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REMARKS ON PICKANDS' THEOREM

BY

ZBIGNIEW MICHNA (WROCŁAW)

To Professor Tomasz Rolski
who has been dedicating his life to science

Abstract. In this article we present the Pickands theorem and his double sum method. We follow Piterbarg's proof of this theorem. Since his proof relies on general lemmas, we present a complete proof of Pickands' theorem using the Borell inequality and Slepian lemma. The original Pickands' proof is rather complicated and is mixed with upcrossing probabilities for stationary Gaussian processes. We give a lower bound for Pickands constant. Moreover, we review equivalent definitions, simulations and bounds of Pickands constant.

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1. INTRODUCTION

James Pickands III (see Pickands [18] and [19]) gave a smart and sophisticated way of finding the asymptotic behavior of the probability

$$\mathbb{P}\big(\sup_{t\in\mathbf{T}} X(t) > u\big)$$

as $u \to \infty$, where X is a Gaussian process. More precisely, for $t \in [0,p]$ let $X(t)$ be a continuous stationary Gaussian process with expected value $\mathbb{E}X(t) = 0$ and covariance

$$r(t) = \mathbb{E}\big(X(t+s)X(s)\big) = 1 - |t|^{\alpha} + o(|t|^{\alpha}),$$

where $0 < \alpha \leqslant 2$. Furthermore, we assume that $r(t) < 1$ for all $t > 0$. Then

$$\mathbb{P}\big(\sup_{t\in[0,p]} X(t) > u\big) = H_{\alpha}\, p\, u^{2/\alpha}\, \Psi(u)\big(1+o(1)\big),$$

where H_α is a positive and finite constant (Pickands constant), and $\Psi(u)$ is the tail of standard normal distribution. The classical definition of *Pickands constant* is given by the limit

$$H_\alpha = \lim_{T\to\infty} \frac{1}{T} \mathbb{E} \exp\big(\sup_{t\in[0,T]} Z_\alpha(t)\big),$$

where

$$Z_\alpha(t) = \sqrt{2} B_\alpha(t) - |t|^\alpha, \tag{1.1}$$

and B_α is a fractional Brownian motion with Hurst parameter $\alpha/2$ that is a centered Gaussian process with the covariance $\mathbb{E}\big(B_\alpha(t)B_\alpha(s)\big) = \frac{1}{2}(|t|^\alpha + |s|^\alpha - |t-s|^\alpha)$, where $t, s \in \mathbb{R}$ and $0 < \alpha \leqslant 2$.

First we prove the Pickands theorem following Piterbarg's proof and ideas which are based on the Borell inequality and Slepian lemma. Lemma 2.5 below is slightly different than Lemma D.2 in Piterbarg [20], that is, the constant before the exponent depends on T.

The original Pickands' proof is rather complicated and is mixed with upcrossing probabilities for Gaussian stationary processes. In his paper this theorem is a lemma (see Pickands [19]). Moreover, Qualls and Watanabe [21] proved the Pickands theorem in a more general setting, that is, for a regularly varying covariance function. The proof of Pickands' theorem uses the elementary Bonferroni inequality, and therefore this idea of proof is called the *double sum method*; however, in the literature the Bonferroni inequality appears in a stronger version. In this paper we present a sharper version of the Bonferroni inequality which has an impact on some lower bounds of Pickands constant (see Dębicki et al. [12] and Shao [22]). Finally, we review equivalent definitions, simulations and bounds of Pickands constant.

2. LEMMAS AND AUXILIARY THEOREMS

In the paper we will consider real-valued stochastic processes and fields. Let us write

$$\Psi(u) = 1 - \Phi(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-s^2/2}\, ds,$$

and notice that

$$\Psi(u) = \frac{1}{\sqrt{2\pi} u} e^{-u^2/2} \big(1 + o(1)\big) \tag{2.1}$$

as $u \to \infty$.

LEMMA 2.1. *Let* (X_1, X_2) *be a Gaussian vector with values in* $\mathbb{R}^2$ *with* $\mathbb{E}X_1 = m_1$, $\mathbb{E}X_2 = m_2$, $\mathbf{Var}\, X_1 = \sigma_1^2$, $\mathbf{Var}\, X_2 = \sigma_2^2$ *and* $\rho = \mathbf{Cov}(X_1, X_2)$. *Then*

$$X_2 = aX_1 + Z,$$

where

$$a = \frac{\rho}{\sigma_1^2},$$

and Z *is independent of* X_1 *and is normally distributed with mean* $m_2 - \alpha m_1$ *and variance*

$$\sigma_2^2 - \frac{\rho^2}{\sigma_1^2}.$$

The next lemma can easily be proved by mathematical induction.

LEMMA 2.2 (Bonferroni's inequality [4]). *Let* $(\Omega, \mathcal{S}, \mathbb{P})$ *be a probability space, and* $A_1, A_2, \ldots, A_n \in \mathcal{S}$ *for* $n \geqslant 2$. *Then*

$$\mathbb{P}\big(\bigcup_{i=1}^{n} A_i\big) \geqslant \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{1 \leqslant i < j \leqslant n} \mathbb{P}(A_i \cap A_j).$$

Using the above Bonferroni inequality, we obtain a sharper lower bound of Pickands constant than in Dębicki et al. [12] (twice bigger). We skip the proof which goes the same way as in Dębicki et al. [12].

THEOREM 2.1. *We have*

$$H_\alpha \geqslant \frac{\alpha}{2^{2+2/\alpha}\Gamma\left(\frac{1}{\alpha}\right)}.$$

The following theorem plays a crucial role in extremes of Gaussian processes.

THEOREM 2.2 (Slepian's inequality [23]). *Let Gaussian fields* $X(t)$ *and* $Y(t)$ *be separable, where* $t \in \mathbf{T}$, *and* $\mathbf{T}$ *is an arbitrary parameter set. Moreover, we assume that the covariance functions* $r_X(t,s) = \mathbb{E}\big(X(t) - \mathbb{E}X(t)\big)\big(X(s) - \mathbb{E}X(s)\big)$ *and* $r_Y(t,s) = \mathbb{E}\big(Y(t) - \mathbb{E}Y(t)\big)\big(Y(s) - \mathbb{E}Y(s)\big)$ *satisfy*

$$r_X(t,t) = r_Y(t,t),$$

$$r_X(t,s) \leqslant r_Y(t,s)$$

for all $t, s \in \mathbf{T}$, *and their expected values fulfill*

$$\mathbb{E}X(t) = \mathbb{E}Y(t)$$

for all $t \in \mathbf{T}$. *Then for any* u

$$\mathbb{P}(\sup_{t \in \mathbf{T}} X_t < u) \leqslant \mathbb{P}(\sup_{t \in \mathbf{T}} Y_t < u).$$

The most important tool in the theory of Gaussian processes (see Borell [5] or, e.g., Adler and Taylor [1]) is the following inequality.

THEOREM 2.3 (Borell's inequality). *Let $X(t)$ be a centered a.s. bounded Gaussian field, where $t \in \mathbf{T}$, and $\mathbf{T}$ is an arbitrary parameter set. Then*

$$\mathbb{E} \sup_{t\in\mathbf{T}} X(t) = m < \infty, \qquad \sup_{t\in\mathbf{T}} \mathbf{Var}\, X(t) = \sigma^2 < \infty,$$

and for all $w \geqslant m$

$$\mathbb{P}\big(\sup_{t\in\mathbf{T}} X(t) > w\big) \leqslant \exp\left(-\frac{(w-m)^2}{2\sigma^2}\right).$$

In the rest of the paper we will tacitly assume that $0 < \alpha \leqslant 2$. The next lemma can be found in Piterbarg [20]. Following ideas from Piterbarg [20], we provide its detailed proof.

LEMMA 2.3. *Let $\chi(t)$ be a continuous Gaussian field, where $t = (t_1, t_2) \in \mathbb{R}^2$ with $\mathbb{E}\chi(t) = -|t_1|^\alpha - |t_2|^\alpha$ and $\mathbf{Cov}\big(\chi(t), \chi(s)\big) = |t_1|^\alpha + |t_2|^\alpha + |s_1|^\alpha + |s_2|^\alpha - |t_1 - s_1|^\alpha - |t_2 - s_2|^\alpha$, $s = (s_1, s_2)$, and let $X(t)$ be a continuous homogeneous Gaussian field, where $t = (t_1, t_2) \in \mathbb{R}^2$ with expected value $\mathbb{E}X(t) = 0$ and covariance*

$$r(t) = \mathbb{E}\big(X(t+s)X(s)\big) = 1 - |t_1|^\alpha - |t_2|^\alpha + o(|t_1|^\alpha + |t_2|^\alpha).$$

Then for any compact set $\mathbf{T} \subset \mathbb{R}^2$

$$\mathbb{P}\big(\sup_{t\in u^{-2/\alpha}\mathbf{T}} X(t) > u\big) = \Psi(u) H(\mathbf{T})\big(1 + o(1)\big)$$

as $u \to \infty$, where

$$H(\mathbf{T}) = \mathbb{E} \exp\big(\sup_{t\in\mathbf{T}} \chi(t)\big) < \infty\,.$$

REMARK 2.1. *The continuity of the field $\chi(t)$ follows from the Sudakov, Dudley and Fernique theorem* (see, e.g., Piterbarg [20]).

P r o o f. Let $u > 0$; then

$$\mathbb{P}\big(\sup_{t\in u^{-2/\alpha}\mathbf{T}} X(t) > u\big) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-v^2/2}\, \mathbb{P}\big(\sup_{t\in u^{-2/\alpha}\mathbf{T}} X(t) > u | X(0) = v\big)\, dv.$$

Substituting $v = u - w/u$, we obtain the right-hand side of this equality in the form

$$\frac{1}{\sqrt{2\pi}u} e^{-u^2/2} \int\limits_{-\infty}^{\infty} e^{w - w^2/(2u^2)}\, \mathbb{P}\big(\sup_{t\in u^{-2/\alpha}\mathbf{T}} X(t) > u | X(0) = u - w/u\big)\, dw.$$

Let us put

$$\chi_u(t) = u\big(X(u^{-2/\alpha}t) - u\big) + w.$$

Thus, let us rewrite the last integral (without the function before the integral, which is $\Psi(u)$ as $u \to \infty$) as

$$\int_{-\infty}^{\infty} e^{w - w^2/(2u^2)}\, \mathbb{P}\big(\sup_{t \in \mathbf{T}} \chi_u(t) > w | X(0) = u - w/u\big)\, dw. \tag{2.2}$$

Next, compute the expected value and variance of the distribution $\chi_u(t)$ under the condition $X(0) = u - w/u$ (this distribution is Gaussian by Lemma 2.1). By Lemma 2.1, we get

$$\mathbb{E}\big(\chi_u(t)|X(0)\big) = u\mathbb{E}\big(X(u^{-2/\alpha}t)|X(0)\big) - u^2 + w = uaX(0) - u^2 + w,$$

where $a = r(u^{-2/\alpha}t)$. Hence

$$\begin{aligned} ex(u,t) &:= \mathbb{E}\big(\chi_u(t)|X(0) = u - w/u\big) \\ &= -u^2[1 - r(u^{-2/\alpha}t)] + w[1 - r(u^{-2/\alpha}t)], \end{aligned} \tag{2.3}$$

which, by the assumption on the covariance r, tends to $-|t_1|^\alpha - |t_2|^\alpha$ as $u \to \infty$. Now, let us calculate the variance

$$\begin{aligned} \mathbf{Var}\,\big(\chi_u(t)|X(0) = u - w/u\big) &= u^2\mathbf{Var}\,\big(X(u^{-2/\alpha}t)|X(0) = u - w/u\big) \\ &= u^2\mathbf{Var}\,(Z) = u^2\big(1 - r^2(u^{-2/\alpha}t)\big), \end{aligned} \tag{2.4}$$

where Z is a suitable random variable from Lemma 2.1. By the assumption on the covariance r, the variance in (2.4) tends to $2(|t_1|^\alpha + |t_2|^\alpha)$ as $u \to \infty$. Similarly we compute

$$\begin{aligned} &\mathbf{Var}\,\big(\chi_u(t) - \chi_u(s)|X(0) = u - w/u\big) \\ &\qquad = u^2\mathbf{Var}\,\big(X(u^{-2/\alpha}t) - X(u^{-2/\alpha}s)|X(0) = u - w/u\big), \end{aligned}$$

which, by Lemma 2.1, equals

$$u^2\big[\mathbf{Var}\,\big(X(u^{-2/\alpha}t) - X(u^{-2/\alpha}s)\big) - [r(u^{-2/\alpha}t) - r(u^{-2/\alpha}s)]^2\big].$$

Thus we get

$$\begin{aligned} &\mathbf{Var}\,\big(\chi_u(t) - \chi_u(s)|X(0) = u - w/u\big) \\ &\qquad = u^2\Big[2\big[1 - r\big(u^{-2/\alpha}(t-s)\big)\big] - [r(u^{-2/\alpha}t) - r(u^{-2/\alpha}s)]^2\Big], \end{aligned}$$

and one can estimate

$$\begin{aligned}\mathbf{Var}\,\big(\chi_u(t)-\chi_u(s)|X(0)=u-w/u\big) &\leqslant 2u^2\big[1-r\big(u^{-2/\alpha}(t-s)\big)\big]\\ &=2(|t_1-s_1|^\alpha+|t_2-s_2|^\alpha)+u^2o(u^{-2}[|t_1-s_1|^\alpha+|t_2-s_2|^\alpha])\\ &=(|t_1-s_1|^\alpha+|t_2-s_2|^\alpha)\big(2+o(1)\big),\end{aligned}$$

where $o(1)\to 0$ if $u\to\infty$ or $|t_1-s_1|\to 0$ and $|t_2-s_2|\to 0$. Hence

$$\mathbf{Var}\,\big(\chi_u(t)-\chi_u(s)|X(0)=u-w/u\big)\leqslant 3(|t_1-s_1|^\alpha+|t_2-s_2|^\alpha) \tag{2.5}$$

for u sufficiently large and t,s belonging to any bounded set of $\mathbb{R}^2$. One can also show that the covariance of $\chi_u(t)$ and $\chi_u(s)$ under the condition $X(0)=u-\frac{w}{u}$ tends to $|t_1|^\alpha+|t_2|^\alpha+|s_1|^\alpha+|s_2|^\alpha-|t_1-s_1|^\alpha-|t_2-s_2|^\alpha$. Thus the finite-dimensional distributions of the field $\chi_u(t)$ under the condition $X(0)=u-\frac{w}{u}$ converge to the finite-dimensional distributions of $\chi(t)$, and, by (2.5), the distribution of the field $\chi_u(t)$ under the condition $X(0)=u-\frac{w}{u}$ is tight, which implies that the field $\chi_u(t)$ under the condition $X(0)=u-\frac{w}{u}$ converges weakly to $\chi(t)$ as $u\to\infty$.

By weak convergence,

$$\mathbb{P}\big(\sup_{t\in\mathbf{T}}\chi_u(t)>w|X(0)=u-w/u\big)\to\mathbb{P}\big(\sup_{t\in\mathbf{T}}\chi(t)>w\big) \tag{2.6}$$

as $u\to\infty$. Since the process $\chi_u(t)$ under the condition $X(0)=u-\frac{w}{u}$ is continuous on $\mathbf{T}$, we infer by the Borell theorem (Theorem 2.3) that

$$\mathbb{E}\Big(\sup_{t\in\mathbf{T}}\big(\chi_u(t)-ex(u,t)\big)\big|X(0)=u-w/u\Big)\leqslant m<\infty,$$

$$\sup_{t\in\mathbf{T}}\mathbf{Var}\,\big(\chi_u(t)|X(0)=u-w/u\big)\leqslant\sigma^2<\infty,$$

where, by (2.3), (2.4) and (2.6), m and σ^2 depend only on α, and

$$\mathbb{P}\Big(\sup_{t\in\mathbf{T}}\big(\chi_u(t)-ex(u,t)\big)>w|X(0)=u-w/u\Big)\leqslant\exp\left(\frac{-(w-m)^2}{2\sigma^2}\right) \tag{2.7}$$

for all $w\geqslant m$ for sufficiently large u. Since

$$\begin{aligned}&\mathbb{P}\Big(\sup_{t\in\mathbf{T}}\big(\chi_u(t)-m\big)>w\big|X(0)=u-w/u\Big)\\ &\qquad\qquad\leqslant\mathbb{P}\Big(\sup_{t\in\mathbf{T}}\big(\chi_u(t)-ex(u,t)\big)>w\big|X(0)=u-w/u\Big),\end{aligned}$$

by (2.7) we have

$$\mathbb{P}\big(\sup_{t\in\mathbf{T}}\chi_u(t) > w|X(0) = u - w/u\big) \leqslant \exp\left(\frac{-(w-2m)^2}{2\sigma^2}\right). \tag{2.8}$$

Then, using (2.8) and the dominated convergence theorem, we get

$$\mathbb{E}\big[\exp\big(\sup_{t\in\mathbf{T}}\chi_u(t)\big)\big|X(0) = u - w/u\big] \to \mathbb{E}\big[\exp\big(\sup_{t\in\mathbf{T}}\chi(t)\big)\big]$$

as $u \to \infty$ and $\mathbb{E}\big[\exp\big(\sup_{t\in\mathbf{T}}\chi(t)\big)\big] < \infty$. Thus, taking into account (2.2), we get the assertion. ∎

COROLLARY 2.1. *If* $\mathbf{T} = [a, b] \times [c, d]$, *then*

$$H(\mathbf{T}) \leqslant \lceil b-a\rceil\, \lceil d-c\rceil\, H([0,1]\times[0,1]),$$

where $\lceil x\rceil$ *is the smallest integer greater than or equal to* x.

Proof. We augment our rectangle to the rectangle with sides of length $\lceil b-a\rceil$ and $\lceil d-c\rceil$. This rectangle can be divided into $\lceil b-a\rceil\, \lceil d-c\rceil$ unit squares. By the homogeneity of the random field X we get the assertion. ∎

Reducing one dimension in the previous lemma, we get the following

LEMMA 2.4. *Let* $Z_\alpha(t)$ *be the process defined in* (1.1), *and* $X(t)$ *be a continuous centered Gaussian process with covariance*

$$r(t) = \mathbb{E}\big(X(t+s)X(s)\big) = 1 - |t|^\alpha + o(|t|^\alpha).$$

Then for any $T > 0$

$$\mathbb{P}\big(\sup_{t\in[0,\,u^{-2/\alpha}T]} X(t) > u\big) = \Psi(u)H(T)\big(1+o(1)\big)$$

as $u \to \infty$, *where*

$$H(T) = \mathbb{E}\exp\big(\sup_{t\in[0,\,T]} Z_\alpha(t)\big) < \infty. \tag{2.9}$$

Proof. The proof goes the same way as the proof of Lemma 2.3. ∎

COROLLARY 2.2. *For* $T > 0$ *we have*

$$H(T) \leqslant \lceil T\rceil\, H([0,1]).$$

The next lemma is different than Lemma D.2 in Piterbarg [20], that is, the constant before the exponent depends on T.

LEMMA 2.5. *Let $0 < \epsilon < 1/2$ and $0 < \epsilon^\alpha < 1/2$, and $1 - 2|t|^\alpha \leqslant r(t) \leqslant 1 - \frac{1}{2}|t|^\alpha$ for all $t \in [0, \epsilon]$, where $X(t)$ is defined in Lemma 2.4. Then for $T > 0$, $t_0 > T$ and u sufficiently large*

$$\mathbb{P}\big(\sup_{t\in [0,\, u^{-2/\alpha}T]} X(t) > u, \sup_{t\in [u^{-2/\alpha}t_0,\, u^{-2/\alpha}(t_0+T)]} X(t) > u\big) \leqslant C(\alpha, t_0, T)\, \Psi(u),$$

where

$$C(\alpha, t_0, T) = 4\lceil D\,T\rceil\, \lceil D\,(t_0+T)\rceil \exp\big(-\tfrac{1}{8}(t_0-T)^\alpha\big) H([0,1]\times[0,1]),$$

and $D = (2\sqrt{2}/\sqrt{7})^{2/\alpha} 16^{1/\alpha}$.

REMARK 2.2. *Let us notice that the assumption $r(t) = 1 - |t|^\alpha + o(|t|^\alpha)$ implies that there exists $\epsilon > 0$ such that $1 - 2|t|^\alpha \leqslant r(t) \leqslant 1 - \frac{1}{2}|t|^\alpha$ for all $t \in [0, \epsilon]$.*

Proof. Let us consider a Gaussian field $Y(t,s) = X(t) + X(s)$. Then

$$\mathbb{P}\big(\sup_{t\in A} X(t) > u, \sup_{t\in B} X(t) > u\big) \leqslant \mathbb{P}\big(\sup_{(t,s)\in A\times B} Y(t,s) > 2u\big), \tag{2.10}$$

where $A = [0,\, u^{-2/\alpha}T]$ and $B = [u^{-2/\alpha}t_0,\, u^{-2/\alpha}(t_0+T)]$. Let us notice that

$$\sigma^2(t,s) = \mathbf{Var}\, Y(t,s) = 2 + 2r(t-s) = 4 - 2\big(1 - r(t-s)\big). \tag{2.11}$$

By the assumptions of the lemma, for $|t-s| \leqslant \epsilon$ we have

$$\frac{1}{2}|t-s|^\alpha \leqslant 1 - r(t-s) \leqslant 2|t-s|^\alpha,$$

which gives

$$4 - 4|t-s|^\alpha \leqslant \sigma^2(t,s) \leqslant 4 - |t-s|^\alpha.$$

Thus, for sufficiently large u we get

$$\inf_{(t,s)\in(A\times B)} \sigma^2(t,s) \geqslant 4 - 4 \sup_{(t,s)\in(A\times B)} |t-s|^\alpha \geqslant 4 - 4\epsilon^\alpha > 2, \tag{2.12}$$

where in the last inequality we used the assumption of the lemma. Similarly for sufficiently large u we obtain

$$\begin{aligned} \sup_{(t,s)\in(A\times B)} \sigma^2(t,s) &\leqslant 4 - \inf_{(t,s)\in(A\times B)} |t-s|^\alpha \\ &\leqslant 4 - |u^{-2/\alpha}(t_0-T)|^\alpha = 4 - u^{-2}(t_0-T)^\alpha. \end{aligned} \tag{2.13}$$

Let us put

$$Y^*(t,s) = \frac{Y(t,s)}{\sigma(t,s)},$$

where $\sigma(t,s)$ is defined in (2.11). Let us estimate the right-hand side of (2.10). Thus, for sufficiently large u we have

$$\text{(2.14)}\quad \mathbb{P}\big(\sup_{(t,s)\in A\times B} Y(t,s) > 2u\big) = \mathbb{P}\left(\exists (t,s)\in A\times B : \frac{Y(t,s)}{\sigma(t,s)} > \frac{2u}{\sigma(t,s)}\right)$$
$$\leqslant \mathbb{P}\left(\sup_{(t,s)\in A\times B} Y^*(t,s) > \frac{2u}{\sqrt{4-u^{-2}(t_0-T)^\alpha}}\right),$$

where in the last line we used (2.13). Let us compute the following expectation for $(t,s)\in A\times B$ and $(t_1,s_1)\in A\times B$. We have

$$\mathbb{E}[Y^*(t,s)-Y^*(t_1,s_1)]^2 = \mathbb{E}\left[\frac{Y(t,s)-Y(t_1,s_1)}{\sigma(t,s)} + \frac{Y(t_1,s_1)}{\sigma(t,s)} - \frac{Y(t_1,s_1)}{\sigma(t_1,s_1)}\right]^2$$
$$\leqslant 2\mathbb{E}\left[\frac{Y(t,s)-Y(t_1,s_1)}{\sigma(t,s)}\right]^2 + 2\left[\frac{1}{\sigma(t,s)} - \frac{1}{\sigma(t_1,s_1)}\right]^2 \mathbb{E}Y^2(t_1,s_1) =: I_1,$$

where in the last inequality we used the relation $(a+b)^2 \leqslant 2a^2+2b^2$. Continuing the computation, we get

$$\begin{aligned}
I_1 &\leqslant \frac{2}{\inf_{(t,s)\in A\times B}\sigma^2(t,s)}\,\mathbb{E}\left[Y(t,s)-Y(t_1,s_1)\right]^2 \\
&\quad + 2\left[\frac{1}{\sigma(t,s)} - \frac{1}{\sigma(t_1,s_1)}\right]^2 \sigma^2(t_1,s_1) \\
&= \frac{2}{\inf_{(t,s)\in A\times B}\sigma^2(t,s)}\,\mathbb{E}\left[Y(t,s)-Y(t_1,s_1)\right]^2 + 2\left[\frac{\sigma(t_1,s_1)-\sigma(t,s)}{\sigma(t,s)}\right]^2 \\
&\leqslant \frac{2}{\inf_{(t,s)\in A\times B}\sigma^2(t,s)}\left[\mathbb{E}\left[Y(t,s)-Y(t_1,s_1)\right]^2 + [\sigma(t_1,s_1)-\sigma(t,s)]^2\right] =: I_2.
\end{aligned}$$

Using (2.12) for sufficiently large u, we get

$$\begin{aligned}
I_2 &\leqslant \mathbb{E}\left[Y(t,s)-Y(t_1,s_1)\right]^2 + [\sigma(t_1,s_1)-\sigma(t,s)]^2 \\
&= \mathbb{E}[X(t)-X(t_1)+X(s)-X(s_1)]^2 + [\sigma(t_1,s_1)-\sigma(t,s)]^2 \\
&\leqslant 2\mathbb{E}[X(t)-X(t_1)]^2 + 2\mathbb{E}[X(s)-X(s_1)]^2 + [\sigma(t_1,s_1)-\sigma(t,s)]^2 =: I_3,
\end{aligned}$$

where in the last inequality we used again the relation $(a+b)^2 \leqslant 2a^2+2b^2$. Continuing the reasoning, we see that

$$\begin{aligned}
I_3 &= 2\mathbb{E}[X(t)-X(t_1)]^2 + 2\mathbb{E}[X(s)-X(s_1)]^2 \\
&\quad + \sigma^2(t_1,s_1) - 2\sigma(t_1,s_1)\sigma(t,s) + \sigma^2(t,s) \\
&= 2\mathbb{E}[X(t)-X(t_1)]^2 + 2\mathbb{E}[X(s)-X(s_1)]^2 \\
&\quad + \mathbb{E}Y^2(t_1,s_1) - 2\sqrt{\mathbb{E}Y^2(t_1,s_1)\mathbb{E}Y^2(t,s)} + \mathbb{E}Y^2(t,s) =: I_4.
\end{aligned}$$

By Schwarz's inequality we obtain

$$\begin{aligned}I_4 &\leqslant 2\mathbb{E}[X(t)-X(t_1)]^2+2\mathbb{E}[X(s)-X(s_1)]^2\\&\quad+\mathbb{E}Y^2(t_1,s_1)-2\mathbb{E}[Y(t_1,s_1)Y(t,s)]+\mathbb{E}Y^2(t,s)\\&=2\mathbb{E}[X(t)-X(t_1)]^2+2\mathbb{E}[X(s)-X(s_1)]^2+\mathbb{E}[Y(t,s)-Y(t_1,s_1)]^2\\&=2\mathbb{E}[X(t)-X(t_1)]^2+2\mathbb{E}[X(s)-X(s_1)]^2\\&\quad+\mathbb{E}[X(t)-X(t_1)+X(s)-X(s_1)]^2=:I_5.\end{aligned}$$

Using the inequality $(a+b)^2 \leqslant 2a^2+2b^2$, we get

$$I_5 \leqslant 4\mathbb{E}[X(t)-X(t_1)]^2+4\mathbb{E}[X(s)-X(s_1)]^2. \tag{2.15}$$

For $|t-t_1| \leqslant \epsilon$ we have

$$\mathbb{E}[X(t)-X(t_1)]^2 = 2-2r(|t-t_1|) \leqslant 4|t-t_1|^\alpha, \tag{2.16}$$

where in the last inequality we used the assumption of the lemma. Thus, by (2.15) and (2.16), we have for $(t,s)\in A\times B$ and $(t_1,s_1)\in A\times B$ and u sufficiently large

$$\mathbb{E}[Y^*(t,s)-Y^*(t_1,s_1)]^2 \leqslant 16[|t-t_1|^\alpha+|s-s_1|^\alpha]. \tag{2.17}$$

Since $\mathbb{E}[Y^*(t,s)]^2=1$, by (2.17) we get

$$\mathbb{E}[Y^*(t,s)Y^*(t_1,s_1)] \geqslant 1-8|t-t_1|^\alpha-8|s-s_1|^\alpha. \tag{2.18}$$

Let us define the random field

$$Z(t,s)=\frac{1}{\sqrt{2}}\big(\eta_1(t)+\eta_2(s)\big), \tag{2.19}$$

where η_1 and η_2 are independent Gaussian stationary processes with $\mathbb{E}\eta_1(t)=\mathbb{E}\eta_2(t)=0$ and $\mathbb{E}[\eta_i(t)\eta_i(s)]=\exp(-32|t-s|^\alpha)$ for $i=1,2$. Hence

$$\begin{aligned}\mathbb{E}[Z(t,s)Z(t_1,s_1)] &= \frac{1}{2}\big(\mathbb{E}[\eta_1(t)\eta_1(t_1)]+\mathbb{E}[\eta_2(s)\eta_2(s_1)]\big)\\&=\frac{1}{2}[\exp(-32|t-t_1|^\alpha)+\exp(-32|s-s_1|^\alpha)]\\&\leqslant 1-8|t-t_1|^\alpha-8|s-s_1|^\alpha\end{aligned} \tag{2.20}$$

for sufficiently small $|t-t_1|$ and $|s-s_1|$, by the fact that $e^{-x}\leqslant 1-\frac{1}{2}x$ for sufficiently small and positive x. Thus, by (2.18) and (2.20) we obtain

$$\mathbb{E}[Y^*(t,s)Y^*(t_1,s_1)] \geqslant \mathbb{E}[Z(t,s)Z(t_1,s_1)] \tag{2.21}$$

for sufficiently small $|t - t_1|$ and $|s - s_1|$. Hence, by the Slepian inequality, we have for large u

$$\text{(2.22)} \qquad \mathbb{P}\big(\sup_{(t,s)\in A\times B} Y^*(t,s) > u^*\big) \leqslant \mathbb{P}\big(\sup_{(t,s)\in A\times B} Z(t,s) > u^*\big),$$

where

$$u^* = \frac{2u}{\sqrt{4 - u^{-2}(t_0 - T)^\alpha}}$$

(see (2.14)). Let us put

$$\eta(t,s) = Z\left(\frac{t}{16^{1/\alpha}}, \frac{s}{16^{1/\alpha}}\right);$$

then

$$\text{(2.23)} \qquad \mathbb{P}\big(\sup_{(t,s)\in A\times B} Z(t,s) > u^*\big) = \mathbb{P}\big(\sup_{(t,s)\in A'\times B'} \eta(t,s) > u^*\big),$$

where $A' = [0,\, u^{-2/\alpha}T16^{1/\alpha}]$ and $B' = [u^{-2/\alpha}t_0 16^{1/\alpha},\, u^{-2/\alpha}(t_0+T)16^{1/\alpha}]$. Notice that $\eta(t,s)$ satisfies the assumptions of Lemma 2.3 (for the field X). For

$$u \geqslant u_0 = \left[\frac{t_0 - T}{\epsilon}\right]^{\alpha/2}$$

we get

$$\frac{u^*}{u} = \frac{2}{\sqrt{4 - u^{-2}(t_0 - T)^\alpha}} \leqslant \frac{2}{\sqrt{4 - u_0^{-2}(t_0 - T)^\alpha}} = \frac{2}{\sqrt{4 - \epsilon^\alpha}} < \frac{2\sqrt{2}}{\sqrt{7}},$$

where in the last inequality we used the assumption of the lemma that $\epsilon^\alpha < \frac{1}{2}$. Thus, we get $A' \subset \big[0,\, \big(u^*\frac{\sqrt{7}}{2\sqrt{2}}\big)^{-2/\alpha}T16^{1/\alpha}\big]$ and $B' \subset \big[0,\, \big(u^*\frac{\sqrt{7}}{2\sqrt{2}}\big)^{-2/\alpha}(t_0+T)16^{1/\alpha}\big]$. Let us define $\mathbf{T} = \big[0,\, \big(\frac{\sqrt{7}}{2\sqrt{2}}\big)^{-2/\alpha}T16^{1/\alpha}\big] \times \big[0,\, \big(\frac{\sqrt{7}}{2\sqrt{2}}\big)^{-2/\alpha}(t_0+T)16^{1/\alpha}\big]$. Hence

$$\text{(2.24)} \qquad \begin{aligned} \mathbb{P}\big(\sup_{(t,s)\in A'\times B'} \eta(t,s) > u^*\big) &\leqslant \mathbb{P}\big(\sup_{(t,s)\in (u^*)^{-2/\alpha}\mathbf{T}} \eta(t,s) > u^*\big) \\ &= \Psi(u^*)H(\mathbf{T})\big(1 + o(1)\big) \end{aligned}$$

as $u \to \infty$, where in the last line we used Lemma 2.3. Since $\frac{1}{1-x} \geqslant 1 + x$ for $x < 1$, we get for sufficiently large u

$$(u^*)^2 = \frac{4u^2}{4 - u^{-2}(t_0 - T)^\alpha} \geqslant u^2\big[1 + \tfrac{1}{4}u^{-2}(t_0 - T)^\alpha\big] = u^2 + \tfrac{1}{4}(t_0 - T)^\alpha \geqslant u^2.$$

Thus, using (2.1), we deduce that for sufficiently large u

$$\Psi(u^*) \leqslant 2\Psi(u)\exp\big(-\tfrac{1}{8}(t_0-T)^\alpha\big).$$

Hence, by (2.24) it follows that for sufficiently large u

$$(2.25)\quad \mathbb{P}\big(\sup_{(t,s)\in A'\times B'}\eta(t,s)>u^*\big)$$
$$\leqslant 2\Psi(u)\exp\big(-\tfrac{1}{8}(t_0-T)^\alpha\big)H(\mathbf{T})\big(1+o(1)\big)\leqslant 4\Psi(u)\exp\big(-\tfrac{1}{8}(t_0-T)^\alpha\big)H(\mathbf{T}).$$

From Corollary 2.1 we obtain

(2.26)
$$H(\mathbf{T})\leqslant H([0,1]\times[0,1])\left\lceil\left(\frac{\sqrt{7}}{2\sqrt{2}}\right)^{-2/\alpha}T16^{1/\alpha}\right\rceil\left\lceil\left(\frac{\sqrt{7}}{2\sqrt{2}}\right)^{-2/\alpha}(t_0+T)16^{1/\alpha}\right\rceil.$$

Thus, using (2.10), (2.14), (2.22), (2.23), (2.25) and (2.26), we get the assertion of the lemma. ■

3. PICKANDS' THEOREM

THEOREM 3.1 (Pickands' theorem). *Let $X(t)$, where $t\in[0,p]$, be a continuous stationary Gaussian process with expected value $\mathbb{E}X(t)=0$ and covariance*

$$r(t)=\mathbb{E}\big(X(t+s)X(s)\big)=1-|t|^\alpha+o(|t|^\alpha).$$

Furthermore, we assume that $r(t)<1$ for all $t>0$. Then

$$\mathbb{P}\big(\sup_{t\in[0,p]}X(t)>u\big)=H_\alpha\,p\,u^{2/\alpha}\,\Psi(u)\big(1+o(1)\big)\quad \textit{as } u\to\infty,$$

where

$$H_\alpha=\lim_{T\to\infty}\frac{H(T)}{T}$$

is positive and finite (Pickands constant), and $H(T)$ is defined in (2.9).

P r o o f. Put

$$\Delta_k=[ku^{-2/\alpha}T,(k+1)u^{-2/\alpha}T],$$

where $k\in\mathbb{N}$ and $T\geqslant p$, and $N_p=\lfloor p/(u^{-2/\alpha}T)\rfloor$. Thus,

$$\mathbb{P}\big(\sup_{t\in[0,p]}X(t)>u\big)\leqslant\sum_{k=0}^{N_p}\mathbb{P}\big(\sup_{t\in\Delta_k}X(t)>u\big)$$
$$=(N_p+1)\mathbb{P}\big(\sup_{t\in\Delta_0}X(t)>u\big),$$

where in the last equality we use stationarity of the process X. Thus, by Lemma 2.4, we get

$$\limsup_{u\to\infty} \frac{\mathbb{P}\big(\sup_{t\in[0,p]} X(t) > u\big)}{u^{2/\alpha}\Psi(u)} \leqslant \frac{p}{T}\, H(T). \tag{3.1}$$

Let us estimate our probability from below. We have

$$\begin{aligned}
&\mathbb{P}\big(\sup_{t\in[0,p]} X(t) > u\big) \geqslant \mathbb{P}\Big(\bigcup_{k=0}^{N_p-1} \{\sup_{t\in\Delta_k} X(t) > u\}\Big) \\
&\geqslant N_p\, \mathbb{P}\big(\sup_{t\in\Delta_0} X(t) > u\big) - \sum_{0\leqslant i<j\leqslant N_p-1} \mathbb{P}\big(\sup_{t\in\Delta_i} X(t) > u,\ \sup_{t\in\Delta_j} X(t) > u\big),
\end{aligned} \tag{3.2}$$

where in the last inequality we applied Lemma 2.2. Let us consider the last double sum (that is why the method is called a *double sum method*)

$$\begin{aligned}
\Sigma_2 &= \sum_{0\leqslant i<j\leqslant N_p-1} \mathbb{P}\big(\sup_{t\in\Delta_i} X(t) > u,\ \sup_{t\in\Delta_j} X(t) > u\big) \\
&= \sum_{k=1}^{N_p-1} (N_p - k)\mathbb{P}\big(\sup_{t\in\Delta_0} X(t) > u,\ \sup_{t\in\Delta_k} X(t) > u\big) \\
&\leqslant N_p\, \mathbb{P}\big(\sup_{t\in\Delta_0} X(t) > u,\ \sup_{t\in\Delta_1} X(t) > u\big) \\
&\quad + N_p \sum_{k=2}^{N_{\epsilon/4}-1} \mathbb{P}\big(\sup_{t\in\Delta_0} X(t) > u,\ \sup_{t\in\Delta_k} X(t) > u\big) \\
&\quad + N_p \sum_{k=N_{\epsilon/4}}^{N_p-1} \mathbb{P}\big(\sup_{t\in\Delta_0} X(t) > u,\ \sup_{t\in\Delta_k} X(t) > u\big).
\end{aligned}$$

Let us denote the last three terms by A_1, A_2 and A_3, respectively. We will show that these terms are negligible after dividing them by $u^{2/\alpha}\Psi(u)$ and passing with $u \to \infty$ and $T \to \infty$. Moreover, bounds on them justify that the Pickands constant is well defined.

First, let us consider A_3 and take u such that $u^{-2/\alpha}T \leqslant \epsilon/16$. Then it is easy to see that the distance of the intervals Δ_0 and Δ_k is at least $\epsilon/4$ in A_3. Hence, in A_3 (for k from A_3), for $(t, s) \in \Delta_0 \times \Delta_k$ we have

$$\begin{aligned}
\mathbf{Var}\,\big(X(t) + X(s)\big) &= 2 + 2r(t - s) = 4 - 2\big(1 - r(t - s)\big) \\
&\leqslant 4 - 2 \inf_{s\geqslant\epsilon/4} \big(1 - r(s)\big) = 4 - \delta < 4,
\end{aligned} \tag{3.3}$$

where $\delta = 2\inf_{s\geqslant\epsilon/4} \big(1 - r(s)\big) > 0$ (by the assumptions on $r(t)$). Let us observe that $X(t) + X(s)$ is a continuous Gaussian field on $[0, T] \times [0, T]$, which implies by Theorem 2.3 that

$$\mathbb{E} \sup_{(t,s)\in\Delta_0\times\Delta_k} \big(X(t)+X(s)\big) \leqslant m; \tag{3.4}$$

consequently, by (3.3) and (3.4), we get

$$\mathbb{P}\big(\sup_{t\in\Delta_0} X(t)>u,\ \sup_{t\in\Delta_k} X(t)>u\big) \leqslant \mathbb{P}\big(\sup_{(t,s)\in\Delta_0\times\Delta_k} X(t)+X(s)>2u\big)$$

$$\leqslant \exp\left(-\frac{(2u-m)^2}{2(4-\delta)}\right) = \exp\left(-\frac{(u-m/2)^2}{2(1-\delta/4)}\right) \leqslant \exp\left(-\frac{1}{2}\left(\frac{u-m/2}{1-\delta/8}\right)^2\right),$$

where in the last inequality we used the fact that $1-\delta/4 \leqslant (1-\delta/8)^2$. Hence

$$\begin{aligned} \limsup_{u\to\infty} \frac{A_3}{N_p\Psi(u)} &\leqslant \limsup_{u\to\infty} \frac{N_p^2 \exp\left(-\frac{1}{2}\left(\frac{u-m/2}{1-\delta/8}\right)^2\right)}{N_p\Psi(u)} \\ &= \lim_{u\to\infty} \left\lfloor \frac{p}{u^{-2/\alpha}T} \right\rfloor \sqrt{2\pi}\, u \exp\left(-\frac{1}{2}\left(\frac{u-a/2}{1-\delta/8}\right)^2 + \frac{1}{2}u^2\right) \\ &= 0, \end{aligned} \tag{3.5}$$

where the second line follows from (2.1) and the fact that $1-\delta/8<1$ (by assumption, $r(t)<1$ for $t>0$).

Now, let us consider A_2. For $k\geqslant 2$ we have from Lemma 2.5 (C_1 and C_2 are constants depending on α)

$$\begin{aligned} &\mathbb{P}\big(\sup_{t\in\Delta_0} X(t)>u,\ \sup_{t\in\Delta_k} X(t)>u\big) \\ &\qquad \leqslant C_1 \lceil C_2T\rceil \lceil C_2(k+1)T\rceil \exp\big(-\tfrac{1}{8}(k-1)^\alpha T^\alpha\big)\Psi(u). \end{aligned}$$

Thus

$$A_2 \leqslant C_1\lceil C_2T\rceil \Psi(u) N_p \sum_{k=2}^{N_{\epsilon/4}-1} \lceil C_2(k+1)T\rceil \exp\big(-\tfrac{1}{8}(k-1)^\alpha T^\alpha\big).$$

Let us estimate $\sum_{k=2}^{N_{\epsilon/4}-1} \lceil C_2(k+1)T\rceil \exp\big(-\frac{1}{8}(k-1)^\alpha T^\alpha\big)$. We have

$$\begin{aligned}
&\sum_{k=2}^{N_{\epsilon/4}-1} \lceil C_2(k+1)T\rceil \exp\big(-\tfrac{1}{8}(k-1)^\alpha T^\alpha\big)\\
&\quad\leqslant \sum_{k=2}^{\infty} \lceil C_2(k+1)T\rceil \exp\big(-\tfrac{1}{8}(k-1)^\alpha T^\alpha\big)\\
&\quad\leqslant \lceil C_2T\rceil \sum_{k=2}^{\infty} (k+1) \exp\big(-\tfrac{1}{8}(k-1)^\alpha T^\alpha\big)\\
&\quad= \lceil C_2T\rceil \sum_{k=1}^{\infty} (k+2) \exp\big(-\tfrac{1}{8}k^\alpha T^\alpha\big)\\
&\quad\leqslant 3\,\lceil C_2T\rceil \sum_{k=1}^{\infty} k \exp\big(-\tfrac{1}{8}k^\alpha T^\alpha\big)\\
&\quad\leqslant 3\,\lceil C_2T\rceil \exp\big(-\tfrac{1}{8}T^\alpha\big) + 3\,\lceil C_2T\rceil \int_1^\infty s \exp\big(-\tfrac{1}{8}s^\alpha T^\alpha\big)\,ds =: L_1,
\end{aligned}$$

where the last inequality is valid for $T^\alpha > 8/\alpha$ (then the function under the integral sign is decreasing for $s > 1$). Substituting $t = \frac{1}{8}s^\alpha T^\alpha$ and continuing the computations (from now on, C will be any positive constant depending on α and its values can change from line to line), we see that

$$L_1 \leqslant C\,\lceil T\rceil \exp\big(-\tfrac{1}{8}T^\alpha\big) + \frac{C\,\lceil T\rceil}{T^2} \int_{T^\alpha/8}^\infty t^{2/\alpha-1}\exp(-t)\,dt =: L_2.$$

Using the property of the incomplete gamma function, i.e.,

$$\int_u^\infty s^w e^{-s}\,ds = u^w e^{-u}\big(1+O(1/u)\big) \quad \text{as } u\to\infty,$$

where $w \in \mathbb{R}$, we get

$$L_2 \leqslant C\,\lceil T\rceil \exp\big(-\tfrac{1}{8}T^\alpha\big)\big(1+O(T^{-\alpha})\big)$$

for $T^\alpha > 8/\alpha$. Thus we obtain

$$A_2 \leqslant C\,\lceil T\rceil^2\,\Psi(u)N_p \exp\big(-\tfrac{1}{8}T^\alpha\big)\big(1+O(T^{-\alpha})\big),$$

which yields

$$\limsup_{u\to\infty} \frac{A_2}{\Psi(u)N_p} \leqslant C\,\lceil T\rceil^2 \exp\big(-\tfrac{1}{8}T^\alpha\big)\big(1+O(T^{-\alpha})\big). \tag{3.6}$$

Now, let us consider the term A_1. Thus

$$
\begin{aligned}
(3.7)\quad \mathbb{P}\big(\sup_{t\in\Delta_0} X(t)>u,\ \sup_{t\in\Delta_1} X(t)>u\big)
&\leqslant \mathbb{P}\big(\sup_{t\in\Delta_0} X(t)>u,\ \sup_{t\in u^{-2/\alpha}[T,\,T+\sqrt{T}]} X(t)>u\big)\\
&\quad+\mathbb{P}\big(\sup_{t\in\Delta_0} X(t)>u,\ \sup_{t\in u^{-2/\alpha}[T+\sqrt{T},\,2T+\sqrt{T}]} X(t)>u\big)\\
&\leqslant \mathbb{P}\big(\sup_{t\in u^{-2/\alpha}[T,\,T+\sqrt{T}]} X(t)>u\big)\\
&\quad+\mathbb{P}\big(\sup_{t\in\Delta_0} X(t)>u,\ \sup_{t\in u^{-2/\alpha}[T+\sqrt{T},\,2T+\sqrt{T}]} X(t)>u\big)\\
&= \mathbb{P}\big(\sup_{t\in[0,\,u^{-2/\alpha}\sqrt{T}]} X(t)>u\big)\\
&\quad+\mathbb{P}\big(\sup_{t\in\Delta_0} X(t)>u,\ \sup_{t\in u^{-2/\alpha}[T+\sqrt{T},\,2T+\sqrt{T}]} X(t)>u\big).
\end{aligned}
$$

First let us consider the second term in the last equality of (3.7). By Lemma 2.5 we have

$$
\begin{aligned}
&\mathbb{P}\big(\sup_{t\in\Delta_0} X(t)>u,\ \sup_{t\in u^{-2/\alpha}[T+\sqrt{T},\,2T+\sqrt{T}]} X(t)>u\big)\\
&\qquad\leqslant 4\lceil C\,T\rceil\,\lceil C\,(2T+\sqrt{T})\rceil \exp\big(-\tfrac{1}{8}\,T^{\alpha/2}\big)H([0,1]\times[0,1])\Psi(u).
\end{aligned}
$$

The first term of the last equality in (3.7) can be estimated by Lemma 2.4:

$$
\mathbb{P}\big(\sup_{t\in[0,\,u^{-2/\alpha}\sqrt{T}]} X(t)>u\big)=\Psi(u)H(\sqrt{T})\big(1+o(1)\big).
$$

Hence we obtain

$$
\begin{aligned}
(3.8)\quad &\mathbb{P}\big(\sup_{t\in\Delta_0} X(t)>u,\ \sup_{t\in\Delta_1} X(t)>u\big)\\
&\leqslant \Psi(u)H(\sqrt{T})\big(1+o(1)\big)+C\lceil T\rceil\,\lceil 2T+\sqrt{T}\rceil \exp\big(-\tfrac{1}{8}\,T^{\alpha/2}\big)\Psi(u)\\
&\leqslant \Psi(u)\lceil\sqrt{T}\rceil H(1)\big(1+o(1)\big)+C\lceil T\rceil\,\lceil 2T+\sqrt{T}\rceil \exp\big(-\tfrac{1}{8}\,T^{\alpha/2}\big)\Psi(u),
\end{aligned}
$$

where in the last inequality we used Corollary 2.2. Thus we get

$$
(3.9)\quad \limsup_{u\to\infty}\frac{A_1}{N_p\Psi(u)}\leqslant\lceil\sqrt{T}\rceil H(1)+C\lceil T\rceil\,\lceil 2T+\sqrt{T}\rceil\exp\big(-\tfrac{1}{8}\,T^{\alpha/2}\big).
$$

Now, consider the lower bound

$$
\liminf_{u\to\infty}\frac{\mathbb{P}\big(\sup_{t\in[0,p]}X(t)>u\big)}{p\,u^{2/\alpha}\Psi(u)}=\liminf_{u\to\infty}\frac{\mathbb{P}\big(\sup_{t\in[0,p]}X(t)>u\big)}{N_pT\Psi(u)},
$$

which by Lemma 2.4, (3.2), (3.5), (3.6) and (3.9) is greater than or equal to

$$\begin{aligned}(3.10)\qquad f(T) = \frac{H(T)}{T} &- \frac{C\,\lceil T\rceil^2}{T}\exp\big(-\tfrac{1}{8}T^{\alpha}\big)\big(1+O(T^{-\alpha})\big)\\ &- \frac{\lceil\sqrt{T}\rceil}{T}H(1) - C\frac{\lceil T\rceil}{T}\,\lceil 2T+\sqrt{T}\rceil\exp\big(-\tfrac{1}{8}\,T^{\alpha/2}\big).\end{aligned}$$

Let us assume that $\limsup_{T\to\infty} H(T)/T > 0$; then by (3.1) and (3.10) we get

$$\begin{aligned}\frac{H(T)}{T} &\geqslant \limsup_{u\to\infty}\frac{\mathbb{P}\big(\sup_{t\in[0,1]}X(t)>u\big)}{u^{2/\alpha}\Psi(u)} \geqslant \liminf_{u\to\infty}\frac{\mathbb{P}\big(\sup_{t\in[0,1]}X(t)>u\big)}{u^{2/\alpha}\Psi(u)}\\ &\geqslant \limsup_{S\to\infty} f(S) = \limsup_{S\to\infty}\frac{H(S)}{S},\end{aligned}$$

which implies

$$\infty > \liminf_{T\to\infty}\frac{H(T)}{T} \geqslant \limsup_{T\to\infty}\frac{H(T)}{T} > 0,$$

and $\lim_{T\to\infty} H(T)/T$ exists and is finite and positive.

It remains to prove that

$$\limsup_{T\to\infty}\frac{H(T)}{T} > 0.$$

Let us put $D = \bigcup_{j=0}^{\infty}\Delta_{2j}\cap[0,1]$. Then

$$\mathbb{P}\big(\sup_{t\in[0,1]}X(t)>u\big) \geqslant \mathbb{P}\big(\sup_{t\in D}X(t)>u\big).$$

Applying the Bonferroni inequality for the set D (see Lemma 2.2 and (3.2)) and using Lemma 2.4 and the bound for A_2 and (3.5) (note that A_1 disappears by the definition of the set D), we get

$$\begin{aligned}\frac{H(T)}{T} &\geqslant \limsup_{u\to\infty}\frac{\mathbb{P}\big(\sup_{t\in[0,1]}X(t)>u\big)}{u^{2/\alpha}\Psi(u)}\\ &\geqslant \frac{H(S)}{2S} - \frac{C\,\lceil S\rceil^2}{S}\exp\big(-\tfrac{1}{8}S^{\alpha}\big)\big(1+O(S^{-\alpha})\big)\\ &= S^{-1}\Big(\frac{H(S)}{2} - C\,\lceil S\rceil^2\exp\big(-\tfrac{1}{8}S^{\alpha}\big)\big(1+O(S^{-\alpha})\big)\Big),\end{aligned}$$

which is positive for sufficiently large S because $H(S)$ is an increasing function of S and $C\,\lceil S\rceil^2\exp\big(-\frac{1}{8}S^{\alpha}\big)\big(1+O(S^{-\alpha})\big)$ tends to zero when $S\to\infty$. ■

4. AN OVERVIEW OF THE RESULTS ON PICKANDS CONSTANT

The values of Pickands constants are known only for $\alpha = 1$ and $\alpha = 2$, that is, $H_1 = 1$ in the Brownian motion case ($B_1(t)$ is the standard Brownian motion) and $H_2 = 1/\sqrt{\pi}$ in the generate case ($B_2(t) = t\mathcal{N}$, where $\mathcal{N}$ is a standard normal random variable).

There are also equivalent definitions of Pickands constant. In Berman [3], using the theory of sojourn times, it is proven that

$$H_\alpha = \int_{0^+}^{\infty} \frac{1}{x}\, dG(x),$$

where G is the distribution function of the random variable

$$L = \int_{-\infty}^{\infty} \mathbb{I}\{Z_\alpha(s) + \eta > 0\} ds,$$

Z_α is defined in (1.1), and η is a unit mean exponentially distributed random variable independent of Z_α. In Hüsler [15] and Albin and Choi [2] it is shown that

$$H_\alpha = \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} \mathbb{P}\big(\sup_{k\geqslant 1} Z_\alpha(\epsilon k) + \eta \leqslant 0\big),$$

where k is an integer, Z_α is defined in (1.1), and η is a unit mean exponentially distributed random variable which is independent of Z_α. A similar form of Pickands constant is given in Dieker and Yakir [13] where

$$H_\alpha = \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} \mathbb{P}\big(\sup_{k\in\mathbb{Z}} Z_\alpha(\epsilon k) = 0\big).$$

A quite useful representation of Pickands constant is

$$H_\alpha = \mathbb{E}\left(\frac{\sup_{t\in\mathbb{R}} e^{Z_\alpha(t)}}{\int_{-\infty}^{\infty} e^{Z_\alpha(t)}\, dt}\right) \tag{4.1}$$

and its discrete form

$$H_\alpha = \mathbb{E}\left(\frac{\sup_{t\in\mathbb{R}} e^{Z_\alpha(t)}}{\epsilon \sum\limits_{k\in\mathbb{Z}} e^{Z_\alpha(\epsilon k)}}\right) \tag{4.2}$$

for any $\epsilon > 0$, which can be found in Dieker and Yakir [13].

One of the first attempts to simulate the Pickands constant is made in Shao [22]. In that article some estimators of Pickands constants and the simulation results are provided. Another attempt of simulation of the Pickands constant is described in Michna [17] where the estimator of the probability

$$\mathbb{P}\Big(\sup_{t>0} \big(B_\alpha(t) - t\big) > u\Big) \tag{4.3}$$

for $\alpha > 1$ is found by using the change of measure technique. More precisely, the process $B_\alpha(t)$ with a positive linear drift hits a level $u > 0$ a.s., that is, $\sigma_a = \inf\{t > 0 : B_\alpha(t) + at > u\}$ is finite a.s., where $a > 0$, which enables us to determine numerically its value, and due to the change of measure technique the expectation of a certain functional of σ_a equals the probability (4.3) (see Michna [17], Proposition 2). Thus, using the Monte Carlo method, it is possible to estimate the probability (4.3). Then, comparing simulation with the exact asymptotic behavior of the probability (4.3) (see Hüsler and Piterbarg [16]), which contains Pickands constant, allows us to find numerically Pickands constant. However, the discretization step (simulation of $B_\alpha(t)$ by using the Cholesky factorization needs a lot of computer memory) taken in the simulation of Michna [17] is too big, which gives some aberrations for α close to one. This is noticed in the article of Burnecki and Michna [6] where the discretization step is sufficiently small to stabilize the simulation and there the values of Pickands constant are closer to one for $\alpha \downarrow 1$. Moreover, in the paper of Dębicki [8] it is proven that Pickands constant H_α as a function of α is continuous in its whole domain.

In Dieker and Yakir [13] a simulation of Pickands constant is conducted. They used the following estimator of Pickands constant (compare it with (4.1) and (4.2)):

$$H_\alpha^\epsilon(T) = \mathbb{E}\left(\frac{\max_{-T/\epsilon \leqslant k \leqslant T/\epsilon} e^{Z_\alpha(\epsilon k)}}{\epsilon \sum\limits_{-T/\epsilon \leqslant k \leqslant T/\epsilon} e^{Z_\alpha(\epsilon k)}}\right),$$

where k is an integer, $\epsilon > 0$ is sufficiently small, and $T > 0$ is sufficiently large (in Dieker and Yakir [13], $\epsilon = 1/2^{18}$ and $T = 128$). Some upper and lower bounds for Pickands constant can be found in Dieker and Yakir [13] but those bounds are not given in an explicit form, and therefore they are simulated by using the Monte Carlo method. Moreover, the simulated values of Pickands constant are for $\alpha \in [0.7, 2]$. In the literature there was a conjecture stating that $H_\alpha = 1/\Gamma(\alpha)$ (see Dębicki and Mandjes [11]), which is due to K. Breitung. It is easy to notice that Breitung's hypothesis is true for $\alpha = 1$ and $\alpha = 2$. Although the simulation of Dieker and Yakir [13] gives "a strong evidence that this conjecture is not correct", that is, "the confidence interval and error bounds are well above the curve for α in the range 1.6–1.8".

The first attempt to find bounds of Pickands constants is the work of Shao [22] where one can find lower and upper bounds of Pickands constants and the asymptotic behavior of Pickands constants when $\alpha \downarrow 0$. We should notice that Breitung's hypothesis on Pickands constant fulfills this asymptotic. The results of Shao [22] use the Slepian and Borell inequalities. In Dębicki et al. [12], a lower bound of Pickands constant is given (its modification is in this article). Dębicki and Kisowski [10] provide very ingenious upper bounds of Pickands constants based on some ordering of generalized Pickands constants (see Dębicki [7]). However, none of the bounds disproves Breitung's hypothesis. This is done in a clever way in Harper

[14], that is, it is shown that for α sufficiently close to zero Pickands constant is greater than Breitung's hypothesis.

There are generalizations of Pickands constants which open new paths to investigate and show interesting relations with the extremal index of max-stable stationary processes (see Dębicki [7] and Dębicki and Hashorva [9]).

REFERENCES

[1] R. J. Adler and J. E. Taylor, *Random Fields and Geometry*, Springer, New York 2007.
[2] J. M. P. Albin and H. Choi, *A new proof of an old result by Pickands*, Electron. Commun. Probab. 15 (2010), pp. 339–345.
[3] S. M. Berman, *Sojourns and Extremes of Stochastic Processes*, The Wadsworth & Brooks/Cole Statistics/Probability Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove 1992.
[4] C. E. Bonferroni, *Teoria statistica delle classi e calcolo delle probabilità* (in Italian), Pubbl. d. R. Ist. Super. di Sci. Econom. e Commerciali di Firenze 8 (1936), pp. 1–62.
[5] C. Borell, *The Brunn–Minkowski inequality in Gauss space*, Invent. Math. 30 (1975), pp. 207–216.
[6] K. Burnecki and Z. Michna, *Simulation of Pickands constant*, Probab. Math. Statist. 22 (2002), pp. 193–199.
[7] K. Dębicki, *Ruin probability for Gaussian integrated processes*, Stochastic Process. Appl. 98 (2002), pp. 151–174.
[8] K. Dębicki, *Some properties of generalized Pickands constants*, Teor. Veroyatn. Primen. 50 (2) (2005), pp. 396–404.
[9] K. Dębicki and E. Hashorva, *On extremal index of max-stable stationary processes*, this fascicle, pp. 299–317.
[10] K. Dębicki and P. Kisowski, *A note on upper estimates for Pickands constants*, Statist. Probab. Lett. 78 (2008), pp. 2046–2051.
[11] K. Dębicki and M. Mandjes, *Open problems in Gaussian fluid queueing theory*, Queueing Syst. 68 (2011), pp 267–273.
[12] K. Dębicki, Z. Michna, and T. Rolski, *Simulation of the asymptotic constant in some fluid models*, Stoch. Models 19 (2003), pp. 407–423.
[13] A. B. Dieker and B. Yakir, *On asymptotic constants in the theory of extremes for Gaussian processes*, Bernoulli 20 (3) (2014), pp. 1600–1619.
[14] A. J. Harper, *Pickands' constant H_α does not equal $1/\Gamma(1/\alpha)$, for small α*, Bernoulli 23 (1) (2017), pp. 582–602.
[15] J. Hüsler, *Extremes of a Gaussian process and the constant H_α*, Extremes 2 (1999), pp. 59–70.
[16] J. Hüsler and V. I. Piterbarg, *Extremes of a certain class of Gaussian processes*, Stochastic Process. Appl. 83 (1999), pp. 257–271.
[17] Z. Michna, *On tail probabilities and first passage times for fractional Brownian motion*, Math. Methods Oper. Res. 49 (1999), pp. 335–354.
[18] J. Pickands III, *Upcrossing probabilities for stationary Gaussian processes*, Trans. Amer. Math. Soc. 145 (1969), pp. 51–73.
[19] J. Pickands III, *Asymptotic properties of the maximum in a stationary Gaussian process*, Trans. Amer. Math. Soc. 145 (1969), pp. 75–86.
[20] V. I. Piterbarg, *Asymptotic Methods in the Theory of Gaussian Processes and Fields*, Transl. Math. Monogr., Vol. 148, AMS, Providence 1996.
[21] C. Qualls and H. Watanabe, *Asymptotic properties of Gaussian processes*, Ann. Math. Statist. 43 (1972), pp. 580–596.

[22] Q. Shao, *Bounds and estimators of a basic constant in extreme value theory of Gaussian processes*, Statist. Sinica 6 (1996), pp. 245–257.

[23] D. Slepian, *The one-sided barrier problem for the Gaussian noise*, Bell Syst. Tech. J. 41 (1962), pp. 463–501.

Zbigniew Michna
Department of Mathematics and Cybernetics
Wrocław University of Economics
53-345 Wrocław, Poland
E-mail: zbigniew.michna@ue.wroc.pl

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