Hilbert space obtained by completing Γ_0 with respect to this new inner product $\langle \cdot, \cdot \rangle_\pi$. It follows from Theorem 3.2 that T is a symmetric operator in Γ_π.

REMARK 3.2. *As shown at the end of Section* 2.4, *for each α we have a Hilbert space $\Gamma_\pi^{(\alpha)}$ on which $T^{(\alpha)}$ is symmetric. As Γ_π is a direct sum of $\Gamma_\pi^{(\alpha)}$ with weight p_α^{-1}, the truncated direct sum T remains symmetric.*

The following result extends the Karlin–McGregor formula to the truncated direct sum of one-mode interacting Fock spaces, see Theorem 2.1.

THEOREM 3.3. *Assume that $T^{(\alpha)}$ is a self-adjoint operator in $\Gamma_\pi^{(\alpha)}$ for all α. Then there exists an $N \times N$ positive-definite matrix-valued measure $[\mu_{\gamma\gamma'}]$ such that*

$$(3.14) \qquad (T^m)_{(\alpha,i),(\beta,j)} = \frac{1}{\pi(\beta,j)} \sum_{\gamma,\gamma'=1}^{N} \int_{-\infty}^{+\infty} x^m Q_{\alpha,i}^{(\gamma)}(x) Q_{\beta,j}^{(\gamma')}(x) \mu_{\gamma\gamma'}(dx).$$

Proof. First we note that T is a self-adjoint operator in Γ_π. In fact, by assumption the direct sum $\tilde{T} = T^{(1)} \oplus \ldots \oplus T^{(N)}$ is a self-adjoint operator in Γ_π, and so is T since it is a sum of $\tilde{T}$ and a bounded self-adjoint operator determined by joint parameters, see (3.1). Let

$$(3.15) \qquad T = \int_{-\infty}^{+\infty} x dE(x)$$

be the spectral decomposition and define

$$\mu_{\gamma\gamma'}(dx) = p_\gamma p_{\gamma'} \langle \Phi_0^{(\gamma)}, E(dx)\Phi_0^{(\gamma')} \rangle_\pi, \qquad \gamma, \gamma' = 1, 2, \ldots, N.$$

It is easy to verify that $[\mu_{\gamma\gamma'}(dx)]$ is an $N \times N$ positive-definite matrix-valued measure (for any Borel set S, the $N \times N$ matrix $[\mu_{\gamma\gamma'}(S)]$ is Hermitian with non-negative eigenvalues). By definition we have

$$(T^m)_{(\alpha,i),(\beta,j)} = \langle \Phi_i^{(\alpha)}, T^m \Phi_j^{(\beta)} \rangle = \pi(\alpha,i) \langle \Phi_i^{(\alpha)}, T^m \Phi_j^{(\beta)} \rangle_\pi.$$

Then, by Theorem 3.1, we have

$$\begin{aligned}(T^m)_{(\alpha,i),(\beta,j)} &= \frac{1}{\pi(\beta,j)} \sum_{\gamma,\gamma'=1}^{N} p_\gamma p_{\gamma'} \langle Q_{\alpha,i}^{(\gamma)}(T)\Phi_0^{(\gamma)}, T^m Q_{\beta,j}^{(\gamma')}(T)\Phi_0^{(\gamma')} \rangle_\pi \\ &= \frac{1}{\pi(\beta,j)} \sum_{\gamma,\gamma'=1}^{N} p_\gamma p_{\gamma'} \langle \Phi_0^{(\gamma)}, T^m Q_{\alpha,i}^{(\gamma')}(T) Q_{\beta,j}^{(\gamma')}(T)\Phi_0^{(\gamma')} \rangle_\pi.\end{aligned}$$

The last inner product is expressed in terms of the spectral decomposition (3.15) and we obtain (3.14). ■

3.4. Generating functions. We now discuss how to obtain $N \times N$ positive-definite matrix-valued measure $[\mu_{\gamma\gamma'}]$ in Theorem 3.3. Specializing parameters in (3.14), we obtain

$$(T^m)_{(\alpha,0),(\beta,0)} = \frac{1}{p_\beta} \int_{-\infty}^{+\infty} x^m \mu_{\alpha\beta}(dx), \tag{3.16}$$

where $\alpha, \beta = 1, 2, \ldots, N$ and $m = 1, 2, \ldots$ Hence $\mu_{\alpha\beta}$ is determined by the generating functions:

$$G_{(\alpha,0),(\beta,0)}(T; z) = z^{-1} g_{(\alpha,0),(\beta,0)}(z^{-1}), \tag{3.17}$$

$$g_{(\alpha,0),(\beta,0)}(T; z) = \sum_{m=0}^{\infty} (T^m)_{(\alpha,0),(\beta,0)} z^m. \tag{3.18}$$

Our desire is to express $G_{(\alpha,0),(\beta,0)}(T; z)$ in terms of the generating functions for $T^{(\alpha)}$:

$$g^{(\alpha)}(z) = g_{00}(T^{(\alpha)}; z) = \sum_{m=0}^{\infty} (T^{(\alpha)m})_{00} z^m,$$
$$G^{(\alpha)}(z) = G_{00}(T^{(\alpha)}; z) = z^{-1} g_{00}(T^{(\alpha)}; z^{-1}).$$

However, the direct formula seems too lengthy to write down in a concise form. We state an inductive formula instead.

THEOREM 3.4. *The notation and assumptions being as in Theorem* 3.3, *we have*

$$G_{(\alpha,0),(\alpha,0)}(T; z) = \frac{G^{(\alpha)}(z)}{1 - p_\alpha G^{(\alpha)}(z) \sum\limits_{\gamma,\gamma' \neq \alpha} p_\gamma G_{(\gamma',0),(\gamma,0)}(T \backslash T^{(\alpha)}; z)}, \tag{3.19}$$

and for $\alpha \neq \beta$

$$G_{(\alpha,0),(\beta,0)}(T; z) = p_\alpha G_{(\alpha,0),(\alpha,0)}(T; z) \sum_{\gamma \neq \alpha} G_{(\gamma,0),(\beta,0)}(T \backslash T^{(\alpha)}; z). \tag{3.20}$$

Here $T \backslash T^{(\alpha)}$ *stands for the truncated direct sum of* $T^{(1)}, \ldots, T^{(N)}$ *except* $T^{(\alpha)}$.

Proof. We consider $F_{(\alpha,0),(\alpha,0)}(T; m)$, for the definition see (2.4). The corresponding walks are divided into two classes: the first class consists of walks confined in $\Gamma^{(\alpha)}$, and the second consists of those never passing $\Gamma^{(\alpha)}$ except the initial and terminal $(\alpha, 0)$. Thus,

$$F_{(\alpha,0),(\alpha,0)}(T; m) = F_{00}(T^{(\alpha)}; m) + \sum_{\gamma,\gamma' \neq \alpha} p_\alpha p_\gamma \big((T \backslash T^{(\alpha)})^{m-2}\big)_{(\gamma',0),(\gamma,0)},$$

where the second term is understood to be zero for $m = 1$. Multiplying by z^m and taking the sum over $m = 1, 2, \dots$, we obtain

$$
\begin{aligned}
f_{(\alpha,0),(\alpha,0)}(T; z) &= \sum_{m=1}^{\infty} F_{(\alpha,0),(\alpha,0)}(T; m) z^m \\
&= f_{00}(T^{(\alpha)}; z) + p_\alpha z^2 \sum_{\gamma,\gamma' \neq \alpha} p_\gamma g_{(\gamma',0),(\gamma,0)}(T \backslash T^{(\alpha)}; z).
\end{aligned}
$$

Then (3.19) follows easily by using the general formula (2.5). Similarly, starting with

$$
F_{(\alpha,0),(\beta,0)}(T; m) = \sum_{\gamma \neq \alpha} p_\alpha \big((T \backslash T^{(\alpha)})^{m-1} \big)_{(\gamma,0),(\beta,0)},
$$

one may derive (3.20) with no difficulty. ■

3.5. Examples. For two one-mode interacting Fock spaces

$$
(\Gamma^{(\alpha)}, \{\Phi_n^{(\alpha)}\}, T^{(\alpha)}), \qquad \alpha = 1, 2,
$$

we consider their truncated direct sum T. By a direct consequence from Theorem 3.4 we have

$$
\begin{aligned}
G_{(1,0),(1,0)}(T; z) &= \frac{G^{(1)}}{1 - p_1 p_2 G^{(1)} G^{(2)}}, \quad G_{(1,0),(2,0)}(T; z) = \frac{p_1 G^{(1)} G^{(2)}}{1 - p_1 p_2 G^{(1)} G^{(2)}}, \\
G_{(2,0),(1,0)}(T; z) &= \frac{p_2 G^{(1)} G^{(2)}}{1 - p_1 p_2 G^{(1)} G^{(2)}}, \quad G_{(2,0),(2,0)}(T; z) = \frac{G^{(2)}}{1 - p_1 p_2 G^{(1)} G^{(2)}},
\end{aligned}
$$

where the variable z is omitted on the right-hand sides.

For illustrating the computation we consider the *one-mode free Fock space*, i.e., the one-mode interacting Fock space associated with $T \in \mathfrak{T}_\infty$ with the following parameters:

$$
a_0 = a_1 = \ldots = 1, \qquad b_0 = b_1 = \ldots = 0, \qquad c_1 = c_2 = \ldots = 1.
$$

For the relevant properties of free Fock space, see, e.g., [11], [12], [17].

Let $\Gamma^{(1)}$ and $\Gamma^{(2)}$ be one-mode interacting Fock spaces and T be their truncated direct sum with joint parameters $p_1 > 0$ and $p_2 > 0$. It is known that

$$
\begin{aligned}
g^{(1)}(z) = g^{(2)}(z) &= \sum_{m=0}^{\infty} \frac{(2m)!}{(m+1)!m!} z^{2m} = \frac{1 - \sqrt{1 - 4z^2}}{2z^2}, \\
G^{(1)}(z) = G^{(2)}(z) &= \frac{z - \sqrt{z^2 - 4}}{2}.
\end{aligned}
$$

By the above explicit formulae we obtain

$$G_{(1,0),(1,0)}(T;z) = G_{(2,0),(2,0)}(T;z) = \frac{1}{2}\,\frac{(1-p_1p_2)z-(1+p_1p_2)\sqrt{z^2-4}}{(1+p_1p_2)^2-p_1p_2z^2}\,, \tag{3.21}$$

$$G_{(1,0),(2,0)}(T;z) = \frac{p_1}{2}\,\frac{z^2-2(1+p_1p_2)-z\sqrt{z^2-4}}{(1+p_1p_2)^2-p_1p_2z^2}\,, \tag{3.22}$$

$$G_{(2,0),(1,0)}(T;z) = \frac{p_2}{2}\,\frac{z^2-2(1+p_1p_2)-z\sqrt{z^2-4}}{(1+p_1p_2)^2-p_1p_2z^2}\,. \tag{3.23}$$

It is known that (3.21) is the Stieltjes transform of the Kesten distribution with parameters $(1+p_1p_2, 1)$, which we denote by κ. For $0 < p_1p_2 \leqslant 1$, κ is absolutely continuous with respect to the Lebesgue measure and has the density function

$$\frac{1+p_1p_2}{2\pi}\,\frac{\sqrt{4-x^2}}{(1+p_1p_2)^2-p_1p_2x^2}\,, \qquad |x| \leqslant 2.$$

For $p_1p_2 > 1$, κ is the sum of the absolutely continuous measure defined by the above function and two atoms:

$$\frac{p_1p_2-1}{2p_1p_2}(\delta_{-\xi}+\delta_{\xi}), \qquad \xi = \frac{1+p_1p_2}{\sqrt{p_1p_2}}\,.$$

Thus, μ_{11} and μ_{22} in (3.16) are determined as

$$\mu_{11} = p_1\kappa, \qquad \mu_{22} = p_2\kappa.$$

Moreover, we can obtain μ_{12} and μ_{21} explicitly by computation. Consequently,

$$[\mu_{\alpha\beta}] = \begin{bmatrix} p_1\kappa & \dfrac{p_1x}{1+p_1p_2}\,\kappa \\ \dfrac{p_2x}{1+p_1p_2}\,\kappa & p_2\kappa \end{bmatrix}.$$

We know that the Kesten distribution with parameters $(2, 1)$ coincides with the arcsine law with mean zero and variance two. This happens when $p_1 = p_2 = 1$. In this case, the truncated direct sum is essentially reduced to the one-dimensional random walk and the appearance of the arcsine law is well known, e.g., [13], Section 4.

REMARK 3.3. The term "Kesten distribution" is after [14]. See [12], Chapter 4, for its precise definition and relevant properties.

A new operation for probability distributions is defined through the truncated direct sum of one-mode interacting Fock spaces. Given two probability distributions ν_1 and ν_2, we associate the interacting Fock spaces $\Gamma^{(1)}$ and $\Gamma^{(2)}$, see Theorem 2.1. After truncation we obtain a map $(\nu_1, \nu_2) \mapsto (\mu_{11}/p_1, \mu_{22}/p_2)$. For instance, as we have observed in the above example, a pair of the Kesten distributions (κ, κ) is obtained from a pair of semicircle laws $\nu_1 = \nu_2$.

Finally, we record the expression of $G_{(\alpha,0),(\beta,0)}(T; z)$ for the truncated direct sum of three one-mode interacting Fock spaces:

$$G_{(1,0),(1,0)}(T; z) = \frac{G^{(1)}(1 - p_2 p_3 G^{(2)} G^{(3)})}{1 - 2p_1 p_2 p_3 G^{(1)} G^{(2)} G^{(3)} - \sum\limits_{1 \leqslant \alpha < \beta \leqslant 3} p_\alpha p_\beta G^{(\alpha)} G^{(\beta)}},$$

$$G_{(1,0),(2,0)}(T; z) = \frac{p_1 G^{(1)} G^{(2)}(1 + p_3 G^{(3)})}{1 - 2p_1 p_2 p_3 G^{(1)} G^{(2)} G^{(3)} - \sum\limits_{1 \leqslant \alpha < \beta \leqslant 3} p_\alpha p_\beta G^{(\alpha)} G^{(\beta)}},$$

where the variable z is omitted on the right-hand sides. The expression for a general $G_{(\alpha,0),(\beta,0)}(T; z)$ is obtained by permutating the indices.

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