PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 1, Fasc. 1 (1980), p. 49-57

CENTRAL LIMIT THEOREM IN $D[0, \infty)$ FOR BREAKDOWN PROCESSES

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Abstract. Main results are given in Theorems 1-3. Theorem 1 asserts that a mixture of independent random elements of (D, d) satisfying the Central Limit Theorem in (D, d) also satisfies the Central Limit Theorem in (D, d). Theorem 2 determines an upper bound for $P\{X(t_1) = X(t_2) \neq X(t)\}$, where $t_1 \leqslant t \leqslant t_2$, $t_2 - t_1$ is small and X is a breakdown process. Theorem 3 gives sufficient conditions under which a breakdown process satisfies the Central Limit Theorem in (D, d).

1. Introduction. A stochastic process X assuming values 0 and 1 at any time is called the *binary process.* The class of binary processes plays an important role in reliability theory. The assumption that the value 0 is taken on if a component of a system is functioning and the value 1 if the component is failed allows us to use the class of binary processes for describing the behaviour of a system. The first moment at which a system fails is an important characteristic. If the state of a system is described by the number of failed components, i.e. by a sum of binary processes, then such a moment can be defined as the first passage time into some set. Thus an investigation of the asymptotic behaviour of sums of binary processes seems to be important. A subclass of the class of binary processes, called *breakdown processes,* is considered in Section 3. For those processes we give sufficient conditions under which the normalized sums of independent and identically distributed breakdown processes converge in distribution to a Gaussian process.

2. Central Limit Theorem for mixture of independent random elements of (D, d). Denote by D the space of all real-valued right continuous functions on $[0, \infty)$ which have left-hand limits in $(0, \infty)$. We consider D with metric

d defined in [5] where it was shown that (D, d) is a complete separable metric space. Let $\tilde{D}$ denote the space of D-valued right continuous functions on $[0, \infty)$ which have left-hand limits in $(0, \infty)$. The space $\tilde{D}$ with metric $\tilde{d}$ defined in [6] is a complete separable metric space. By D_0 we denote the set of those elements of D which are non-negative and non-decreasing. Let $(\tilde{D}^m, \tilde{d}^m)$ and (D_0^m, d^m) denote the Cartesian product of m copies of the spaces $(\tilde{D}, \tilde{d})$ and (D_0, d), respectively. Define the mapping τ of $\tilde{D} \times D_0$ in $\tilde{D}$ by $\tau(x, v) = x \circ v$, where $(x \circ v)(t) = x(v(t))$ for $x \in \tilde{D}$, $v \in D_0$, $t \geqslant 0$. In [6] it has been shown that τ is a measurable and continuous mapping on $\tilde{C} \times C_0$, where $\tilde{C}$ and C_0 are subsets of the sets of continuous functions belonging to $\tilde{D}$ and D_0, respectively. By $\boldsymbol{\tau}$ we denote the mapping of $\tilde{D}^m \times D_0^m$ in $\tilde{D}^m$ defined by

$$\boldsymbol{\tau}(x, v) = \big(\tau(x_1, v_1), \tau(x_2, v_2), \ldots, \tau(x_m, v_m)\big),$$

where $x = (x_1, x_2, \ldots, x_m) \in \tilde{D}$ and $v = (v_1, v_2, \ldots, v_m) \in D_0^m$. Hence $\boldsymbol{\tau}$ is continuous on $\tilde{C}^m \times C_0^m$, where $\tilde{C}^m$ and C_0^m denote the Cartesian product of m copies of $\tilde{C}$ and C_0, respectively.

A random element Y of (D, d) is said to *satisfy the Central Limit Theorem* (CLT) in (D, d) if there exists a Gaussian process Z with sample paths in D which is the limit in distribution in (D, d) of the sequence $\{\zeta_n\}$, where

$$\zeta_n(t) = \frac{1}{\sqrt{n}}\big(Y_1(t) + Y_2(t) + \ldots + Y_n(t) - \mathrm{E}Y_1(t) - \mathrm{E}Y_2(t) - \ldots - \mathrm{E}Y_n(t)\big),$$

and $Y_1, Y_2, \ldots$ are independent copies of Y on the same probability space.

Let $\delta = (\delta_1, \delta_2, \ldots, \delta_m)$ be the random vector of R^m the components δ_i of which take on the values 0 or 1 and $\delta_1 + \delta_2 + \ldots + \delta_m = 1$. Put $p_i = P\{\delta_i = 1\}$ for $0 < p_i < 1$, $i = 1, 2, \ldots, m$.

THEOREM 1(1). *Let $Y_1, Y_2, \ldots, Y_m$ be independent random elements of (D, d) satisfying the CLT in (D, d). Then*

$$Y = \sum_{i=1}^{m} \delta_i Y_i$$

satisfies the CLT in (D, d).

Proof. Let

$$\xi_n = \sum_{k=1}^{n} \sum_{i=1}^{m} \delta_{k,i} Y_{k,i}, \quad n \geqslant 1,$$

where $\{Y_{k,i}, k \geqslant 1\}$, $i = 1, 2, \ldots, m$, and $\{\boldsymbol{\delta}_k = (\delta_{k,1}, \delta_{k,2}, \ldots, \delta_{k,m}), k \geqslant 1\}$ are independent sequences of independent copies of Y_i, $i = 1, 2, \ldots, m$, and of

(1) Theorem 1 was suggested by Prof. C. Ryll-Nardzewski.

$\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_m)$, respectively. Notice that ξ_n has the same distribution as the random element

$$\tilde{\xi}_n = \sum_{i=1}^{m} \sum_{k=1}^{v_{n,i}} Y_{k,i},$$

where $v_n = (v_{n,1}, v_{n,2}, \ldots, v_{n,m})$ is a random vector in R^m with multinomial distribution, i.e.

$$P\{v_n = (k_1, k_2, \ldots, k_m)\} = \frac{n!}{k_1! k_2! \ldots k_m!} p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m},$$

and v_n is independent of $Y_{n,i}$, $i = 1, 2, \ldots, m$, $n \geqslant 1$.

Write

$$A_i(t) = \mathrm{E}\, Y_{n,i}(t), \quad t \geqslant 0, \quad A = (A_1, A_2, \ldots, A_m), \quad \bar{Y}_{n,i} = Y_{n,i} - A_i,$$

$$\tilde{Y}_{n,i}(s) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[ns]} \bar{Y}_{j,i}, \quad \tilde{v}_{n,i}(s) = \frac{s v_{n,i}}{n}, \quad s \geqslant 0,$$

$$\tilde{Y}_n = (\tilde{Y}_{n,1}, \tilde{Y}_{n,2}, \ldots, \tilde{Y}_{n,m}), \quad \tilde{v}_n = (\tilde{v}_{n,1}, \tilde{v}_{n,2}, \ldots, \tilde{v}_{n,m}),$$

$$V_{n,i} = \frac{1}{\sqrt{n}} (v_{n,i} - np_i), \quad V_n = (V_{n,1}, V_{n,2}, \ldots, V_{n,m}).$$

Note that $\tilde{Y}_{n,i}, \tilde{v}_{n,i}, \tilde{Y}_n, \tilde{v}_n$ are random elements of $(\tilde{D}, \tilde{d}), (D_0, d), (\tilde{D}^m, \tilde{d}^m)$ and (D_0^m, d^m), respectively, and $A_i \in D$, $A \in D^m$.

Using Theorem 1 from [2] we infer that $\{\tilde{Y}_{n,i}\}$ converges in distribution in $(\tilde{D}, \tilde{d})$ to a Gaussian random element $\mathscr{W}_i$ of $(\tilde{D}, \tilde{d})$. Furthermore, $\mathscr{W}_i$ is a homogeneous random element of $(\tilde{D}, \tilde{d})$ with independent increments having continuous paths with probability one and such that $\mathscr{W}_i(0) = 0$ and $\mathscr{W}_i(1)$ has the same distribution as the limit in distribution of

$$\left\{\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \bar{Y}_{j,i}\right\}.$$

Hence $\tilde{Y}_n \xrightarrow{\mathrm{D}} \mathscr{W} = (\mathscr{W}_1, \mathscr{W}_2, \ldots, \mathscr{W}_m)$ in $(\tilde{D}^m, \tilde{d}^m)$. By the law of large numbers and the Central Limit Theorem for multinomial distribution we have

$$v_n \to v \text{ a.e.} \quad \text{and} \quad V_n \xrightarrow{\mathrm{D}} N = (N_1, N_2, \ldots, N_m),$$

where $v(t) = (p_1 t, p_2 t, \ldots, p_m t)$ and N is a Gaussian random vector in R^m the expected value of which is the zero vector. From the separability of the metric spaces $\tilde{D}^m$, D_0^m and R^m it follows that $(\tilde{Y}_n, \tilde{v}_n, V_n)$ is a random element of $\tilde{D}^m \times D_0^m \times R^m$ with the product topology. Hence and from properties of τ we obtain

$$(1) \qquad (\tau(\tilde{Y}_n, v_n), V_n) \xrightarrow{\mathrm{D}} (\tau(\mathscr{W}, v), N)$$

in $\tilde{D}^m \times R^m$ with the product topology. Since the mapping + is continuous on $\tilde{C} \times D$ (see [6]), so (1) yields

$$(2) \qquad \sum_{i=1}^{m} \tau(\tilde{Y}_{n,i}, \tilde{v}_{n,i})(1) + \sum_{i=1}^{m} V_{n,i} A_i \xrightarrow{D} \sum_{i=1}^{m} \sqrt{p_i}\, \mathscr{W}_i(1) + \sum_{i=1}^{m} N_i A_i$$

in (D, d). The left-hand side of (2) is equal to the random element

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{m} \sum_{j=1}^{v_{n,i}} \bar{Y}_{j,i} + \frac{1}{\sqrt{n}} \sum_{i=1}^{m} (v_{n,i} - np_i) A_i$$

which at t is equal to

$$\frac{1}{\sqrt{n}} \left(\xi_n(t) - \mathrm{E}\xi_n(t)\right).$$

Hence we obtain the assertion of Theorem 1.

Note that the arguments above prove Theorem 1 also for $m = \infty$.

Remark 1. For $m = 2$ the covariance function r of the limiting Gaussian process is of the form

$$r(s, t) = pr_1(s, t) + (1-p) r_2(s, t) + \eta(s)\eta(t) p(1-p), \quad 0 \leqslant s \leqslant t < \infty,$$

where r_1 and r_2 are the covariance functions of Y_1 and Y_2, respectively, and $\eta(t) = \mathrm{E}\, Y_1(t) - \mathrm{E}\, Y_2(t)$, $t \geqslant 0$.

3. Breakdown processes. Let $\{u_n, n \geqslant 0\}$, $\{v_n, n \geqslant 1\}$, $\{u'_n, n \geqslant 1\}$ and $\{v'_n, n \geqslant 0\}$ be independent sequences of positive random variables which have no atoms at zero and are defined on the same probability space. Assume that u_0 and v'_0 have distribution functions (d.f.'s) G_0 and F_0, respectively, u_n, u'_n $(n \geqslant 1)$ have d.f. G and v_n, v'_n $(n \geqslant 1)$ have d.f. F.

Let

$$Z_0 = u_0, \quad Z_n = Z_{n-1} + v_n + u_n, \ n \geqslant 1,$$

$$Z'_0 = 0, \quad Z'_n = Z'_{n-1} + v'_{n-1} + u'_n, \ n \geqslant 1.$$

Define the process X_0 setting $X_0(\omega, t) = 1$ if there exists an $n \geqslant 0$ such that $Z_n(\omega) \leqslant t < Z_n(\omega) + v_{n+1}(\omega)$ and setting $X_0(\omega, t) = 0$ if $t \leqslant u_0(\omega)$ or if there exists an $n \geqslant 0$ such that $Z_n(\omega) + v_{n+1}(\omega) \leqslant t < Z_{n+1}(\omega)$. Analogically, define the process X_1 setting $X_1(\omega, t) = 1$ if there exists an $n \geqslant 0$ such that $Z'_n(\omega) \leqslant t < Z'_n(\omega) + v'_n(\omega)$ and setting $X_1(\omega, t) = 0$ if there exists an $n \geqslant 0$ such that $Z'_n(\omega) + v'_n(\omega) \leqslant t < Z'_{n+1}(\omega)$. Hence X_0 and X_1 are random elements of (D, d).

Let δ be a random variable taking on the values 1 and 0 with probability p and $1-p$, respectively, $0 < p < 1$. Define the process X setting $X = \delta X_1 + (1-\delta) X_0$. In reliability theory, X_0, X_1 and X are known as breakdown processes (see [3], Section 7).

For each $s \geqslant 0$ define $\gamma(s)$ as the time to the nearest change (to the right of s) of a state of the process X, i.e.

$$\gamma(s) = \inf\{t-s, t > s, X(s) \neq X(t)\}.$$

Similarly, for each $s \geqslant 0$ define $\gamma_0(s)$ and $\gamma_1(s)$ for X_0 and X_1, respectively. All those random variables have no atoms at zero.

Let

$$H' = \sum_{n=0}^{\infty} F^{*n} * G^{*\prime}, \quad H_{0,0} = G_0 * F * H', \quad H_{0,1} = G_0 * H',$$

$$H_{1,0} = F_0 * H', \quad H_{,1} = F_0 * G * H', \quad H_0 = (1-p)H_{0,0} + pH_{1,0},$$

$$H_1 = (1-p)H_{0,1} + pH_{1,1}, \quad H = H_0 + H_1,$$

where $*$ denotes the convolution operation of d.f.'s. All the functions are non-negative and non-decreasing.

LEMMA 1. *For* $0 \leqslant s \leqslant t < \infty$, $y \geqslant 0$ *and* $i = 0, 1$ *we have*

$$P\{\gamma_i(s) \leqslant y\} = \int_s^{s+y} (1-G(s+y-u))\,dH_{i,0}(u) + \int_s^{s+y} (1-F(s+y-u))\,dH_{i,1}(u),$$

$$P\{\gamma(s) \leqslant y\} = \int_s^{s+y} (1-G(s+y-u))\,dH_0(u) + \int_s^{s+y} (1-F(s+y-u))\,dH_1(u).$$

Proof. Put $p_{i,j}(s, y) = P\{X_i(s) = j, \gamma_i(s) > y\}$ for $i, j = 0, 1$. Note that

$$\begin{aligned} p_{0,1}(s, y) &= P\{\exists_{n\geqslant 0}\, Z_n \leqslant s < Z_n + v_{n+1}, \gamma_0(s) > y\} \\ &= \sum_{n\geqslant 0} P\{Z_n \leqslant s < Z_n + v_{n+1}, Z_n + v_{n+1} - s > y\} \\ &= \sum_{n\geqslant 0} \int_0^s P\{u \leqslant s < u + v_{n+1}, u + v_{n+1} - s > y\}\,dP\{Z_n \leqslant u\} \\ &= \int_0^s 1 - F(s+y-u)\,dH_{0,1}(u). \end{aligned}$$

Now

$$\begin{aligned} p_{0,0}(s, y) &= P\{s < Z_0, Z_0 - s > y\} + \\ &\quad + P\{\exists_{n\geqslant 0}\, Z_n + v_n \leqslant s < Z_n + v_{n+1} + u_{n+1}, Z_n + v_{n+1} + u_{n+1} - s > y\} \\ &= 1 - G_0(s+y) + \sum_{n\geqslant 0} \int_0^s (1-G(s+y-u))\,dP\{Z_n + v_{n+1} \leqslant u\} \\ &= 1 - G_0(s+y) + \int_0^s (1-G(s+y-u))\,dH_{0,0}(u). \end{aligned}$$

Analogically we obtain

$$p_{1,0}(s, y) = \int_0^s (1-G(s+y-u))\,dH_{1,0}(u),$$

$$p_{1,1}(s, y) = 1-F_0(s+y)+\int_0^s (1-F(s+y-u))\,dH_{1,1}(u).$$

Notice that

$$(3)\quad P\{\gamma_0(s) > y\} = \int_0^s (1-F(s+y-u))\,dH_{0,1}(u)+ \int_0^s (1-G(s+y-u))\,dH_{0,0}(u)+1-G_0(s+y),$$

$$(4)\quad P\{\gamma_1(s) > y\} = \int_0^s (1-F(s+y-u))\,dH_{1,1}(u)+ \int_0^s (1-G(s+y-u))\,dH_{1,0}(u)+1-F_0(s+y).$$

Setting $y = 0$ in (3) and (4) and using the fact that

$$P\{\gamma_0(s) > 0\} = P\{\gamma_1(s) > 0\} = 1$$

we obtain

$$\int_0^s (1-F(s-u))\,dH_{i,1}(u)+\int_0^s (1-G(s-u))\,dH_{i,0}(u) = \begin{cases} G_0(s) & \text{if } i = 0,\\ F_0(s) & \text{if } i = 1.\end{cases}$$

Now, putting the so-calculated $G_0(s+y)$ and $F_0(s+y)$ in (3) and (4), respectively, we obtain the assertion of Lemma 1 for $i = 0, 1$. The second assertion follows from the first one and from the definition of X.

COROLLARY 1. *For* $0 \leqslant s \leqslant t < \infty$ *and* $i = 0, 1$ *we have*

$$P\{\gamma_i(s) \leqslant t-s\} \leqslant H_i(t)-H_i(s), \qquad P\{\gamma(s) \leqslant t-s\} \leqslant H(t)-H(s).$$

Define a function $\tilde{F}$ by

$$\tilde{F}(t) = \max\{G_0(t), F_0(t), G(t), F(t)\}, \qquad t \geqslant 0.$$

LEMMA 2. *For* $0 \leqslant t_1 \leqslant t \leqslant t_2$ *and* $i = 0, 1$ *we have*

$$(5)\qquad P\{X_i(t_1) = X_i(t_2) \neq X_i(t)\} \leqslant P\{\gamma_i(t_1) < t-t_1\}\tilde{F}(t_2-t_1),$$

$$(6)\qquad P\{X(t_1) = X(t_2) \neq X(t)\} \leqslant P\{\gamma(t_1) < t-t_1\}\tilde{F}(t_2-t_1).$$

Proof. We prove (5) for $i = 0$. The other assertions are proved in a similar way.

Define random variables s_n and S_n by $s_{2n} = u_n$, $s_{2n+1} = v_{n+1}$ and $S_n = s_0 + s_1 + \ldots + s_{n-1}$, $n \geqslant 1$. Let

$$N(t) = \begin{cases} 0 & \text{if } s_0 > t, \\ \max\{k\colon S_{k-1} \leqslant t\} & \text{if } s_0 \leqslant t. \end{cases}$$

If $X_0(t_1) = X_0(t_2) \neq X(t)$ for $t_1 < t < t_2$, then $N(t_1) < N(t) < N(t_2)$. Thus

$$(7) \quad P\{X_0(t_1) = X_0(t_2) \neq X_0(t)\} \leqslant P\{S_{N(t_1)} < t, S_{N(t)} < t_2, N(t) - N(t_1) > 0\}$$

$$= \int_0^{t-t_1} P\{S_{N(t)} < t_2, N(t) - N(t_1) > 0 \mid S_{N(t_1)} - t_1 = u\} \, dP\{S_{N(t_1)} - t_1 \leqslant u\}.$$

Since $S_{N(t_1)} - t_1 = \gamma_0(t_1)$, the right-hand side of (7) is equal to

$$\int_0^{t-t_1} P\{S_{N(t)} < t_2, N(t) - N(t_1) > 0 \mid \gamma_0(t_1) = u\} \, dP\{\gamma_0(t_1) \leqslant u\}$$

$$= \int_0^{t-t_1} P\{S_{N(t)} - S_{N(t_1)} < t_2 - t_1 - u, N(t) - N(t_1) > 0 \mid \gamma_0(t_1) = u\} \, dP\{\gamma_0(t_1) \leqslant u\}.$$

Note that the integrand does not exceed

$$(8) \quad P\{S_{N(t)} - S_{N(t_1)} < t_2 - t_1, N(t) - N(t_1) > 0 \mid \gamma_0(t_1) = u\}$$

$$= \sum_{k=1}^{\infty} \sum_{k_1=0}^{k-1} P\{S_k - S_{k_1} < t_2 - t_1 \mid N(t) = k, N(t_1) = k_1, \gamma_0(t_1) = u\} \times$$

$$\times P\{N(t) = k, N(t_1) = k_1 \mid \gamma_0(t_1) = u\}$$

$$\leqslant \sum_{k=1}^{\infty} \sum_{k_1=0}^{k-1} \tilde{F}(t_2 - t_1) \, P\{N(t) = k, N(t_1) = k_1 \mid \gamma_0(t_1) = u\} \leqslant \tilde{F}(t_2 - t_1).$$

Hence in view of (7) and (8) we obtain (5) for $i = 0$.

THEOREM 2. *If*

$$\lim_{t \to 0} \tilde{F}(t) t^{-\alpha} < \infty \quad \textit{for some } \alpha > 0,$$

then for each $c > 0$ *there exists a number* b *such that for* $t_1 \leqslant t \leqslant t_2$ *and* $i = 0, 1$ *the following inequalities hold*:

$$(9) \quad P\{X_i(t_1) = X_i(t_2) \neq X_i(t)\} \leqslant b\big(\tilde{H}_i(t_2) - \tilde{H}_i(t_1)\big)^{2\beta_1},$$

$$(10) \quad P\{X(t_1) = X(t_2) \neq X(t)\} \leqslant b\big(\tilde{H}(t_2) - \tilde{H}(t_1)\big)^{2\beta_1},$$

where $\beta_1 = \min(1, \alpha)$, $\tilde{H}_i(t) = H_i(t) + t$, $\tilde{H}(t) = H(t) + t$.

Proof. There exist numbers u and $b_1, u \leqslant c$, such that $\tilde{F}(t) \leqslant b_1 t^{\alpha}$ for $t \leqslant u$. Hence for c there exists a constant b_2 such that $\tilde{F}(t) \leqslant b_2 t^{\alpha}$ for

$t \leqslant c$. By Lemma 2 and Corollary 1, for $0 \leqslant t_1 \leqslant t \leqslant t_2 \leqslant c$ we have

(11) $$P\{X_i(t_1) = X_i(t_2) \neq X_i(t)\} \leqslant b_2\big(H_i(t_2)-H_i(t_1)\big)(t_2-t_1)^{\alpha}.$$

Now there exists a number b such that the right-hand side of (11) does not exceed

$$b\big(H_i(t_2)-H_i(t_1)(t_2-t_1)\big)^{\beta_1} \leqslant b\big(\tilde{H}_i(t_2)-H_i(t_1)\big)^{2\beta_1}.$$

This completes the proof of Theorem 2.

4. CLT for breakdown processes. From the definition of $X = \delta X_1 + (1-\delta)X_0$ and Theorem 1 we get

COROLLARY 2. *If X_1 and X_0 satisfy the CLT in (D, d), then X satisfies the CLT in (D, d).*

The following technical lemma will be used in the sequel:

LEMMA 3. *If u and v are random variables such that $|u| \leqslant 2$, $|v| \leqslant 2$, $\mathrm{E}u^2 \leqslant a$, $\mathrm{E}v^2 \leqslant a$ and $\mathrm{E}u^2v^2 \leqslant a^{2\alpha}$ for some $\alpha > 0$, then*

$$\mathrm{E}(u-\mathrm{E}u)^2(v-\mathrm{E}v)^2 \leqslant Aa^{\beta},$$

where $\beta = \min(2\alpha, 3/2)$, $A = 90\,B^{\beta_0-\beta}$, $\beta_0 = 2\max(1, \alpha)$, and B is such that $B > 1$ and $16\,a/B < 1$.

Proof. Calculating $(u-\mathrm{E}u)^2(v-\mathrm{E}v)^2$, taking the expectation and next using the Schwarz inequality, we see that $\mathrm{E}(u-\mathrm{E}u)^2(v-\mathrm{E}v)^2$ does not exceed

(12) $$\mathrm{E}u^2v^2 + 3\mathrm{E}u^2\,\mathrm{E}v^2 + 2\mathrm{E}v^2(\mathrm{E}u^4)^{1/2} + 2\mathrm{E}u^2(\mathrm{E}v^4)^{1/2} + 4(\mathrm{E}u^2\,\mathrm{E}v^2\,\mathrm{E}u^4\,\mathrm{E}v^4)^{1/2}.$$

By assumptions we infer that (12) does not exceed

$$\begin{aligned} a^{2\alpha}+7a^2+16a^{3/2} &\leqslant B^{\beta_0}\left[\left(\frac{a}{B}\right)^2+\left(\frac{7a}{B}\right)^2+\left(\frac{16a}{B}\right)^{3/2}\right] \\ &\leqslant B^{\beta_0}\left[\left(\frac{a}{B}\right)^{\beta}+\left(\frac{7a}{B}\right)^{\beta}+\left(\frac{16a}{B}\right)^{\beta}\right] \leqslant Aa^{\beta}B^{\beta_0-\beta}. \end{aligned}$$

This completes the proof.

THEOREM 3. *If*

$$\varlimsup_{t\to 0} \tilde{F}(t)\,t^{-\alpha} < \infty \quad \textit{for some } \alpha > \frac{1}{2}$$

and H_0 and H_1 are continuous, then X_0 and X_1 satisfy the CLT in (D, t).

Proof. Note that the following equalities are true for $0 \leqslant t_1 \leqslant t \leqslant t_2 < \infty$ and $i = 0, 1$:

(13) $$\begin{aligned} \mathrm{E}\big(X_i(t)-X_i(t_1)\big)^2 &= P\{X_i(t_1) \neq X_i(t)\} \leqslant P\{\gamma_i(t_1) < t-t_1\}, \\ \mathrm{E}\big(X_i(t)-X_i(t_1)\big)^2\big(X_i(t_2)-X_i(t)\big)^2 &= P\{X_i(t_1) = X_i(t_2) \neq X_i(t)\}. \end{aligned}$$

By (13) and Corollary 1 we have

$$\mathrm{E}\big(X_i(t)-\mathrm{E}X_i(t)-X_i(t_1)+\mathrm{E}X_i(t_1)\big)^2 \leqslant \mathrm{E}\big(X_i(t)-X_i(t_1)\big)^2 \leqslant H_i(t)-H_i(t_1).$$

Now Lemma 3 yields, for $0 \leqslant t_1 \leqslant t \leqslant t_2 \leqslant c$,

$$\mathrm{E}\big(X_i(t)-\mathrm{E}X_i(t)-X_i(t_1)+\mathrm{E}X_i(t_1)\big)^2\big(X_i(t_2)-\mathrm{E}X_i(t_2)-X_i(t)+\mathrm{E}X_i(t)\big)^2 \leqslant A\big(H_i(t_2)+t_2-H_i(t_1)-t_1\big)^{\beta},$$

where A depends on c and $\beta = \min(2\alpha, 3/2)$. Now using Theorem 2 from [4] we infer that X_i $(i = 0, 1)$ satisfy the CLT in $D[0, c]$, $c > 0$, with the Skorohod topology. Hence, in view of Theorem 3′ in [5], we conclude that X_i $(i = 0, 1)$ satisfy the CLT in (D, d). This completes the proof of Theorem 3.

Using the arguments analogical to those in [1] (see p. 151-153) it can be shown that if F_0, G_0, F and G are continuous, then X does not satisfy the CLT in D endowed with the topology of uniform convergence on compacta in $[0, \infty)$. It is a consequence of the fact that

$$\frac{1}{\sqrt{n}}(X_1+X_2+\ldots+X_n),$$

with X_i being independent copies of X, is a random element of (D, d) but not a random element of D with the topology of uniform convergence on compacta in $[0, \infty)$.

The author wishes to thank Professor C. Ryll-Nardzewski for his helpful comments and Dr. T. Byczkowski for discussions and critical reading of the manuscript.

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Received on 22. 6. 1979