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GELFAND–RAIKOV REPRESENTATIONS OF COXETER GROUPS ASSOCIATED WITH POSITIVE DEFINITE NORM FUNCTIONS

BY

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Abstract. The main purpose of the paper is to study the type of Gelfand–Raikov representations of Coxeter groups (W, S) for the special positive definite functions coming from the deformed Poisson (Haagerup) positive definite functions $q^{L(w)}$ for some special length (norm) functions L on Coxeter groups W.

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1. INTRODUCTION

In recent investigations of non-commutative, free probability and type B-free probability, important roles are taken by the representations and the geometry of Cayley graphs of the symmetry groups $\mathfrak{S}_n$, type B-symmetry groups and, in general, by Coxeter groups (W, S) – see the papers: [3]–[6], [8]–[10], [12], [13], and [16].

In the last papers connected with the free infinite divisibility, the first main step was done looking for the length function on $\mathfrak{S}_n$, a so-called *block length function*: for w in $\mathfrak{S}_n$ and $s_k = (k, k+1)$, $\|w\| :=$ cardinality of different Coxeter generators s_{i_j} in an irreducible decomposition $w = s_{i_1} s_{i_2} \ldots s_{i_k}$; see [1], [2], and [5]–[7].

The norm functions: $\|w\|_s$ and $\|w\| := \sum_{s \in S} \|w\|_s$, are the main objects of our paper. We will study the positive definite functions like $f_{q,s}(w) = q^{\|w\|_s}$, $f_q(w) = q^{\|w\|}$ and related positive definite functions $f_{Q,F}$ and f_Q (for the definition, cf. (2.3) below) of the Coxeter group (W, S). We are interested in the following:

PROBLEM 1. Are Gelfand–Raikov representations associated with the non-central positive definite functions $f_{q,s}, f_q, f_{Q,F}$, and f_Q irreducible or factorial? What are their structures?

PROBLEM 2. What does it mean the infinite divisibility of f for Gelfand–Raikov representation π_f for $f = f_{q,s}$?

To answer these questions we will work with the Gelfand–Raikov (GNS) construction of representations coming from the given positive definite, normalized functions on W and the structure of those representations like: inducing representations, cyclic representations, factoriality for special situations (see Sections 3–6 below). The main results for Coxeter groups are given in Section 7 and the next sections. One of the more interesting results is Theorem 8.1, saying that for finite Coxeter groups, the Gelfand–Raikov representation related to our length function, the function $f_q = \prod_{s\in S} f_{q,s}$, is equivalent to the left regular representation of the group W.

Also some structure results are presented in Theorem 10.1 in the case of the affine Weyl (Coxeter) group, and in Theorem 11.1 for compact hyperbolic Coxeter groups. Our representations in typical cases are: either the left regular ones $\mathcal{L}_W$ or direct sums of trivial representations $\mathbf{1}_W$ and those induced from certain parabolic subgroups (containing $\mathcal{L}_W$).

2. THE s-SEMINORM $\|w\|_s$ ON A COXETER GROUP AND PROBLEM SETTING

DEFINITION 2.1. A *Coxeter group* (W, S) is defined by a set S, $|S| \leqslant \infty$, of generators and a set of fundamental relations of the form

$$(ss')^{m(s,s')} = e \quad (s, s' \in S), \tag{2.1}$$
$$m(s,s') = m(s',s) \in \{\infty, 1, 2, \ldots\}, \quad m(s,s') = 1 \text{ if and only if } s = s',$$

where e denotes the identity element of W. The order $|S|$ is called a *rank* of W. A subgroup $\langle J\rangle$ generated by a subset J of S is called a *parabolic* subgroup of W.

Our fundamental reference for Coxeter groups is the book of Humphreys [21].

Prepare an undirected graph Γ with S as vertex set, joining vertices s and s' by an edge whenever $m(s,s') \geqslant 3$, labelled $m(s,s')$ if $m(s,s') \geqslant 4$. We call Γ a *Coxeter graph* of (W,S). A Coxeter group is called *irreducible* if its Coxeter graph is connected. In the case where S is finite, as a geometric representation of (W,S), we prepare a vector space V over $\boldsymbol{R}$ with basis $\{\alpha_s; s \in S\}$, and a symmetric bilinear form B on V given by

$$B(\alpha_s, \alpha_{s'}) := -\cos\frac{\pi}{m(s,s')}. \tag{2.2}$$

Then $B(\alpha_s,\alpha_s) = 1$, $B(\alpha_s,\alpha_{s'}) \leqslant 0$ $(s \neq s')$. For each $s \in S$, define a reflection σ_s on V by $\sigma_s(\lambda) := \lambda - 2B(\alpha_s,\lambda)\alpha_s$ $(\lambda \in V)$. Then the order of $\sigma_s\sigma_{s'}$ is exactly $m(s,s')$ for $s, s' \in S$. This geometric representation σ is not necessarily faithful. When B is positive definite (resp. positive semidefinite), we call Γ *positive definite* (resp. *positive semidefinite*). For an irreducible (W,S), its graph Γ is positive definite if and only if W is finite, and Γ is positive semidefinite (but not positive definite) if and only if W is an affine Weyl group. All finite irreducible Coxeter groups are given in [21], Sections 2.4 and 6.4.

DEFINITION 2.2. Let (W,S) be a Coxeter group. For $w \in W$, let $|w|$ be the length of w with respect to S, that is, the length of a reduced expression of w by means of elements in S. For a fixed $s \in S$, a length function $\|w\|_s$ $(w \in W)$, called an *s-seminorm*, is defined as $\|w\|_s = 1$ or 0 according as a reduced expression of w contains s or not. Put $\|w\| := \sum_{s\in S}\|w\|_s$ $(w \in W)$.

For $0 \leqslant r \leqslant 1$, the function $f_r^0(w) := r^{|w|}$ $(w \in W)$ is positive definite on W (cf. [5] and [10]). For $Q := (q_s)_{s\in S}$, $0 \leqslant q_s \leqslant 1$, and $F \subset S$, put for $w \in W$

$$f_{Q,F}(w) := \prod_{s\in F} f_{q_s,s}(w) \quad \text{with } f_{q,s}(w) := q^{\|w\|_s},$$
$$f_Q(w) := f_{Q,S}(w) = \prod_{s\in S} f_{q_s,s}(w). \tag{2.3}$$

Then $f_{Q,F}$ and f_Q are positive definite functions on W (cf. [12]), and called *norm functions*. The case of the functions f_r^0, especially for $W = \mathfrak{S}_\infty := \lim_{n\to\infty}\mathfrak{S}_n = \bigcup_{2\leqslant n<\infty}\mathfrak{S}_n$, has been treated in [17], together with the case of $f_q := f_{Q,S}$ with $q_s = q$ (for all $s \in S$), in connection with asymptotic theory of characters or invariant positive definite functions (cf., e.g., [18]–[20] and [22]).

Now we are concerned with functions defined in (2.3).

Note that, for $q_1,\ q_2$ such that $0 < q_i \leqslant 1$, $q_1q_2 = q$, we have $f_{q,s}(w) = f_{q_1,s}(w)\cdot f_{q_2,s}(w)$. In particular, if $0 < q < 1$, then $f_{q,s} = (f_{q^{1/m},s})^m$ for any $m \geqslant 2$, and so it is *infinitely divisible*.

We are interested in Problems 1 and 2 stated in the Introduction.

3. GELFAND–RAIKOV REPRESENTATIONS, POSITIVE DEFINITE FUNCTIONS

3.1. Gelfand–Raikov representation. Let G be a discrete group, $\mathcal{P}(G)$ the set of all positive definite functions on G, and $\mathcal{P}_1(G)$ the subset consisting of $f \in \mathcal{P}(G)$ normalized as $f(e) = 1$ at the unit element $e \in G$. Let $\mathfrak{F}(G)$ be the space of all functions on G which vanish outside of a finite number of elements, and consider it as a G-module by left translations of $g \in G$. A function F on G defines a linear functional on $\mathfrak{F}(G)$ by $F(\psi) := \sum_{g\in G} F(g)\psi(g)$ $(\psi \in \mathfrak{F}(G))$. In particular, a positive definite function f gives a positive semidefinite inner product on $\mathfrak{F}(G)$ by $\langle\varphi,\psi\rangle'_f := f(\varphi * \psi^*)$, where $(\varphi*\psi)(g) := \sum_{h\in G}\varphi(h^{-1}g)\psi(h)$, $\psi^*(g) = \overline{\psi(g^{-1})}$. Then this gives a G-invariant, non-negative definite inner product in $\mathfrak{F}(G)$, and a seminorm $\|\varphi\|_f := \sqrt{\langle\varphi,\varphi\rangle'_f}$. Let J_f be the kernel of this inner product. Then we get on the quotient space $\mathfrak{F}(G)/J_f$ a positive definite inner product $\langle\cdot,\cdot\rangle_f$, and then completing it, we obtain a Hilbert space $\mathfrak{H}_f$. For $\varphi \in \mathfrak{F}(G)$, its image in $\mathfrak{H}_f$ is denoted by φ^f; then $\|\varphi^f\|_f = \|\varphi\|_f$.

On $\mathfrak{H}_f$, we get a unitary representation π_f of G with a cyclic vector $\boldsymbol{v}_f := \delta_e^f$ the image of the delta function $\delta_e \in \mathfrak{F}(G)$ and such that

$$f(g) = \langle\pi_f(g)\boldsymbol{v}_f,\boldsymbol{v}_f\rangle_f, \quad \langle\pi_f(g)\boldsymbol{v}_f,\pi_f(h)\boldsymbol{v}_f\rangle_f = f(h^{-1}g). \tag{3.1}$$

We call π_f a *Gelfand–Raikov representation* (*GR representation* in short) associated with f, since Gelfand and Raikov [15] gave this construction in 1943, and thus proved for the first time that any locally compact group has sufficiently many unitary representations.

3.2. Cyclic vectors corresponding to a positive definite function. In this paper, a representation of a discrete group G is usually assumed to be *unitary*. A cyclic representation π with a specified cyclic vector $\boldsymbol{v}$ is denoted by $(\pi, \boldsymbol{v})$, similarly to $(\pi_f, \boldsymbol{v}_f)$. This means that we have also specified the positive definite function $f(g) = \langle \pi(g)\boldsymbol{v}, \boldsymbol{v}\rangle$ associated with $\boldsymbol{v}$. When π_i $(i = 1, 2)$ are both cyclic, and unit vectors $\boldsymbol{v}_i \in V(\pi_i)$ $(i = 1, 2)$ define the same positive definite function $\langle \pi_1(g)\boldsymbol{v}_1, \boldsymbol{v}_1\rangle_{V(\pi_1)} = \langle \pi_2(g)\boldsymbol{v}_2, \boldsymbol{v}_2\rangle_{V(\pi_2)}$, then we write $(\pi_1, \boldsymbol{v}_1) \cong (\pi_2, \boldsymbol{v}_2)$.

LEMMA 3.1. *Let π_i $(i = 1, 2)$ be a cyclic representation of G with cyclic vectors $\boldsymbol{v}_i$, respectively. Assume that there exists a G-module homomorphism $\Phi : V(\pi_1) \to V(\pi_2)$ such that $\Phi(\boldsymbol{v}_1) = \boldsymbol{v}_2$ and that the associated positive definite functions f_i $(i = 1, 2)$, defined by $f_i(g) := \langle \pi_i(g)\boldsymbol{v}_i, \boldsymbol{v}_i\rangle_{V(\pi_i)}$ $(i = 1, 2; g \in G)$, coincide with each other. Then Φ is necessarily unitary.*

LEMMA 3.2. *Let π be a cyclic representation of G. For two unit cyclic vectors $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$ for π, the corresponding positive definite functions are the same as $f(g) = \langle \pi(g)\boldsymbol{v}_a, \boldsymbol{v}_a\rangle = \langle \pi(g)\boldsymbol{v}_b, \boldsymbol{v}_b\rangle$ if and only if there exists a unitary intertwining operator U of π such that $U\boldsymbol{v}_a = \boldsymbol{v}_b$.*

3.3. Partial orders for positive definite functions and subrepresentations. For $f_1, f_2 \in \mathcal{P}(G)$, we define a partial order $f_1 \lessapprox f_2$ if $\langle \varphi, \varphi\rangle'_{f_1} \leqslant \langle \varphi, \varphi\rangle'_{f_2}$ for any $\varphi = \sum_{1\leqslant i\leqslant n} c_i\delta_{g_i} \in \mathfrak{F}(G)$ with $c_i \in \boldsymbol{C}$, or

$$\sum_{1\leqslant i,j\leqslant n} c_i\overline{c_j} f_1(g_j^{-1}g_i) \leqslant \sum_{1\leqslant i,j\leqslant n} c_i\overline{c_j} f_2(g_j^{-1}g_i).$$

Note that $f_1 \lessapprox f_2$ is equivalent to that $f_2 - f_1$ is again positive definite. We also define $f_1 \preccurlyeq f_2$ if $f_1 \lessapprox af_2$ for some $a > 0$.

LEMMA 3.3. *Suppose that $f_1 \lessapprox f_2$ for $f_1, f_2 \in \mathcal{P}(G)$. Then there exists a natural G-module homomorphism P_{f_1,f_2} from π_{f_2} onto π_{f_1} such that $\varphi^{f_2} \to \varphi^{f_1}$ $(\varphi \in \mathfrak{F}(G))$.*

P r o o f. Since $\langle \varphi, \varphi\rangle'_{f_1} \leqslant \langle \varphi, \varphi\rangle'_{f_2}$ $(\varphi \in \mathfrak{F}(G))$, we have $J_{f_1} \supset J_{f_2}$, and there exists a self-adjoint positive operator A on $\mathfrak{H}_{f_2}$ such that $\langle A\varphi^{f_2}, \varphi^{f_2}\rangle_{f_2} = \langle \varphi, \varphi\rangle'_{f_1} = \langle \varphi^{f_1}, \varphi^{f_1}\rangle_{f_1}$, and that A commutes with $\pi_{f_2}(g)$ $(g \in G)$. Take $B = \sqrt{A}$; then it also commutes with $\pi_{f_2}(g)$ and $\langle B\varphi^{f_2}, B\varphi^{f_2}\rangle_{f_2} = \langle \varphi^{f_1}, \varphi^{f_1}\rangle_{f_1}$. From this we see that the image $B(\mathfrak{H}_{f_2})$ is isomorphic to $\mathfrak{H}_{f_1}$. Denote by Q this natural isomorphism; then $P_{f_1,f_2} := QB$ satisfies $P_{f_1,f_2}(\varphi^{f_2}) = \varphi^{f_1}$ $(\varphi \in \mathfrak{F}(G))$, and gives a surjective map from $\mathfrak{H}_{f_2}$ onto $\mathfrak{H}_{f_1}$. It intertwines π_{f_2} with π_{f_1}. ■

COROLLARY 3.1. *Suppose $f_1 \preccurlyeq f_2$ for $f_1, f_2 \in \mathcal{P}(G)$. Then there exists a natural G-homomorphic map P_{f_1,f_2} from $\mathfrak{H}_{f_2}$ onto $\mathfrak{H}_{f_1}$ defined by $P_{f_1,f_2}(\varphi^{f_2}) = \varphi^{f_1}$, which intertwines π_{f_2} with π_{f_1}. Furthermore, suppose $f_1 \preccurlyeq f_2$ and $f_2 \preccurlyeq f_1$ at the same time. Then P_{f_1,f_2} gives a natural G-module isomorphism between π_{f_2} and π_{f_1}.*

4. THE CASE OF s-SEMINORM AND POSITIVE DEFINITE FUNCTION $f_{q_s,s}$

4.1. Decomposition of positive definite functions $f_{q_s,s}$. Denote by W_s the subgroup of W consisting of elements w with $\|w\|_s = 0$. Then it is also a Coxeter group with the set of generators $S_s := S \setminus \{s\}$, and for $0 \leqslant q_s \leqslant 1$,

$$f_{q_s,s}(w) = q_s^{\|w\|_s} = \begin{cases} 1 & \text{if } w \in W_s, \\ q_s & \text{if } w \in W \setminus W_s, \end{cases} \tag{4.1}$$

$$f_{q_s,s} = q_s 1_W + (1 - q_s) X_{W_s}, \tag{4.2}$$

where 1_W denotes the constant function equal to 1 on W, and X_{W_s} is the trivial character of W_s extended as 0 on $W \setminus W_s$. These formulas are valid for $q_s = 0$ with $0^0 = 1$.

Note that the formula (4.2) proves that $f_{q_s,s}$ is positive definite on W, since the function X_{W_s} is a diagonal matrix element of the induced representation $\Pi_s := \mathrm{Ind}_{W_s}^{W} \mathbf{1}_{W_s}$ of the trivial representation $\mathbf{1}_{W_s}$ of W_s, as seen in Lemma 4.1 below.

4.2. Induced representations and GR representations. In general, let G be a discrete group and H its subgroup, and denote by $\mu_{G/H}$ a G-invariant measure on G/H. For a unitary representation ρ_H of H, its induced representation $\Pi := \mathrm{Ind}_H^G \rho_H$ is defined as follows. Let $\mathcal{V}'$ be a space of functions φ on G with values in the space $V(\rho_H)$ of ρ_H satisfying

$$\varphi(gh) = \rho_H(h)^{-1}\big(\varphi(g)\big) \quad (h \in H, g \in G), \tag{I-1}$$

$$\|\varphi\|^2 = \int_{G/H} |\varphi(g)|^2 d\mu_{G/H}(gH) < \infty. \tag{I-2}$$

Dividing $\mathcal{V}'$ by the kernel of the inner product and then completing, we get a Hilbert space $\mathcal{V}$. The operator $\Pi(g_0)$ is defined by $\Pi(g_0)\varphi(g) := \varphi(g_0^{-1}g)$ $(g \in G)$.

As positive definite functions on W, 1_W is the character of the trivial representation $\mathbf{1}_W$, and X_{W_s} is a matrix element of Π_s.

LEMMA 4.1. *The function X_{W_s} on W is a diagonal matrix element of $\Pi_s = \mathrm{Ind}_{W_s}^{W} \mathbf{1}_{W_s}$, corresponding to a cyclic vector X_{W_s} in $V(\Pi_s)$.*

Let us note that the GR representation π_f associated with f is characterized, modulo equivalence, as a cyclic representation containing a unit vector $\boldsymbol{v}_f$ such that $\langle \pi_f(g)\boldsymbol{v}_f, \boldsymbol{v}_f\rangle = f(g)$.

Consider the direct sum $\pi_s := \mathbf{1}_W \oplus \Pi_s$ of two representations, and take a unit vector $\boldsymbol{v}_{1,s} \in V(\pi_s)$ given by

$$\boldsymbol{v}_{1,s} = \sqrt{q_s} \cdot 1_W \oplus \sqrt{1-q_s} \cdot X_{W_s}. \tag{4.3}$$

LEMMA 4.2. *The cyclic subrepresentation π'_s of $\pi_s = \mathbf{1}_W \oplus \Pi_s$ on the subspace $\langle \pi_s(W)\boldsymbol{v}_{1,s}\rangle$ generated by $\boldsymbol{v}_{1,s}$ is unitarily equivalent to the GR representation $\pi_{f_{q_s,s}}$. A W-isomorphism from π'_s to $\pi_{f_{q_s,s}}$, which maps the unit cyclic vector $\boldsymbol{v}_{1,s} \in V(\pi'_s)$ to the unit cyclic vector $\boldsymbol{v}_{f_{q_s,s}} \in V(\pi_{f_{q_s,s}})$, is unique.*

5. GR REPRESENTATIONS $\pi_{f_{Q,F}}$ AND INDUCED ONES $\mathrm{Ind}_{W_{F'}}^W \mathbf{1}_{W_{F'}}$

5.1. Isomorphism of $\pi_{f_{Q,F}}$ into the direct sum $\bigoplus_{F'\subset F} \mathrm{Ind}_{W_{F'}}^W \mathbf{1}_{W_{F'}}$. For a subset F of S, put $W_F := \langle S \setminus F\rangle$ the subgroup generated by $S \setminus F$. From a property of Coxeter groups we have the following.

LEMMA 5.1. *For two subsets F_1, F_2 of S, we have $W_{F_1} \cap W_{F_2} = W_{F_1\cup F_2}$. For a subset B of W, denote by X_B the indicator function of B. Then $X_{W_{F_1}} \cdot X_{W_{F_2}} = X_{W_{F_1\cup F_2}}$.*

Let $Q = (q_s)_{s\in S}, 0 \leqslant q_s \leqslant 1$ $(s \in S)$. For a subset $F \subset S$, we have a product formula as $f_Q = f_{Q,F} \cdot f_{Q,S\setminus F}$. If F is finite, applying Lemma 5.1, we have the following expression of the positive definite function $f_{Q,F}$:

$$\begin{gathered} f_{Q,F} = \prod_{s\in F} \big(q_s 1_W + (1-q_s)X_{W_s}\big) = \sum_{F'\subset F} c_{Q;F,F'} X_{W_{F'}}, \\ c_{Q;F,F'} := \prod_{s\in F\setminus F'} q_s \prod_{t\in F'} (1-q_t). \end{gathered} \tag{5.1}$$

For a finite subset $F \subset S$, consider the direct sum $\pi_{(F)}$ of quasi-regular representations $\Pi_{F'} := \mathrm{Ind}_{W_{F'}}^W \mathbf{1}_{W_{F'}}$, induced from subgroups $W_{F'} = \langle S \setminus F'\rangle$ with $F' \subset F$, as

$$\pi_{(F)} := \bigoplus_{F'\subset F} \Pi_{F'}, \quad V(\pi_{(F)}) := \bigoplus_{F'\subset F} V(\Pi_{F'}), \tag{5.2}$$

and take a unit vector $\boldsymbol{w}_{Q,F}$ of $V(\pi_{(F)})$ depending on Q as

$$\boldsymbol{w}_{Q,F} := \sum_{F'\subset F}^{\oplus} d_{Q;F,F'} X_{W_{F'}}. \tag{5.3}$$

Here, $\Pi_\emptyset = \mathbf{1}_W$ for $F' = \emptyset$, since $W_\emptyset = \langle S\rangle = W$. The diagonal matrix element of $\pi_{(F)}$ with respect to $\boldsymbol{w}_{Q,F}$ is

$$\langle \pi_{(F)}(g)\boldsymbol{w}_{Q,F}, \boldsymbol{w}_{Q,F}\rangle_{V(\pi_{(F)})} = \sum_{F'\subset F} (d_{Q;F,F'})^2 X_{W_{F'}}(g) \quad (g \in W).$$

PROPOSITION 5.1. *Let F be a finite subset of S. Take the direct sum $\pi_{(F)} = \bigoplus_{F' \subset F} \Pi_{F'}, \Pi_{F'} = \mathrm{Ind}_{W_{F'}}^{W} \mathbf{1}_{W_{F'}}$, and a unit vector $\boldsymbol{w}_{Q,F} \in V(\pi_{(F)})$ given by*

$$\text{(5.4)} \qquad \boldsymbol{w}_{Q,F} = \sum_{F' \subset F}^{\oplus} d_{Q;F,F'} X_{W_{F'}}, \quad d_{Q;F,F'} = \sqrt{c_{Q;F,F'}} \quad (F' \subset F).$$

Then the cyclic subrepresentation $\pi'_{Q,F}$ of $\pi_{(F)}$ spanned by $\boldsymbol{w}_{Q,F}$ is associated with the positive definite function $f_{Q,F}$. Or, for $\pi'_{Q,F} := \pi_{(F)}|_{\langle \pi_{(F)}(G)\boldsymbol{w}_{Q,F}\rangle}$,

$$\text{(5.5)} \qquad (\pi_{f_{Q,F}}, \boldsymbol{v}_{f_{Q,F}}) \cong (\pi'_{Q,F}, \boldsymbol{w}_{Q,F}).$$

5.2. Orders among positive definite functions on W.

LEMMA 5.2. (i) *For a subset $F \subset S$, assume that $W_F = \langle S \setminus F \rangle$ is finite. Then the positive definite function X_{W_F} on W is dominated by $X_{W_S} = X_{\{e\}} = \delta_e$ as $X_{W_F} \lessapprox |W_F| \cdot \delta_e$.*

(ii) *For two subsets $F_1 \subset F_2 \subset S$, assume that $[W_{F_1} : W_{F_2}] < \infty$. Then $X_{W_{F_1}} \preccurlyeq X_{W_{F_2}}$ or, more exactly, $X_{W_{F_1}} \lessapprox |W_{F_1}/W_{F_2}| \cdot X_{W_{F_2}}$.*

(iii) *For $F_1 \subset F_2 \subset S$, let us assume that $F_2 \setminus F_1$ is finite. If $q_s \neq 0$ for $s \in F_2 \setminus F_1$, then $f_{Q,F_1} \preccurlyeq f_{Q,F_2}$. More generally, let $F_{21}^0 := \{s \in F_2 \setminus F_1; q_s = 0\}$. Then $f_{Q,F_2} = f_{Q,F_2 \setminus F_{21}^0}$ and $f_{Q,F_1} \cdot X_{W_{F_2 \setminus (F_1 \cup F_{21}^0)}} \preccurlyeq f_{Q,F_2}$.*

Proof. (i) For $\varphi \in \mathfrak{F}(W)$,

$$\langle \varphi, \varphi \rangle_{X_{W_F}} = \sum_{g,h \in W} \varphi(g) X_{W_F}(h^{-1}g)\overline{\varphi(h)} \leqslant |W_F| \sum_{g \in W} |\varphi(g)|^2 = |W_F| \langle \varphi, \varphi \rangle_{\delta_e}.$$

(ii) Similarly to (i).

(iii) This comes from the following:

$$f_{Q,F_2} = f_{Q,F_1} \prod_{s \in F_2 \setminus F_1} \big(q_s 1_W + (1-q_s) X_{W_s}\big) = f_{Q,F_1} \sum_{F' \subset F_2 \setminus F_1} c_{Q;F_2 \setminus F_1, F'} X_{W_{F'}}.$$

Thus the proof is complete. ■

LEMMA 5.3. *Assume W is finite. Then any positive definite function f on W is dominated by $f_0 := \delta_e$. More exactly,*

$$\text{(5.6)} \qquad f \lessapprox f(e)|W| \cdot f_0, \quad \|\varphi\|_f \leqslant \sqrt{f(e)|W|} \cdot \|\varphi\|_{\ell^2(W)} \ \ (\varphi \in \mathfrak{F}(W)).$$

5.3. Isomorphism of $\pi_{f_{Q,F}}$ with $\mathrm{Ind}_{W_F}^{W} \mathbf{1}_{W_F}$ for an F finite.

THEOREM 5.1. *Let F be a finite subset of S. Assume that $0 \leqslant q_s < 1$ $(s \in F)$ for $Q = (q_s)_{s \in S}$ and that the subgroup $\langle F \rangle \subset W$ generated by F is finite.*

(i) *The GR representation* $(\pi_{f_{Q,F}}, \boldsymbol{v}_{f_{Q,F}})$ *is isomorphic to* $(\Pi_F, \boldsymbol{u}_{Q,F})$, *where* $\Pi_F = \mathrm{Ind}_{W_F}^W \mathbf{1}_{W_F}$ *and* $\boldsymbol{u}_{Q,F}$ *is a cyclic vector which corresponds to* $\boldsymbol{v}_{f_{Q,F}}$ *under a surjective* W*-module isomorphism* Ψ_F *from* $V(\pi_{f_{Q,F}})$ *onto* $V(\mathrm{Ind}_{W_F}^W \mathbf{1}_{W_F})$.

(ii) *Let us consider* $\langle F\rangle$*-cyclic subspaces spanned by* $\pi_{f_{Q,F}}(\langle F\rangle)\boldsymbol{v}_{f_{Q,F}}$ *and* $\Pi_F(\langle F\rangle)\boldsymbol{u}_{Q,F}$ *respectively. Then, as cyclic representations of* $\langle F\rangle$,

$$(\pi_{f_{Q,F}}|_{\langle F\rangle}, \boldsymbol{v}_{f_{Q,F}}) \cong (\Pi_F|_{\langle F\rangle}, \boldsymbol{u}_{Q,F}) \cong (\mathcal{L}_{\langle F\rangle}, \boldsymbol{u}'_{Q,F}). \tag{5.7}$$

Here the second "$\cong$*" means that* $\Pi_F|_{\langle F\rangle}$ *on* $\big\langle \Pi_F(\langle F\rangle)\boldsymbol{u}_{Q,F}\big\rangle$, *with the cyclic vector* $\boldsymbol{u}_{Q,F}$, *is equivalent to the left regular representation* $\mathcal{L}_{\langle F\rangle} := \mathrm{Ind}_{\{e\}}^{\langle F\rangle} \mathbf{1}_{\{e\}}$ *on* $\ell^2(\langle F\rangle)$, *with a cyclic vector* $\boldsymbol{u}'_{Q,F} \in \ell^2(\langle F\rangle)$ *corresponding to* $\boldsymbol{u}_{Q,F}$, *such that* $\langle \mathcal{L}_{\langle F\rangle}(g)\boldsymbol{u}'_{Q,F}, \boldsymbol{u}'_{Q,F}\rangle = f_{Q,F}(g)$ $(g \in \langle F\rangle)$.

P r o o f. (i) By Proposition 5.1, we have $(\pi_{f_{Q,F}}, \boldsymbol{v}_{f_{Q,F}}) \cong (\pi'_{Q,F}, \boldsymbol{w}_{Q,F})$, where $\pi'_{Q,F}$ is the cyclic subrepresentation of $\pi_{(F)} = \bigoplus_{F'\subset F} \Pi_{F'}$ generated by $\boldsymbol{w}_{Q,F} = \sum^{\oplus}_{F'\subset F} d_{Q;F,F'} \cdot X_{W_{F'}}$. Note that, under the assumption on (Q, F), we have $d_{Q;F,F} = \big(\prod_{s\in F}(1-q_s)\big)^{1/2} \neq 0$, and so each component $d_{Q;F,F'}X_{W_{F'}}$ of $\boldsymbol{w}_{Q,F}$ is dominated by the principal component $d_{Q;F,F}X_{W_F}$ by Lemma 5.2 (ii). Hence, by Lemma 6.1 below, the cyclic representation $(\pi'_{Q,F}, \boldsymbol{w}_{Q,F})$ is isomorphic to $(\Pi_F, \boldsymbol{u}_{Q,F})$ with a certain cyclic vector $\boldsymbol{u}_{Q,F} \in V(\Pi_F)$ corresponding to $\boldsymbol{w}_{Q,F}$.

(ii) Denote by $f_{\langle F\rangle}$ the restriction $f_{Q,F}|_{\langle F\rangle}$. We apply (i) replacing W by $\langle F\rangle$ and $f_{Q,F}$ by $f_{\langle F\rangle}$. Then we see that $(\pi_{f_{\langle F\rangle}}, \boldsymbol{v}_{f_{\langle F\rangle}}) \cong (\mathcal{L}_{\langle F\rangle}, \boldsymbol{u}'_{Q,F})$. Moreover, the former is naturally isomorphic to $(\pi_{f_{Q,F}}\big|_{\langle F\rangle}, \boldsymbol{v}_{f_{Q,F}})$. ■

6. CYCLIC SUBREPRESENTATIONS AND GR REPRESENTATIONS

6.1. Cyclic subrepresentation of a finite direct sum of representations. Let G be a discrete group, and π_i $(1 \leqslant i \leqslant N)$ be unitary representations of G. Let $\pi = \pi_1 \oplus \ldots \oplus \pi_N$ be their direct sum, and take a non-zero element

$$\boldsymbol{v} := \bigoplus_{1\leqslant i\leqslant N} \boldsymbol{v}_i, \qquad \boldsymbol{v}_i \in V(\pi_i)\ (1 \leqslant i \leqslant N). \tag{6.1}$$

LEMMA 6.1. *Let* $f_i(g) = \langle \pi_i(g)\boldsymbol{v}_i, \boldsymbol{v}_i\rangle_{V(\pi_i)}$ *and assume that* $f_i \preccurlyeq f_N$ *for any* $i < N$. *Then the cyclic subrepresentation* $\pi_{\boldsymbol{v}}$ *of* π *generated by* $\boldsymbol{v}$ *is equivalent to* $\pi_{N,\boldsymbol{v}_N}$: $\pi_{\boldsymbol{v}} \cong \pi_{N,\boldsymbol{v}_N}$, *where* $\pi_{N,\boldsymbol{v}_N}$ *denotes the cyclic subrepresentation of* π_N *generated by* $\boldsymbol{v}_N$. *Put* $f := f_1 + f_2 + \ldots + f_N$. *Then there exists a cyclic vector* $\boldsymbol{w} \in V(\pi_{N,\boldsymbol{v}_N})$ *such that*

$$f(g) = \langle \pi_{N,\boldsymbol{v}_N}(g)\boldsymbol{w}, \boldsymbol{w}\rangle, \qquad (\pi_{\boldsymbol{v}}, \boldsymbol{v}) \cong (\pi_{N,\boldsymbol{v}_N}, \boldsymbol{w}). \tag{6.2}$$

P r o o f. The positive definite function associated with the cyclic representation $\pi_{\boldsymbol{v}}$ with the cyclic vector $\boldsymbol{v}$ is $f = f_1 + \ldots + f_N$, and we have $f \preccurlyeq f_N$ by assumption. On the other hand, $f_N \lessapprox f$. Hence $f_N \preccurlyeq f \preccurlyeq f_N$, and this implies that GR representations π_f and π_{f_N} associated with f and f_N, respectively, are mutually equivalent.

Moreover, $(\pi_{\boldsymbol{v}}, \boldsymbol{v}) \cong (\pi_f, \boldsymbol{v}_f)$ and $(\pi_{N,\boldsymbol{v}_N}, \boldsymbol{v}_N) \cong (\pi_{f_N}, \boldsymbol{v}_{f_N})$. Hence $\pi_{\boldsymbol{v}}$ and $\pi_{N,\boldsymbol{v}_N}$ are mutually equivalent. To get a unitary intertwining operator from $\pi_{\boldsymbol{v}}$ to π_{f_N}, we need to find a cyclic vector $\boldsymbol{w} \in V(\pi_{f_N})$, as asserted in the lemma. ∎

LEMMA 6.2. *Let* $\pi = \pi_1 \oplus \ldots \oplus \pi_N$ *and* $\boldsymbol{v} = \boldsymbol{v}_1 \oplus \ldots \oplus \boldsymbol{v}_N$ $(\boldsymbol{v}_i \in V(\pi_i))$. *Suppose that, for any pair* $\{i, j\}, i \neq j$, $\mathrm{Hom}_G(\pi_i, \pi_j) = \{0\}$, *or there exists no intertwining operator except* 0. *Then the cyclic subrepresentation* $\pi_{\boldsymbol{v}}$ *of* π *generated by* $\boldsymbol{v}$ *is the direct sum of* $\pi_{i,\boldsymbol{v}_i}$ $(1 \leqslant i \leqslant N)$ *as* $\pi_{\boldsymbol{v}} = \sum^{\oplus}_{1 \leqslant i \leqslant N} \pi_{i,\boldsymbol{v}_i}$, *where* $\pi_{i,\boldsymbol{v}_i}$ *denotes the cyclic subrepresentation of* π_i *generated by* $\boldsymbol{v}_i$ *for* $1 \leqslant i \leqslant N$, *and in the case* $\boldsymbol{v}_i = \mathbf{0}$, $\pi_{i,\boldsymbol{v}_i} = \emptyset$ *by definition.*

P r o o f. We may and do assume that $\pi_{i,\boldsymbol{v}_i} = \pi_i$ or $\boldsymbol{v}_i$ generates cyclically the whole π_i for any i. Let P_i be the orthogonal projection of $V(\pi) = \bigoplus_{1 \leqslant i \leqslant N} V(\pi_i)$ onto $V(\pi_i)$, and R be the orthogonal projection onto $V(\pi_{\boldsymbol{v}}) = \langle \pi(G)\boldsymbol{v} \rangle$. Then $R_{ij} := P_j R P_i$ is essentially a G-module homomorphism from $V(\pi_i)$ to $V(\pi_j)$. Therefore, by assumption, $R_{ij} = 0$ for $i \neq j$, and $R = \sum_{1 \leqslant i \leqslant N} R_{ii}$. This means that $V(\pi_{\boldsymbol{v}})$ is the direct sum of $V(\pi_{i,\boldsymbol{v}_i})$ $(1 \leqslant i \leqslant N)$. ∎

In general, let π_i $(i = 1, 2)$ be cyclic unitary representations of G with specified cyclic vectors $\boldsymbol{v}_i$ respectively. We study the structure of the *cyclic part* π'_{12} of $\pi = \pi_1 \oplus \pi_2$ generated by $\boldsymbol{v}'_{12} := \boldsymbol{v}_1 \oplus \boldsymbol{v}_2 \in V(\pi)$. Its representation space $V(\pi'_{12})$ is spanned by $\pi(g)\boldsymbol{v}'_{12} = \pi_1(g)\boldsymbol{v}_1 \oplus \pi_2(g)\boldsymbol{v}_2$ $(g \in G)$. Put

$$\text{(6.3)} \qquad V_i^{(d)} := V(\pi_i) \cap V(\pi'_{12}), \qquad V(\pi_i) = V_i^{(d)} \oplus V_i^{(c)} \quad (i = 1, 2),$$

and $\pi_i^{(d)} := \pi|_{V_i^{(d)}}$ and $\pi_i^{(c)} := \pi|_{V_i^{(c)}}$; moreover, let $\boldsymbol{v}_i = \boldsymbol{v}_i^{(d)} \oplus \boldsymbol{v}_i^{(c)} (i = 1, 2)$ be the decomposition of $\boldsymbol{v}_i$ according to (6.3). Then $\pi_i = \pi_i^{(d)} \oplus \pi_i^{(c)}$.

Put $V^{(d)} := V(\pi_1^{(d)}) \oplus V(\pi_2^{(d)}) \subset V(\pi'_{12})$, let $V^{(c)}$ be the orthogonal complement of $V^{(d)}$ in $V(\pi'_{12})$, and define $\pi^{(d)} := \pi|_{V^{(d)}}, \pi^{(c)} := \pi|_{V^{(c)}}$. Then, $V^{(c)} \subset V(\pi_1^{(c)}) \oplus V(\pi_2^{(c)})$ and

$$\text{(6.4)} \qquad V(\pi'_{12}) = V^{(d)} \oplus V^{(c)} \text{ (in } V(\pi)), \quad \pi'_{12} \cong \pi^{(d)} \oplus \pi^{(c)}.$$

From the definition of $V(\pi_i^{(d)})$ it follows that, for any $\boldsymbol{w}_1 \oplus \boldsymbol{w}_2 \in V^{(c)} \subset V(\pi_1^{(c)}) \oplus V(\pi_2^{(c)})$, the correspondence $T : \boldsymbol{w}_1 \to \boldsymbol{w}_2$, from the first component to the second, is bijective. Moreover, we have $\pi_1^{(c)}(g)\boldsymbol{v}_1^{(c)} \oplus \pi_2^{(c)}(g)\boldsymbol{v}_2^{(c)} \in V^{(c)}$, and so T maps bijectively as

$$T : \pi_1^{(c)}(g)\boldsymbol{v}_1^{(c)} \to \pi_2^{(c)}(g)\boldsymbol{v}_2^{(c)} \quad (g \in G).$$

LEMMA 6.3. *The restriction* $\pi^{(c)}$ *of* π'_{12} *or of* π *onto* $V^{(c)}$ *is equivalent to each of the restrictions* $\pi_i^{(c)}$ *of* π_i *onto* $V_i^{(c)}$ *for* $i = 1, 2$. *The equivalences between them are given by the projections* $V^{(c)} \ni \boldsymbol{w}_1 \oplus \boldsymbol{w}_2 \to \boldsymbol{w}_i \in V_i^{(c)}$. *The cyclic part* π'_{12} *of* $\pi = \pi_1 \oplus \pi_2$ *generated by* $\boldsymbol{v}_1 \oplus \boldsymbol{v}_2$ *is given as*

$$\pi'_{12} = (\pi_1^{(d)} \oplus \pi_2^{(d)}) \oplus \pi^{(c)}, \quad \pi^{(c)} \cong \pi_1^{(c)} \cong \pi_2^{(c)}. \tag{6.5}$$

6.2. The cyclic part of $(\mathrm{Ind}_{H_1}^G \mathbf{1}_{H_1}) \otimes (\mathrm{Ind}_{H_2}^G \mathbf{1}_{H_2})$**.** Let G be a discrete group, and H_i be subgroups of G for $i = 1, 2$. Put $\pi_i = \mathrm{Ind}_{H_i}^G \mathbf{1}_{H_i}$ $(i = 1, 2)$, and $\rho := \pi_1 \otimes \pi_2$.

LEMMA 6.4. *Let* ρ' *be the cyclic subrepresentation of* $\rho = \pi_1 \otimes \pi_2$ *generated by a cyclic vector* $\boldsymbol{w}_1 \otimes \boldsymbol{w}_2$ *with* $\boldsymbol{w}_i := X_{H_i} \in V(\pi_i)$ $(i = 1, 2)$. *Then* ρ' *is canonically equivalent to the induced representation* $\pi := \mathrm{Ind}_{H_1 \cap H_2}^G \mathbf{1}_{H_1 \cap H_2}$, *and* $\boldsymbol{w}_1 \otimes \boldsymbol{w}_2$ *corresponds to the cyclic vector* $X_{H_1 \cap H_2} \in V(\pi)$.

P r o o f. Consider $V(\pi_i)$ as the space of functions φ_i on G such that $\varphi_i(gh_i) = \varphi_i(g)$ $(h_i \in H_i, g \in G)$ with the norm $\|\varphi_i\|^2 = \sum_{g \in G/H_i} |\varphi_i(g)|^2$, where $g \in G/H_i$ means that g runs over a complete set of representatives of G/H_i. In $V(\pi_1) \otimes V(\pi_2)$, we take $\boldsymbol{w}_{12} := \boldsymbol{w}_1 \otimes \boldsymbol{w}_2 = X_{H_1} \otimes X_{H_2}$ and consider the subspace V' spanned by

$$\rho(g_0)\boldsymbol{w}_{12} = X_{g_0 H_1} \otimes X_{g_0 H_2} \quad (g_0 \in G), \tag{6.6}$$

where X_B denotes the indicator function of a subset B of G.

On the other hand, consider a bilinear map Φ' from $V(\pi_1) \times V(\pi_2)$, which assigns to (φ_1, φ_2) a function φ on G given by the product as $\varphi(g) := \varphi_1(g)\varphi_2(g)$: $\Phi'(\varphi_1, \varphi_2) = \varphi$. Then it induces uniquely a linear map Φ'' from $V(\pi_1) \otimes V(\pi_2)$ into a space of functions ψ on G such that $\psi(gh) = \psi(g)$, where $g \in G, h \in H_{12} := H_1 \cap H_2$. Denote by Φ the restriction of Φ'' on the subspace V'. Then $\varphi_{12} := \Phi(\boldsymbol{w}_{12})$ is given by

$$\varphi_{12}(g) = X_{H_1}(g) X_{H_2}(g) = X_{H_1 \cap H_2}(g) \quad (g \in G), \tag{6.7}$$

and $\Phi\big(\rho'(g_0)\boldsymbol{w}_{12}\big) = \pi(g_0)\varphi_{12}$, where

$$\pi(g_0)\varphi_{12}(g) := \varphi_{12}(g_0^{-1} g) = X_{H_1 \cap H_2}(g_0^{-1} g) \quad (g \in G). \tag{6.8}$$

From (6.6) we see that the set of vectors $\{\rho(g_0)\boldsymbol{w}_{12}; g_0 \in G/(H_1 \cap H_2)\}$ gives an orthogonal basis of V'. On the other hand, we see from (6.8) that the set of functions $\{\pi(g_0)\varphi_{12}; g_0 \in G/(H_1 \cap H_2)\}$ gives an orthogonal basis of $\Phi(V')$. This means that Φ is a linear isomorphism from V' onto $\Phi(V')$. Moreover, we see from (6.7) and (6.8) that $\Phi(V') = V(\pi)$, and the representation π on $\Phi(V')$ is nothing but $\mathrm{Ind}_{H_1 \cap H_2}^G \mathbf{1}_{H_1 \cap H_2}$. ∎

NOTE 6.1. On $V(\pi_1) \otimes V(\pi_2)$, Φ is isometric inside of the cyclic subspace V' by Lemma 3.1. But it is not necessarily isometric outside of V' even though it is G-homomorphic.

6.3. The cyclic part of tensor product $\pi_{f_1} \otimes \pi_{f_2}$. For $f_i \in \mathcal{P}_1(G)$ $(i = 1, 2)$, we realize the GR representation π_{f_i} as follows: prepare a standard cyclic vector $\boldsymbol{v}_i$ and a symbolic G-span $\mathfrak{V}_i := \{\pi_{f_i}(g)\boldsymbol{v}_i; g \in G\}$ such that

$$\begin{aligned} f_i(g) &= \langle \pi_{f_i}(g)\boldsymbol{v}_i, \boldsymbol{v}_i\rangle \quad (g \in G), \\ \langle \pi_{f_i}(g)\boldsymbol{v}_i, \pi_{f_i}(h)\boldsymbol{v}_i\rangle &:= f_i(h^{-1}g) \quad (g, h \in G). \end{aligned} \tag{6.9}$$

The representation space $V(\pi_{f_i})$ is a completion of the linear span $\langle \mathfrak{V}_i \rangle$ modulo the kernel of the inner product, and the representation operator $\pi_{f_i}(g_0)$ is induced from the left translation by g_0 as $\pi_{f_i}(g_0)\big(\pi_{f_i}(g)\boldsymbol{v}_i\big) := \pi_{f_i}(g_0 g)\boldsymbol{v}_i$.

Put $f := f_1 f_2$. Consider the tensor product $\pi := \pi_{f_1} \otimes \pi_{f_2}$ on $V(\pi_{f_1}) \otimes V(\pi_{f_2})$, and take a unit vector $\boldsymbol{v}_{12} := \boldsymbol{v}_1 \otimes \boldsymbol{v}_2$. The matrix element associated with $\boldsymbol{v}_{12}$ is

$$\langle \pi(g)\boldsymbol{v}_{12}, \boldsymbol{v}_{12}\rangle = \langle \pi_{f_1}(g)\boldsymbol{v}_1, \boldsymbol{v}_1\rangle \cdot \langle \pi_{f_2}(g)\boldsymbol{v}_2, \boldsymbol{v}_2\rangle = f_1(g)f_2(g) = f(g).$$

LEMMA 6.5. *The cyclic part $\langle \pi(G)\boldsymbol{v}_{12}\rangle \subset V(\pi_{f_1}) \otimes V(\pi_{f_2})$ carries GR representation π_f associated with $f = f_1 f_2$, where $\langle \pi(G)\boldsymbol{v}_{12}\rangle$ denotes the closed linear span of $\pi(G)\boldsymbol{v}_{12}$.*

7. THE CASE OF GR REPRESENTATIONS OF COXETER GROUPS

7.1. GR representations and induced representations. Let (W, S) be a Coxeter group. For a subset $S' \subset S$, denote by $\langle S'\rangle$ the subgroup of W generated by S', and for a subset $F \subset S$, put $W_F := \langle S \setminus F\rangle$ as before. Let us put $S_f := \{s \in S; |W_s| < \infty\}$.

DEFINITION 7.1. An $s \in S$ is called *co-finite* (resp. *co-infinite*) if $|W/W_s| < \infty$ (resp. $|W/W_s| = \infty$). A subset $F \subset S$ is said to be of *infinite type* if $|W_F| = |\langle S \setminus F\rangle| = \infty$.

Note that the induced representation $\Pi_s = \mathrm{Ind}_{W_s}^W \mathbf{1}_{W_s}$ contains or not the trivial representation $\mathbf{1}_W$ according as s is co-finite or co-infinite.

Let $Q = (q_s)_{s\in S}$, $0 \leqslant q_s \leqslant 1$ $(s \in S)$. For a subset $F \subset S$, we have, by the definition in (2.3), $f_{Q,F} = \prod_{s\in F} f_{q_s,s}$ and $f_Q = f_{Q,F} f_{Q,S\setminus F}$.

LEMMA 7.1. *Suppose $F \subset S$ is finite. Then, for $Q = (q_s)_{s\in S}$,*

(7.1) $\pi_{f_{Q,F}} \cong$ *the cyclic part of* $\bigotimes_{s\in F} \pi_{f_{q_s,s}}$ *generated by* $\boldsymbol{v}'_F := \bigotimes_{s\in F} \boldsymbol{v}_{f_{q_s,s}}$;

(7.2) $\pi_{f_Q} \cong$ *the cyclic part of* $\big(\bigotimes_{s\in F} \pi_{f_{q_s,s}}\big) \otimes \pi_{f_{Q,S\setminus F}}$ *generated by* $\boldsymbol{v}'_F \otimes \boldsymbol{v}_{f_{Q,S\setminus F}}$.

LEMMA 7.2. *Suppose $F \subset S$ is finite, and let $Q = (q_s)_{s\in S}$.*

(i) *The GR representation $(\pi_{f_Q}, \boldsymbol{v}_{f_Q})$ is isomorphic to a cyclic subrepresentation of*

$$\bigotimes_{s\in F} (\mathbf{1}_W \oplus \Pi_s) \otimes \pi_{f_{Q,S\setminus F}} \tag{7.3}$$

with a cyclic vector $\big(\bigotimes_{s\in F} \boldsymbol{v}_{1,s}\big) \otimes \boldsymbol{v}_{f_{Q,S\setminus F}}$, where $\boldsymbol{v}_{1,s} \in V(\mathbf{1}_W \oplus \Pi_s)$ is given in (4.3).

(ii) *Put $F_f := F \cap S_f$. Then $(\pi_{f_Q}, \boldsymbol{v}_{f_Q})$ is isomorphic to a cyclic subrepresentation of*

$$\bigotimes_{s\in F\setminus F_f} (\mathbf{1}_W \oplus \Pi_s) \otimes (\mathbf{1}_W \oplus \mathrm{Ind}_{W_{F_f}}^W \mathbf{1}_{W_{F_f}}) \otimes \pi_{f_{Q,S\setminus F}}, \tag{7.4}$$

with a cyclic vector $\big(\bigotimes_{s\in F\setminus F_f} \boldsymbol{v}_{1,s}\big) \otimes \boldsymbol{w}_{F_f} \otimes \boldsymbol{v}_{f_{Q,S\setminus F}}$, where $\boldsymbol{w}_{F_f}$ is a certain cyclic vector of $\mathbf{1}_W \oplus \mathrm{Ind}_{W_{F_f}}^W \mathbf{1}_{W_{F_f}}$.

(iii) *In case of W is finite, the middle term $\mathbf{1}_W \oplus \mathrm{Ind}_{W_{F_f}}^W \mathbf{1}_{W_{F_f}}$ in* (7.4) *with cyclic vector $\boldsymbol{w}_{F_f}$ can be replaced by $\mathrm{Ind}_{W_{F_f}}^W \mathbf{1}_{W_{F_f}}$ with a certain cyclic vector $\boldsymbol{w}'_{F_f}$.*

The following is an extended version of Lemma 6.4 in the case of Coxeter groups. For a finite subset $F \subset S$, assume that Π'_F is the cyclic subrepresentation of $\bigotimes_{s\in F} \Pi_s$ generated by the vector $\bigotimes_{s\in F} X_{W_s}$. Consider a multilinear map

$$\Phi' : \prod_{s\in F} V(\Pi_s) \ni (\varphi_s)_{s\in F} \to \prod_{s\in F} \varphi_s =: \varphi,$$

where $\varphi(w) = \prod_{s\in F} \varphi_s(w)$ $(w \in W)$. Then we get a linear map Φ onto a space of functions on W invariant from the right under $W_F = \bigcap_{s\in F} W_s = \langle S \setminus F\rangle$ as

$$\Phi : \bigotimes_{s\in F} V(\Pi_s) \ni \bigotimes_{s\in F} \varphi_s \to \prod_{s\in F} \varphi_s. \tag{7.5}$$

LEMMA 7.3. *Let $F \subset S$ be finite. The linear map Φ gives a W-isomorphism from the cyclic subrepresentation Π'_F onto the quasi-regular representation $\Pi_F := \mathrm{Ind}_{W_F}^W \mathbf{1}_{W_F}$, and the cyclic vector $\bigotimes_{s\in F} X_{W_s}$ is mapped to the cyclic vector $X_{W_F} \in V(\Pi_F)$.*

7.2. Induced representation Π_s and subrepresentation of $\pi_s = \mathbf{1}_W \oplus \Pi_s$.

LEMMA 7.4. *Suppose $s \in S$ is co-finite, or W/W_s is finite, and $0 < q_s < 1$. Then GR representation π_{f_s} associated with $f_s := f_{q_s,s}$ is equivalent to the induced representation $\Pi_s = \mathrm{Ind}_{W_s}^W \mathbf{1}_{W_s}$, and Π_s contains the trivial representation $\mathbf{1}_W$*

exactly once, *or* $\Pi_s = \mathbf{1}_W \oplus (\mathbf{1}_W)^{\perp}$, *where* $(\mathbf{1}_W)^{\perp}$ *contains* $\mathbf{1}_W$ *no more. Under the isomorphism from* π_{f_s} *to* Π_s, *the unit cyclic vector* $\boldsymbol{v}_{f_s} \in V(\pi_{f_s})$ *is mapped to the following unit cyclic vector* $\boldsymbol{w}_s$ *in* $V(\Pi_s)$: *with* $c_s := |W/W_s|^{-1}$,

$$\boldsymbol{w}_s := \sqrt{q_s c_s + (1-q_s)c_s^2} \cdot 1_W \oplus \sqrt{1-q_s}(X_{W_s} - c_s 1_W), \tag{7.6}$$

so that $(\pi_{f_s}, \boldsymbol{v}_{f_s}) \cong (\Pi_s, \boldsymbol{w}_s)$. *Denote by* P_s *the orthogonal projection of* $V(\Pi_s)$ *onto the subspace carrying the trivial representation* $\mathbf{1}_W$. *Then*

$$\|P_s \boldsymbol{w}_s\| = \sqrt{q_s + (1-q_s)c_s}.$$

LEMMA 7.5. *Let* $s \in S$ *be co-infinite*, *or* $|W/W_s| = \infty$. *Then GR representation* π_{f_s} *associated with* $f_s := f_{q_s,s}$ *is equivalent to* $\pi_s = \mathbf{1}_W \oplus \Pi_s$ *with* $\Pi_s = \mathrm{Ind}_{W_s}^W \mathbf{1}_{W_s}$, *and* Π_s *does not contain the trivial representation* $\mathbf{1}_W$. *The positive definite function* f_s *is the diagonal matrix element corresponding to the reference vector* $\boldsymbol{v}_{1,s} \in V(\pi_s)$ *in* (4.3):

$$(\pi_{f_s}, \boldsymbol{v}_{f_s}) \cong (\mathbf{1}_W \oplus \Pi_s, \boldsymbol{v}_{1,s}). \tag{7.7}$$

8. THE CASE OF FINITE COXETER GROUPS (W, S)

THEOREM 8.1. *Assume* W *is finite*, *and let* $Q = (q_s)_{s \in S}$.

(i) *Suppose* $0 \leqslant q_s < 1$ $(s \in S)$. *Then GR representation* π_{f_Q} *associated with* f_Q *is equivalent to the left regular representation* $\mathcal{L}_W$ *of* W, *or* $(\pi_{f_Q}, \boldsymbol{v}_{f_Q}) \cong (\mathcal{L}_W, \boldsymbol{v}_{Q,S})$, *where* $\boldsymbol{v}_{Q,S} \in \ell^2(W)$ *corresponds to* $\boldsymbol{w}_{Q,F}$ *in* (5.2) *and* (5.3) *for* $F = S$.

(ii) *Suppose* $q_s = 1$ $(s \in F_0), 0 \leqslant q_s < 1$ $(s \in S \setminus F_0)$ *for an* $F_0 \neq \emptyset$. *Then*

$$\pi_{f_Q} \cong \mathrm{Ind}_{\langle F_0 \rangle}^W \mathbf{1}_{\langle F_0 \rangle} \quad \textit{on } \ell^2(W/\langle F_0 \rangle).$$

P r o o f. (i) Apply Proposition 5.1 for $F = S$. Then π_{f_Q} with the cyclic vector $\boldsymbol{v}_{f_Q}$ is realized as the cyclic subrepresentation of $\pi_{(S)}$ associated with the vector $\boldsymbol{w}_{Q,S}$, where $\pi_{(S)} = \bigoplus_{F \subset S} \Pi_F = \mathbf{1}_W \oplus \big(\bigoplus_{\emptyset \subsetneq F \subsetneq S} \mathrm{Ind}_{W_F}^W \mathbf{1}_{W_F} \big) \oplus \mathcal{L}_W$.

We apply Lemma 6.1 for $\pi = \pi_{(S)}$, where the index set $\{1, 2, \ldots, N\}$ is replaced by the set $\{F; F \subset S\}$, and $\boldsymbol{v} = \sum_{1 \leqslant i \leqslant N}^{\oplus} \boldsymbol{v}_i$ is replaced by $\boldsymbol{w}_{Q,S} = \sum_{F \subset S}^{\oplus} \boldsymbol{v}_F$ above. The positive definite function associated with $\boldsymbol{v}_F$ is given by $f_F(g) := (d_{Q;S,F})^2 X_{W_F}(g) = \langle \Pi_F(g)\boldsymbol{v}_F, \boldsymbol{v}_F \rangle$ $(g \in W)$, and $f_S = d_{Q;S,S}\delta_e$, $d_{Q;S,S} \neq 0$. To guarantee that Lemma 6.1 is applicable, we have Lemma 5.2.

(ii) In this case, $f_Q = \prod_{s \in S} f_s = \prod_{s \in S \setminus F_0} f_s \cdot f_{Q,S \setminus F_0}$. We apply Proposition 5.1 for $F := S \setminus F_0$. Note that $W_F = \langle S \setminus F \rangle = \langle F_0 \rangle$. Then, using Lemmas 5.2 and 6.1 as for the assertion (i), we obtain (ii). ∎

The isomorphism between π_{f_Q} and the regular representation $\mathcal{L}_W$ is *twisted* in the sense given in the following theorem. This fact has an important meaning when we consider a limiting process for a growing sequence of Coxeter groups as $(W_n, S_n) \nearrow (W, S), S_n \nearrow S = \bigcup_{n\geqslant 1} S_n$, for instance, in the case of $\mathfrak{S}_n \nearrow \mathfrak{S}_\infty$. Here is the place where we ask the question formulated in our Problem 1, in connection with, e.g., [12] and [18]–[20].

THEOREM 8.2. *Assume W is finite and $0 \leqslant q_s < 1$ $(s \in S)$. Put $C_{f_Q} := \big(f_Q(h^{-1}g)\big)_{g,h\in W}$. Then the matrix C_{f_Q} is Hermitian and* strictly *positive definite. A linear map Ψ_Q from $V(\pi_{f_Q})$ to $\ell^2(W)$,*

$$\Psi_Q : \sum_{g\in W} c_g \pi_{f_Q}(g) \boldsymbol{v}_{f_Q} \to \sum_{g\in W} c_g \delta_g,$$

gives an algebraic isomorphism of cyclic representations $(\pi_{f_Q}, \boldsymbol{v}_{f_Q})$ and $(\mathcal{L}_W, \delta_e)$. Moreover, for $\boldsymbol{v} = \sum_{g\in W} c_g \pi_{f_Q}(g) \boldsymbol{v}_{f_Q} \in V(\pi_{f_Q})$, express $\Psi_Q(\boldsymbol{v}) = \sum_{g\in W} c_g \delta_g \in \ell^2(W)$ as a column vector $\boldsymbol{c} = (c_g)_{g\in W}$. Then $\|\boldsymbol{v}\|^2_{V(\pi_{f_Q})} = \big\|\sqrt{C_{f_Q}}\boldsymbol{c}\big\|^2_{\ell^2(W)}$, where $\sqrt{C_{f_Q}}$ commutes with $\mathcal{L}_W(g)$ $(g \in W)$. In other words, $\sqrt{C_{f_Q}} \cdot \Psi_Q$ is a unitary W-map from $\big(\pi_{f_Q}, V(\pi_{f_Q})\big)$ onto $\big(\mathcal{L}_W, \ell^2(W)\big)$, which maps the cyclic vector $\boldsymbol{v}_{f_Q}$ for π_{f_Q} to the one δ_e for $\mathcal{L}_W$.

9. THE CASE OF INFINITE COXETER GROUPS (W, S)

In this section, we assume that W is infinite.

9.1. GR representations and induced representations.

LEMMA 9.1. (i) *Assume $0 \leqslant q_s < 1$ $(s \in S)$. For a finite subset F of S, the GR representation π_{f_Q} with cyclic vector $\boldsymbol{v}_{f_Q}$ is isomorphic to the cyclic subrepresentation of*

$$\Big(\bigotimes_{\substack{s\in F\\ \text{co-finite}}} \Pi_s\Big) \otimes \Big(\bigotimes_{\substack{s\in F\\ \text{co-infinite}}} (\mathbf{1}_W \oplus \Pi_s)\Big) \otimes \pi_{f_{Q,S\setminus F}}, \tag{9.1}$$

with a cyclic vector $\big(\bigotimes_{s:\,\text{co-finite}} \boldsymbol{w}_s\big) \otimes \big(\bigotimes_{s:\,\text{co-infinite}} \boldsymbol{v}_{1,s}\big) \otimes \boldsymbol{v}_{f_{Q,S\setminus F}}$.

(ii) *Let S be finite and $0 < q_s < 1$ $(s \in S)$. Then $(\pi_{f_Q}, \boldsymbol{v}_{f_Q})$ is isomorphic to a cyclic subrepresentation of $\mathbf{1}_W \oplus \big(\bigoplus_{\emptyset \subsetneq F \subsetneq S,\ \text{infinite type}} \mathrm{Ind}_{W_F}^W \mathbf{1}_{W_F}\big) \oplus \mathcal{L}_W$ containing $\mathbf{1}_W \oplus \mathcal{L}_W$.*

9.2. Intertwining operators among $\Pi_F = \mathrm{Ind}_{W_F}^W \mathbf{1}_{W_F}$. Taking into account Proposition 5.1 and Lemma 6.3, we study here intertwining operators among quasi-regular representations $\Pi_F = \mathrm{Ind}_{W_F}^W \mathbf{1}_{W_F}$ induced from parabolic subgroups $W_F = \langle S \setminus F\rangle$.

9.2.1. *Intertwining operators between $\Pi_i = \mathrm{Ind}_{H_i}^G \mathbf{1}_{H_i}$ $(i = 1, 2)$.* Let G be a discrete group, and H_i $(i = 1, 2)$ its subgroups. The representation spaces $V(\Pi_i)$

consist of functions φ_i on G satisfying

$$\varphi_i(g) = \varphi_i(gh_i) \quad (h_i \in H_i, g \in G), \quad \|\varphi_i\|^2 := \sum_{g \in G/H_i} |\varphi_i(g)|^2 < \infty.$$

Any intertwining operator T from Π_1 to Π_2 is given by a kernel function $K(g, g')$ as follows:

$$T\varphi_1(g) = \sum_{g' \in G/H_i} K(g, g')\varphi_1(g') \quad (g \in G).$$

Put $K'(g) := K(e, g)$ $(g \in G)$. Then $K(g, g') = K'(g^{-1}g')$, and K' satisfies

$$K'(h_2 g h_1) = K'(g) \quad (h_i \in H_i, g \in G),$$
$$\sum_{g \in G/H_1} |K'(g)|^2 < \infty, \qquad \sum_{g \in H_2 \backslash G} |K'(g)|^2 < \infty.$$

Consider the restriction of K' onto a double coset $H_2 g H_1$. If the order of H_2-cosets $H_2 \backslash H_2 g H_1$ is infinite, then K' should be zero on $H_2 g H_1$. Note that

$$h_2 g h_1 \in H_2(h_2' g h_1') \Leftrightarrow h_1 {h_1'}^{-1} \in g^{-1} H_2 g \Leftrightarrow h_1 {h_1'}^{-1} \in g^{-1} H_2 g \cap H_1,$$

etc. Then we have the following criterion:

$$\text{(9.2)} \qquad K'(g) \neq 0 \Rightarrow \begin{cases} |H_2 g H_1 / H_1| = |H_2 / (g H_1 g^{-1} \cap H_2)| < \infty, \\ |H_2 \backslash H_2 g H_1| = |(g^{-1} H_2 g \cap H_1) \backslash H_1| < \infty. \end{cases}$$

9.2.2. *The case of a Coxeter group W and $W_{F_i} = \langle S \setminus F_i \rangle$ $(i = 1, 2)$.* Put $\Pi_i := \Pi_{F_i} = \mathrm{Ind}_{W_{F_i}}^W \mathbf{1}_{W_{F_i}}$ $(i = 1, 2)$. An element $\varphi_i \in V(\Pi_i)$ can be considered as a function on W which is H_i-invariant from the right. A $\varphi_1 \in V(\Pi_1)$ belongs also to $V(\Pi_2)$ only when it is also H_2-invariant from the right, and so invariant under $\langle H_1, H_2 \rangle = W_{F_1 \cap F_2}$ from the right.

LEMMA 9.2. *Let $\Pi_{F_i} = \mathrm{Ind}_{W_{F_i}}^W \mathbf{1}_{W_{F_i}}$ $(i = 1, 2)$. Then their spaces $V(\Pi_{F_1})$ and $V(\Pi_{F_2})$ have a non-trivial intersection (denoted by V_{12}) if and only if*

$$\text{(9.3)} \qquad \begin{aligned} |\langle H_1, H_2 \rangle / H_1| = |W_{F_1 \cap F_2} / W_{F_1}| < \infty, \\ |\langle H_1, H_2 \rangle / H_2| = |W_{F_1 \cap F_2} / W_{F_2}| < \infty, \end{aligned}$$

and in that case Π_{F_1} and Π_{F_2} have a common constituent realized on $V_{12} = \ell^2(W/W_{F_1 \cap F_2})$.

LEMMA 9.3. *There exists a non-zero intertwining operator from Π_{F_1} to Π_{F_2} if and only if there exists a $g \in W$ satisfying*

$$\text{(9.4)} \qquad \begin{aligned} &|W_{F_2} / (g W_{F_1} g^{-1} \cap W_{F_2})| < \infty, \\ &|(g^{-1} W_{F_2} g \cap W_{F_1}) \backslash W_{F_1}| < \infty. \end{aligned}$$

LEMMA 9.4. (i) *The trivial representation* $\mathbf{1}_W$ *is contained in the induced representation* $\Pi_F = \mathrm{Ind}_{W_F}^W \mathbf{1}_{W_F}$ *if and only if* $|W/W_F| < \infty$.

(ii) *The quasi-regular representation* $\Pi_F, F \subsetneqq S$, *is not disjoint with the regular representation* $\mathcal{L}_W$ *if and only if* W_F *is finite. In that case,* Π_F *is isomorphically imbedded into* $\mathcal{L}_W$.

10. THE CASE OF AFFINE WEYL GROUPS

Affine Weyl groups are a kind of infinite Coxeter groups (W, S) generated by affine reflections in Euclidean spaces (cf. the definition in [21], Section 4.2). Irreducible affine Weyl groups are listed in [21], Section 4.7, and their Coxeter graphs are precisely the positive semidefinite ones which are not positive definite (cf. [21], Section 6.5). Note that, for any non-empty $F \subsetneqq S$, the parabolic subgroup $W_F = \langle S \setminus F \rangle$ is a finite Coxeter group. Then, by Lemmas 5.2, 6.1, 9.2, 9.3, and 9.4, we have the following.

THEOREM 10.1. *Let* (W, S) *be an irreducible affine Weyl group, and* $Q = (q_s)_{s \in S}$.

(i) *Assume* $0 < q_s < 1$ $(s \in S)$. *Then* $\pi_{f_Q} \cong \mathbf{1}_W \oplus \mathcal{L}_W$.

(ii) *Assume* $q_s = 0$ $(s \in F_0 \neq \emptyset), 0 < q_s < 1$ $(s \notin F_0)$. *Then* $\pi_{f_Q} \cong \mathcal{L}_W$.

(iii) *Assume* $q_s = 1$ $(s \in F_1 \neq \emptyset), 0 < q_s < 1$ $(s \notin F_1)$. *Then*

$$\pi_{f_Q} \cong \mathbf{1}_W \oplus \mathrm{Ind}_{\langle F_1 \rangle}^W \mathbf{1}_{\langle F_1 \rangle}.$$

(iv) *Assume* $q_s = 0$ $(s \in F_0 \neq \emptyset), q_s = 1$ $(s \in F_1 \neq \emptyset)$, $0 < q_s < 1$ *otherwise. Then*

$$\pi_{f_Q} \cong \mathrm{Ind}_{\langle F_1 \rangle}^W \mathbf{1}_{\langle F_1 \rangle}.$$

11. THE CASE OF HYPERBOLIC COXETER GROUPS

Consider the case where (W, S) is irreducible, of rank n, and the bilinear form on V is non-degenerate. Define a cone C in V by $C := \{\lambda \in V; B(\lambda, \alpha_s) > 0$ $(s \in S)\}$. Such a Coxeter group (W, S) is called *hyperbolic* if B has signature $(n-1, 1)$ and $B(\lambda, \lambda) < 0$ $(\lambda \in C)$; see [21], Section 6.8. An irreducible Coxeter group (W, S) is hyperbolic if and only if the following conditions are satisfied:

(a) B is non-degenerate, but not positive definite;

(b) for each $s \in S$, the Coxeter graph obtained by removing s from Γ is of positive type, or its bilinear form is positive semidefinite.

A hyperbolic Coxeter group (W, S) is called *compact* if the quotient of $O(V)$ by W is compact, where $O(V)$ is the orthogonal group for B. An irreducible Coxeter group (W, S) is compact hyperbolic if and only if it satisfies (a) above and

(c) for each $s \in S$, the Coxeter graph obtained by removing s from Γ is positive definite or W_s is finite.

Hyperbolic Coxeter groups exist only in ranks from 3 to 10, and their numbers are finite in each of the ranks from 4 to 10, as seen in Table 1 below.

TABLE 1. Numbers of irreducible hyperbolic Coxeter groups W

rank W	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
compact hyperbolic	∞	9	5	$\times$	$\times$	$\times$	$\times$	$\times$
non-compact hyperbolic	∞	23	9	12	3	4	4	3

11.1. The case of compact hyperbolic Coxeter groups. By [21], Section 6.8, in this case, for any $s \in S$, the sub-Coxeter group $(W_s, S \setminus \{s\})$ is a finite Coxeter group. Therefore, from Lemmas 6.1, 6.2, 7.5, and 9.1 we get the following.

THEOREM 11.1. *Assume that a Coxeter group (W, S) is compact hyperbolic.*

(i) *Assume for $Q = (q_s)_{s\in S}$, $0 < q_s < 1$ $(s \in S)$. Then $\pi_{f_Q} \cong \mathbf{1}_W \oplus \mathcal{L}_W$.*

(ii) *Assume $q_s = 0$ $(s \in F_0)$ and $0 < q_s < 1$ $(s \notin F_0)$ for an $F_0 \neq \emptyset$. Then $\pi_{f_Q} \cong \mathcal{L}_W$.*

(iii) *Assume $q_s = 1$ $(s \in F_1 \neq \emptyset)$, $0 < q_s < 1$ $(s \notin F_1)$. Then*

$$\pi_{f_Q} \cong \mathbf{1}_W \oplus \mathrm{Ind}_{\langle F_1\rangle}^W \mathbf{1}_{\langle F_1\rangle}.$$

(iv) *Assume $q_s = 0$ $(s \in F_0 \neq \emptyset)$, $q_s = 1$ $(s \in F_1 \neq \emptyset)$, $0 < q_s < 1$ otherwise. Then*

$$\pi_{f_Q} \cong \mathrm{Ind}_{\langle F_1\rangle}^W \mathbf{1}_{\langle F_1\rangle}.$$

11.2. The case of non-compact hyperbolic Coxeter groups. By [21], Section 6.8, in this case, for any $s \in S$, the sub-Coxeter group $(W_s, S \setminus \{s\})$ is a finite or affine Weyl group. We apply Lemmas 6.1, 6.2, 7.5, and 9.1 (ii).

THEOREM 11.2. *Assume that (W, S) is irreducible and non-compact hyperbolic.*

(i) *For $Q = (q_s)_{s\in S}$, $0 < q_s < 1$ $(s \in S)$, the GR representation $(\pi_{f_Q}, \boldsymbol{v}_{f_Q})$ is isomorphic to a cyclic subrepresentation, containing $\mathbf{1}_W \oplus \mathcal{L}_W$, of*

$$\mathbf{1}_W \oplus \Big(\bigoplus_{\substack{s\in S \\ \text{infinite type}}} \mathrm{Ind}_{W_s}^W \mathbf{1}_{W_s}\Big) \oplus \mathcal{L}_W.$$

(ii) *Assume that an infinite type $s \in S$ is unique and denote it by s_0. Then*

$$\pi_{f_Q} \cong \begin{cases} \mathbf{1}_W \oplus \mathrm{Ind}_{W_{s_0}}^W \mathbf{1}_{W_{s_0}} \oplus \mathcal{L}_W & \textit{if } |W/W_{s_0}| = \infty, \\ \mathrm{Ind}_{W_{s_0}}^W \mathbf{1}_{W_{s_0}} \oplus \mathcal{L}_W & \textit{if } |W/W_{s_0}| < \infty. \end{cases}$$

Note that many of non-compact hyperbolic irreducible Coxeter groups, in the complete list in [21], Section 6.9, pp. 142–144, have unique $s \in S$ of infinite type.

Actually, for $n =$ rank $W \geqslant 7$, except two cases for $n = 7$, one for $n = 9$, and two for $n = 10$, all such Coxeter groups have unique s of infinite type. However, it might not be easy to check the condition $|W/W_s| = \infty$.

EXAMPLE 11.1. Irreducible rank 3 Coxeter groups are divided into two cases.

Case 1. The Coxeter graphs are of the form $\circ\overset{m}{\text{—}}\circ\overset{n}{\text{—}}\circ$, $3 \leqslant m \leqslant n \leqslant \infty$. Assume $n < \infty$. Then, except the following cases, the Coxeter group with this graph is compact hyperbolic, and its bilinear form B is of signature $(2, 1)$:

(m, n)	$(3, 3)$	$(3, 4)$	$(3, 5)$	$(3, 6)$	$(4, 4)$
Type of Coxeter group	A_3	B_3	H_3	$\widetilde{G_2}$	$\widetilde{B_2} = \widetilde{C_2}$

Case 2. The Coxeter graphs are triangle with labels $3 \leqslant m \leqslant n \leqslant p$ on three edges. Assume $p < \infty$. Then except only one case of $(m, n, p) = (3, 3, 3)$ for type $\widetilde{A_2}$, all other graphs are for compact hyperbolic Coxeter groups, and Theorem 11.1 is applicable.

In the case where only one component of labels (m, n) or (m, n, p) is ∞, we can apply Theorem 11.2.

EXAMPLE 11.2 (cf. [21], Section 5.1). An example of a non-compact hyperbolic Coxeter group of rank 3 is given as follows: $S = \{s_1, s_2, s_3\}, m(s_1, s_2) = 3, m(s_2, s_3) = \infty, m(s_1, s_3) = 2$. Then the Coxeter group W is isomorphic to $PGL(2, \boldsymbol{Z}) = GL(2, \boldsymbol{Z})/\{\pm 1\}$ by sending the generators s_1, s_2, s_3, respectively, to

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

where the canonical map $GL(2, \boldsymbol{Z}) \to PGL(2, \boldsymbol{Z})$ is denoted by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

In this case, W_F is finite except for $F = \{s_1\}$, where $W_{s_1} = \langle s_2, s_3 \rangle$ is equal to the parabolic subgroup P of upper triangular matrices. Hence we have $|W/W_{s_1}| = \infty$, and for $Q = (q_s)_{s\in S}, 0 \leqslant q_s < 1$ $(s \in S)$, $\pi_{f_Q} \cong \mathbf{1}_W \oplus \mathrm{Ind}_P^W \mathbf{1}_P \oplus \mathcal{L}_W$, by Theorem 11.2 (ii).

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